Garrett Birkhoff and M. K. Bennett

Felix Klein and His "Erlanger Programm"

1. Introduction

Felix Klein's "Erlanger Programm" (E.P.), listed in our references as (Klein 1872), is generally accepted as a major landmark in the mathematics of the nineteenth century. In his obituary biography Courant (1925) termed it "perhaps the most influential and widely read paper in the second half of the nineteenth century." Coolidge (1940, 293) said that it "probably influenced geometrical thinking more than any other work since the time of Euclid, with the exception of Gauss and Riemann."

In a thoughtful recent article, Thomas Hawkins (1984) has challenged these assessments, pointing out that from 1872 to 1890 the E.P. had a very limited circulation; that it was "Lie, not Klein" who developed the theory of continuous groups; that "there is no evidence...that Poincaré ever studied the Programm;" that Killing's classification of Lie algebras (later "perfected by Cartan") bears little relation to the E.P.; and that Study, "the foremost contributor to...geometry in the sense of the Erlanger Programm,...had a strained and distant relationship with Klein."

Our paper should be viewed as a companion piece to the study by Hawkins. In our view, Klein's E.P. did have a major influence on later research, including especially his own. Moreover, Klein was the chief heir of five outstanding Germanic geometers, all of whom died within the decade preceding the E.P.: Möbius (1790-1868), Steiner (1796-1869), von Staudt (1798-1867), Plücker (1801-68), and Clebsch (1833-72).

Klein's close friendship with Lie at the time of the E.P. played an important role in the careers of both men. There is much truth in the frequently expressed idea that Klein's studies of 'discontinuous' groups were in some sense complementary to Lie's theory of 'continuous' groups (Coolidge 1940, 304). Less widely recognized is the fact that in the E.P. and related papers, Klein called attention to basic *global* aspects of geometry, whereas Lie's theorems were purely *local.* After reviewing these and other aspects of Klein's relationship to Lie, we will trace the influence of his ideas on Study, Hilbert, Killing, E. Cartan, and Hermann Weyl.

In discussing these developments, we have tried to follow a roughly chronological order. We hope that this will bring out the well-known evolution of Klein's scientific personality, from that of a brilliant, creative young geometer to that of a farsighted organizer and builder of institutions in his middle years, and finally to that of a mellow elder statesman or "doyen" and retrospective historian of ideas.[1]

2. Klein's Teachers

Klein's E.P. surely owes much to his two major teachers: Plücker and Clebsch. Already as a youth of 17, Klein became an assistant in Plücker's physics laboratory in Bonn. Though primarily a physicist, Plücker was also a very original geometer. Forty years earlier, he had written a brilliant monograph on analytic projective geometry (*Analytisch-geometrische Entwickelungen*, 1828, 1832), which established the use of homogeneous coordinates and the full meaning of duality.

Still more original (and more influential for Klein) was Plücker's "geometry of lines," first proposed in 1846. Plücker proposed taking the self-dual four-dimensional manifold of all *lines in* $\mathbf{R}^3$ as the set of basic "elements" of geometry. In it, the sets of all "points" and of all "planes" can be defined as what today would be called three-dimensional algebraic varieties. Klein's Ph.D. thesis (directed by Plücker) and several of his early papers dealt with this idea.

Shortly before Klein finished his thesis when still only 19, Plücker died. Clebsch, who had just gone to Göttingen from Giessen, invited Klein to join him there soon after. Barely 35 himself, Clebsch became Klein's second great teacher. After making significant contributions to the mechanics of continua (his book on elastic bodies was translated into French and edited by St. Venant), Clebsch had introduced together with his Giessen colleague Gordan (1831-1912) the *invariant theory* of the British mathematicians Boole (1815-64), Cayley (1821-95), and Sylvester (1814-97).

Clebsch assigned to Klein the task of completing and editing the second half of Plücker's work on the geometry of lines (*Neue Geometrie des Raumes gegrundet auf die Betrachtung der geraden Linie als Raumelemente*). As we shall see, Klein also absorbed from Clebsch both the concept of geometric invariant and Clebsch's interest in the "geometric function theory" of Riemann, whose intuitive approach contrasted sharply with the uncompromising rigor of Weierstrass.

Soon after coming to Göttingen from Giessen, Clebsch founded the *Mathematische Annalen* (M.A.) together with Carl Neumann in Leipzig. The first volume of this journal contained six papers by Clebsch and Gordan, along with others by Beltrami, Bessel, Brill, Cayley, Hankel, Jordan, Neumann, Sturm, Weber, and Zeuthen. Of these, Jordan's "Commentaire sur Galois" (151-60) and his paper on Abelian integrals (583-91) are especially relevant for the E.P., as forerunners of his 667-page treatise *Traité des substitutions*, which appeared a year later. In the preface of this great classic on the theory of substitution groups, Jordan thanked Clebsch for explaining how to "attack the geometric problems of Book III, Chap. III, the study of Steiner groups, and the trisection of hyperelliptic functions."

In 1869-70, Klein went to Berlin, against the advice of Clebsch. Although he found its intellectual atmosphere, dominated by Weierstrass and Kronecker, very uncongenial, it was there that Klein met two fellow students who deeply influenced his later development.

The first of these was Stolz. In his (EdM, vol. 1, 149), Klein states that he had been greatly impressed by Cayley's sententious dictum that "descriptive [i.e., projective] geometry is all geometry." Continuing, Klein states:

> In 1869, I had read Cayley's theory of metric projective geometry in Salmon's *Conics*, and heard for the first time about the work of Lobachewsky-Bolyai through Stolz in the winter of 1869-70. Although I understood very little of these indications, I immediately had the idea that there must be a connection between the two. (EdM, vol. 1, 151-52)

Klein then recalls a lecture he gave at Weierstrass's seminar on Cayley's *Massbestimmung* in projective geometry. He kept to himself his vaguely conceived thoughts about the connection of Cayley's work with non-Euclidean geometry, realizing how quickly the meticulous Weierstrass would have dismissed their vagueness, and how much he would have disliked their emphasis on the projective approach to geometry.[2]

Most important, it was during this visit to Berlin that Klein met Lie. In his own words (GMA, vol. 1, 50):

> The most important event of my stay in Berlin was certainly that, toward the end of October, at a meeting of the Berlin Mathematical Society,

> I made the acquaintance of the Norwegian, Sophus Lie. Our work had led us from different points of view finally to the same questions, or at least to kindred ones. Thus it came about that we met every day and kept up an animated exchange of ideas. Our intimacy was all the closer because, at first, we found very little interest in our geometrical concerns in the immediate neighborhood.

3. Five Dazzling Years

The next five years were ones of incredible achievement for the young Klein. During this time he wrote 35 papers and supervised seven Ph.D. theses. It was while carrying on these activities that he wrote his E.P., and so it seems appropriate to recall some of his more important contemporary contacts.

About his continuing interactions with Stolz, Klein states:

> In the summer of 1871, I was again with Stolz.... He familiarized me with the work of Lobachewsky and Bolyai, as well as with that of von Staudt. After endless debates with him, I finally overcame his resistance to my idea that non-Euclidean geometry was part of projective geometry, and published a note about it in...the Math. Annalen (1871). (EdM, vol. 1, 152)

After a digression on the significance of Gaussian curvature, Klein next recalls the background of his second paper on non-Euclidean geometry (Klein 1873). In it, Klein

> investigated the foundations of von Staudt's [geometric] system, and had a first contact with modern axiomatics.... However, even this extended presentation did not lead to a general clarification.... Cayley himself mistrusted my reasoning, believing that a "vicious circle" was buried in it. (EdM, vol. 1, 153)

The connections of Clebsch with Jordan, whose monumental *Traité des substitutions* had just appeared, surely encouraged Klein to study groups and to go to Paris in 1870, where his new friend Lie rejoined him; the two even had adjacent rooms. In Paris, the two friends again talked daily, and also had frequent discussions with Gaston Darboux (1842-1917). There they extended an old theorem of Chasles, which states that orientation-preserving rigid transformations of space are, in general, 'screw motions' of translation along an axis and rotation about it. This theorem had been previously extended by Olinde Rodrigues (see Gray 1980) and Jordan (1869). Klein and Lie wrote two joint notes in Paris, the publication of which in the *Comptes Rendus* was sponsored by Chasles. They also published a third note in the *Berliner Monatshefte* of December 1870; by this time, however, the Franco-Prussian war had forced Klein to return to Germany.

The following year, Klein and Lie published two more joint papers in the *Mathematische Annalen* (GMA XXV and XXVI), of which the second (M.A. 4: 54-84) dealt with "systems of plane curves, transformed into each other by infinitely many permutable linear transformations." Thus it was around this time that Klein and Lie began to give a new direction to geometry, by emphasizing the importance for it of *continuous groups*, about which very little had been published previously (see §8).

It was also in 1871 that Lie's dissertation, *Over en Classe Geometriske Transformatione*, appeared. Originally published in Norwegian, a German translation (by Engel) is in (LGA, vol. 1, XI). It refers repeatedly to Plücker and cites related publications of Klein (LGA, vol. 1, 106, 127, 145, 149). Although seven years older than Klein, Lie had finished his dissertation two years later.

The following year (1872) was especially momentus for Klein. Partly through the influence of Clebsch, he obtained a full professorship at Erlangen when still only 23! It was for his inauguration to this chair that Klein wrote the E.P. Before it was finished, Clebsch died of diptheria at the age of 39, and most of his students moved from Göttingen to Erlangen to work with Klein. These students included Harnack, who later proved a famous theorem in potential theory, and Lindemann, who later published the first proof of the transcendence of π. Klein immediately gave Lindemann the task of writing Clebsch's unpublished lecture notes on geometry, a task that it would take Lindemann fifteen years to complete.

Not only Clebsch and Plücker, but also Möbius, Steiner, and von Staudt had died in the five-year period of 1867-72. Thus by accident, Klein had fallen heir to a great German tradition at age 23. The later sections of our paper will explain how he met this challenge successfully, in a most original and decisive way.

4. The "Erlanger Programm"

The E.P. was above all an affirmation of the key role played by groups in geometry. The most authentic description of its background is contained in Klein's prefatory notes to the relevant section of his *Collected Papers* (GMA, vol. 1, 411-16). There Klein writes:

> In the following third part of the first volume of my *Collected Papers*...are collected those involving the concept of a *continuous transformation group*.
>
> The...first two are joint publications with Lie in the summer of 1870 and the spring of 1871, on "W-curves."[3]
>
> These were followed by my 1871 article "On Line and Metric Geometry".... As in Lie's great work "On Line and Sphere Complexes," new examples of continuous transformation groups were treated in this paper.
>
> In this connection my two papers on non-Euclidean geometry should be cited.[4]
>
> The E.P. itself...was composed in October, 1872. Two circumstances are relevant. First, that Lie visited me for two months beginning September 1. Lie, who on October 1 accompanied me to Erlangen... had daily discussions with me about his new theory of first-order partial differential equations (edited by me and published in the Gött. Nachr. of October 30). Second, Lie entered eagerly into my idea of classifying the different approaches to geometry on a group-theoretic basis.

The E.P. should not be judged as a research paper; it was a semitechnical presentation to the Erlangen philosophical faculty of ideas about geometry that Klein had discussed with Lie; it was primarily the exposition that was Klein's. The paper was published to fulfill an "obligation connected with Klein's appointment to the university" (Rowe 1983). As a new full professor, he was expected to present his colleagues with a printed exposition of some creative work. (In addition, he gave an oral *Antrittsrede* presenting his views on mathematical education; see [Rowe 1985].)

Klein's "Comparative Consideration of Recent Developments in Geometry," which was also intended to impress Clebsch, fulfilled this obligation in a most brilliant and original way. Its first section announced its main theme as follows (E.P., 67): "Given a manifold and a transformation group acting on it, to investigate those properties of figures [*Gebilde*] on that manifold which are invariant under [all] transformations of that group." In today's language, Klein proposed studying the concept of a *homogeneous manifold*: a structure $[M,G]$ consisting of a manifold M and a *group* G acting transitively on M. This contrasts sharply with Riemann's concept of a structure $[M;d]$ consisting of a manifold on which a *metric* $d(p,q)$ is defined by a local *distance* differential $ds^2 = \sum g_{ij} dx_i dx_j$.

Two paragraphs later, Klein restated his proposal in a single terse sentence: "Given a manifold, and a transformation group acting on it, to study its *invariants*." Thus Klein was also proposing to apply to geometry the concept of an 'invariant' that Clebsch, Jordan, and their predecessors had previously applied to algebra, and there only to the full linear group.

Klein began his essay by identifying each of the *continuous groups* of geometric transformations that was associated with some branch of geometry. Naturally, since the Euclidean group of rigid motions ("congruences") was familiar to his readers, he discussed it first, calling it the "*Hauptgruppe*" (chief group). Indeed, the importance of 'free mobility' had already been stressed by Helmholtz and Riemann. He then moved on to the larger projective group, the conformal group generated by inversions, the group of birational transformations leaving invariant the singularities of algebraic varieties, which contained all of the preceding groups, and the (still more general) group of all homeomorphisms leaving the topology of a manifold or space invariant. Curiously, the E.P. failed to mention the affine group and affine geometry, although Klein would later (1908) devote the first section of his chapter on geometric transformations in his *Elementarmathematik vom Höheren Standpunkt aus* to them. His failure to distinguish the group of homeomorphisms from the groups of bijections and of diffeomorphisms is less surprising, since Cantor's discovery that all $\mathbf{R}^n$ are bijective was still two years away.

Klein's audience surely did not notice these lapses. Indeed, his main new concepts, those of 'equivalence' and 'invariance' under a given group of transformations, are still hard to explain to novices today. One difficulty is that invariance and equivalence assume such a bewildering variety of forms depending on the group involved. (See H. Weyl, *The Classical Groups*, 23-26, and G. Birkhoff and S. Mac Lane, *Survey of Modern Algebra*, §9.5, for two more 'modern' attempts to explain these concepts.)

However, the main thrust of the E.P. was very clear. Jordan's *Traité des substitutions* had explained how the concept of a *group* of 'substitutions', properly applied, determines which polynomial equations can be solved by 'radicals'—i.e., by explicit formulas involving rational operations and taking of n-th roots. Klein showed similarly how the concept of a *continuous transformation group* gives an objective basis for classifying geometric theories and theorems. His few readers and listeners must have realized that they were being exposed to new and fundamental perspectives, even if they barely understood them.

5. Klein and Geometry

In 1872, the E.P. was 20 years ahead of its time; it would take at least that long for the new perspectives of Klein and Lie to gain general acceptance. (However, their ideas would take deep root; in fifty years it would become commonplace to refer also to "metric," "projective," "affine," and "conformal" *differential* geometry; see §13.)

In the meantime, Klein's rapid rise to leadership in Germanic geometry was based on his other writings. His preeminence rested on the impression he made on contemporaries, and not on what he might write for posterity. The value of his distinction between the familiar Riemann *sphere* (often associated with conformal geometry) and the *elliptic plane*, consisting of the sphere with opposite points identified, was immediately appreciated.

So was Klein's use of the name "elliptic geometry" for a manifold of constant positive curvature, as distinguished from a "hyperbolic geometry" for one having constant negative curvature, and "parabolic geometry" for Euclidean geometry and its cylindrical and toroidal "space forms" (Kline 1972, 913). (Not long after, du Bois-Reymond made an analogous classification of differential equations into those of "elliptic," "hyperbolic," and "parabolic" type.)

Klein's observations stimulated the British mathematician W. K. Clifford (1873) to call attention to the philosophical difference between local and global homogeneity, which had been overlooked by Riemann and Helmholtz. Clifford also called attention (as did Klein) to the connection between Plücker's line geometry and 'screws' (the Theorem of Chasles). However, he died before he could develop these ideas very far, and Klein would not return to them until 1890 (see §§7 and 8).

Klein also showed the logical incompleteness of von Staudt's pathbreaking introduction of coordinates into axiomatically defined "projective geometries," an observation that stimulated Lüroth and others to clarify the assumptions underlying von Staudt's "algebra of throws."[5]

Although most mathematicians today would consider the E.P. as primarily a contribution to the *foundations* of geometry, Klein did not regard it as such, perhaps because in 1872 the concept of a continuous group was so novel, and even the theory of invariants still a research frontier. His prefatory remarks in his (GMA) about his contributions to the foundations of geometry concentrate on the topics that we have discussed above. By his own reckoning (GMA, vol. 1), Klein published ten papers on the foundations of geometry, fourteen on line geometry (an interest he had inherited from Plücker), and nine on the E.P. (of which the first was with Lie). He also lists (GMA, vol. 2) sixteen papers on intuitive (*Anschauliche*) geometry; in addition, he wrote several books on geometry. In particular, his *Einleitung in die Höhere Geometrie* of 1893 gave a general and quite comprehensive picture of the geometry of the day, and the second volume of his *Elementarmathematik vom Höheren Standpunkt aus* (1908) was also devoted to geometry. But of all his geometrical contributions over the years, Klein himself apparently regarded the E.P. as his "most notable achievement" (Young 1928, v).

6. Klein and Lie

The friendship formed by Klein and Lie in the unwelcoming atmosphere of Berlin in 1869, renewed in Paris in 1870 and again in Erlangen in 1872, proved invaluable for both men and for mathematics. We have already observed that much of the inspiration for the E.P. stemmed from their discussions during these years, and we shall now describe some of the less immediate and more worldly benefits derived by Lie from this friendship.[6]

Whereas Klein had eloquently expressed, within months of conceiving it, his idea that different branches of geometry were associated with invariance under different *groups of transformations* of underlying geometric manifolds, Lie's deeper ideas would mature much more slowly. He would spend the rest of his life in developing the intuitive concept of a 'continuous group of transformations' into a powerful general theory and in applying this concept to geometry and to partial differential equations.

Max Noether, in his obituary article (1900), described Lie's and Klein's 1869-70 sojourn in Berlin and the connections between the E.P. and Lie's later work:

> In the E.P. [we find expressed] for the first time the central role of the [appropriate] transformation group for all geometrical investigations. . . and that, with invariant properties, there is always associated such a group. . . . Lie, who had worked with the most varied groups, but to whom the meaning of classification had remained foreign, found the idea congenial from then on. (Noether 1900, 23-23)

Noether called attention in a footnote to alterations made by Lie in several of his 1872 articles, apparently as a result of Klein's new ideas.

During the decade 1872-82, Lie worked in isolation in Christiania (now Oslo), encouraged almost exclusively by Klein and Adolf Mayer (1839-1908). It was then that he published the striking fact that every finite continuous group ("Lie group" in today's terminology) acting on the line is locally equivalent (ähnlich) to either the translation group of all functions $x \mapsto x+b$, the affine group of all $x \mapsto ax+b$, or the projective group of all $x \mapsto (ax+b)/(cx+d)$, $ad \neq bc$ (LGA, vol. 5, 1-8; Gött. Nachr. 22 [1874]: 529-42). (For an annotated summary, see [Birkhoff 1973, 299-305]. Actually, the theorem is only true locally.)

Lie's thoughts at this time are revealed in a letter of 1873 to Mayer, which states: "I have obtained most interesting results and I expect very many more. They concern an idea whose origin may be found in my earlier works with Klein: namely, to apply the concepts of the theory of substitutions to differential equations" (LGA, vol. 5, 584). Four years later, Lie determined locally (almost) all finite continuous groups acting transitively on a two-dimensional manifold, the next step toward determining all the homogeneous manifolds (or "spaces") envisioned in the E.P.[7]

This was the third in a series of five definitive papers, published in Christiania in the years 1876-79, in which Lie laid the foundations of his theory of continuous groups. The introduction to the first of these states in part (LGA, vol. 5, 9):

> I plan to publish a series of articles, of which the present one is the first, on a new theory that I will call the *theory of transformation groups*. The investigations just mentioned have, as the reader will notice, many points of contact with several mathematical disciplines, especially with the theory of substitutions,[1] with geometry and modern manifold theory,[2] and finally also with the theory of differential equations.[3]
>
> These points of contact establish connections between these former separate fields. . . . I must prepare later articles to present the importance and scope of the new theory.
>
> [1]See Camille Jordan's *Traité des substitutions (Paris, 1870).* Compare also Jordan's investigations of groups of motions.
>
> [2]See various geometric works by Klein and myself, especially Klein's [E.P.], which hitherto has perhaps not been studied sufficiently by mathematicians.
>
> [3]See my investigations on differential equations.

To reach a wider mathematical audience, Lie summarized his new theory of transformation groups in an 88 page paper (Lie 1880) published in German (LGA, vol. 6, I, III). However, even after this, it took another decade for Lie's ideas to be digested and their profound implications seen by most mathematicians. Lie became discouraged, and it seems clear that he owed much to Klein's continuing encouragement and insightful comments during these years (Rowe 1985).

As we shall see in §7, it was thanks largely to Klein's initiative that Lie obtained the cooperation of Friedrich Engel (1868-1941), the coauthor of his magnum opus (and of his posthumous collected works). Without Engel's expository cooperation, the dissemination of Lie's deep new ideas and methods would probably have been much slower.

Lie's relationship with Klein as well as the value for Lie of the latter's extensive correspondence with Mayer (much of it reproduced in [LGA]) has been described by Engel (1900). There Mayer is credited with persuading Clebsch of the value of Lie's new "integration method" (p. 37), and Lie's exchange with Mayer is presented as an example of two mathematicians "making the same discovery independently. . . and almost simultaneously."

Engel went on to describe Lie's "invariant theory of contact transformations," which appeared almost immediately after the E.P. as a completely new theory, entirely due to Lie. On the other hand, like Max Noether, he described Klein's idea that *many* domains of mathematics could be presented as invariant theories of appropriate groups as new and surprising to Lie.

Killing. Klein was also instrumental in establishing contact between Lie and Wilhelm Killing (1847-1923), who was a contemporary of Lie and Klein. As of 1884, Killing was groping toward concepts closely related to those of Lie. Hawkins (1982, §2) has described Klein's benign intervention as follows:

> Shortly after Killing posted a copy of his *Programmschrift* to Felix Klein in July of 1884, Klein informed him that its contents seemed to be closely related to the theory of transformation groups of his friend, Sophus Lie. . . .
>
> Upon learning about Lie, Killing sent him a copy of his *Programmschrift*. . . . Lie apparently did not respond to Killing's overture and definitely did not reciprocate and send Killing some copies of his own

> work. . . . Lie quickly published a note in the *Archiv* [1884] in which he showed how a theorem stated without proof by Killing in [1884] could be derived from some of his previously published results.

Nevertheless, Killing was not discouraged, and with some help from Engel (see §7), he persevered in his effort to classify Lie algebras.

7. Leipzig and Lie Groups

Among the first to appreciate Lie's theory of contact transformations was Adolf Mayer at the University of Leipzig. Mayer had published papers developing Lie's ideas from 1872 on, first on contact transformations and then on Lie's methods for integrating differential equations invariant under a group.

In 1876, Mayer in Leipzig and Klein (then in Munich) had taken over from Carl Neumann the direction of the *Mathematische Annalen*, and in 1882 Klein joined Mayer in Leipzig. In the next year, Engel wrote there his inaugural dissertation *Zur Theorie der Berührungstransformationen* (Teubner, 1883); in the following year, Klein and Mayer sent Engel to Christiania to work with Lie. There Lie could inspire Engel, while Engel could help the lonely and somewhat disorganized Lie to organize and write up his profound discoveries in a systematic and readable form. Engel recalled later that in 1883, apart from his old friends F. Klein and A. Mayer, almost no one was interested in the group theory of which Lie was rightfully very proud. In 1883, only Picard recognized the significance of groups explicitly (*offentlich*) (Engel 1900, 42).[8]

Still later, Engel published (LGA, foreword to vol. 5) a vivid description of his year in Christiania with Lie, with whom he had "two conversations daily." Engel saw that Lie's seventeen published papers gave only a "very incomplete picture of the great buildings which he had in mind," and was inspired to "work with all his strengths" to bring them "as near to completion as possible." Future historians of mathematics should find Engel's careful analysis in this foreword of Lie's publications and Lie's opinions about them invaluable. A similar remark applies to the introduction to (Engel and Faber 1932).

Whereas Engel's inaugural dissertation had been merely competent, his *Habilitationsschrift* of 1885 (Teubner) showed genuine originality. In its introduction, Engel wrote:[9]

> A long stay in Christiania gave me the opportunity, in personal exchange with Sophus Lie, to study in depth [*eingehend*] his theory of continuous transformation groups. Our common goal was to provide a coherent presentation of this theory, my share of the work being essentially only expository. In the process, however, I also occupied myself with some self-contained investigations in this area. My results will be explained in the course of this article. At present, our intended coherent presentation of the theory is not nearly finished... To understand what follows would, therefore, require knowing quite a few of Lie's papers on transformation groups. To minimize this difficulty, a brief summary will be given next to the principal concepts and theorems of Lie's theory, insofar as they will be needed below.

One year later, and to the intense displeasure of Weierstrass, Klein arranged to be replaced by Lie at the University of Leipzig upon his departure for Göttingen. Lie's move to Leipzig, where he was with both Mayer and Engel, proved to be most fruitful. It made it practical for Engel to serve as Lie's disciple, a role that lasted until he finished acting as coeditor with Paul Heegard of Lie's *Collected Works* (LGA) in 1924-34.

Indeed, the first volume of Lie-Engel was completed three years later, and in 1889 Fr. Schur (1856-1932) gave the first rigorous treatment of the *abstract* theory of (local) Lie groups. By 1893, all three volumes of Lie-Engel had appeared, and the (local) theory of Lie groups was firmly established.

In the meantime, Killing published his book *Zur Theorie der Lie'schen Transformationsgruppen* (1886), in which he determined almost all simple Lie algebras (see Hawkins 1982). Its preface states:

> Herr Klein kindly called my attention to the close connection between my investigations and [earlier papers of Lie on finite continuous transformation groups, their treatment by integration methods, and contact transformations].
>
> Now, at the time that my [1884] monograph appeared, I had begun another work, which occupied me longer than I had initially expected. After finishing this... I immediately began to study Lie's work.
>
> There can be no doubt of Lie's priority [as regards many parts of my earlier work]. I can only express my joy that the many-sided researches of Lie have so essentially advanced the general theory [of homogeneous spaces].

Unfortunately, Lie was not happy in Leipzig, and there he became excessively jealous of possible rivals. Thus the introduction of the third volume of Lie-Engel goes out of its way to assert that Lie was "not Klein's student," but that "rather, the contrary was the case."[10] Likewise, in a paper reproduced in (LGA, vol. 2, 472-79), he takes pains to identify gaps in the reasoning of de Tilly, Klein (M.A. 37: 364), Lindemann, Fr. Schur (who had, on the contrary, actually rigorized Lie's somewhat cavalier differentiability assumptions), Helmholtz, and Killing!

A turning point seems to have come when Lie was awarded the first Lobachewsky Prize in 1893 for solving the Riemann-Helmholtz problem (a celebrated problem, which, incidentally, Klein had suggested he work on). Namely, Lie had shown that any n-dimensional Riemannian manifold admitting an $n(n + 1)/2$-parameter group of rigid motions (the "free mobility" condition of Helmholtz) is locally isometric to either Euclidean n-space, the n-sphere, or the n-dimensional "hyperbolic" geometry of Lobachewsky-Bolyai. This classic *local* result of Lie stands in sharp contrast with Klein's continuing concern with the *global* Clifford-Klein problem, to which we will return in §8.

Appropriately, Klein was invited to write a suitable appreciation of Lie's solution for the occasion of the prize presentation, which he did with his usual imaginative, insightful style. His narrative contained, however, one complaint: the presence of an unmotivated, and to Klein unnatural, assumption of differentiability in the foundations of geometry.

Klein's complaint was given a positive interpretation by Hilbert. As the fifth in his famous list of unsolved problems proposed at the 1900 International Mathematical Congress, Hilbert proposed proving that any *continuous* (locally Euclidean) group was in fact an *analytical* group with respect to suitable parameters. Whether this problem should be attributed to Hilbert, Klein, or Lie, its successful solution took another 50 years, and Klein's E.P. was at least one of its indirect sources (see [Birkhoff and Bennett, forthcoming]).

8. Klein and Discontinuous Groups

Much as Lie found in invariance under continuous groups of transformations new ideas for integrating differential equations, Klein found in discontinuous groups new ideas not only for solving algebraic problems (as Jordan had before him), but also (later) for clarifying and extending the 'geometric function theory' originated by Riemann and advanced by

Clebsch. Indeed, already in 1875, Klein had supervised an Erlangen thesis by Harnack on elliptic functions and another by Wedekind in the next year entitled *On the Geometric Interpretation of Binary Forms*. A summary of Wedekind's thesis appeared (M.A. 9: 209-17),[11] immediately preceded by a paper (pp. 183-208) in which Klein used the Schwarz reflection principle to construct Riemann surfaces from regular polygons on the complex sphere. The special case of a rectangle, of course, leads to the elliptic function *sn z*, whose Riemann surface is a torus.

On page 193 of this paper, Klein first associated the regular octahedron with a biquadratic form, and then related the symmetric group of all permutations of five letters to the regular icosahedron as follows: "The 15 planes that pass through the center of the icosahedron and four pairwise antipodal vertices can be divided into 5 triples of orthogonal planes. The 15 lines in which these triples of planes intersect cut the sphere in 30 'doubly counted' points."[12]

Discussions of this paper with Gordan stimulated Klein to write a sequel entitled "On the Icosahedron" (M.A. 12 [1877]: 503-60), in which he "derived the theory of the quintic equation from geometric properties of the icosahedron." This turned out to have interesting connections with work of Jordan (see M.A. 11: 18) and Brioschi (M.A. 13: 109-60),[13] stemming from "Jacobi's berühmte Aufsätze in 3ten and 4ten Bände von Crelle's Journal" (ca. 1830).

Klein followed up these early efforts by a series of papers on elliptic modular functions and related topics during the years 1877-84 (GMA, vol. 3, 3-316). Klein refers to "the special lectures that, during the years 1877-80, I gave on number theory, elliptic functions, and algebraic equations, involving geometric group theory.... The audience included Gierster, Dyck, Bianchi, and Hurwitz" (p. 5). These lecture were given at the Technische Hochschule (now the Technische Universität) in Munich, for which Klein had left Erlangen in 1875. Thus in Munich Klein developed his ideas about discontinuous groups and geometric function theory "in a larger and more responsive circle of workers" (Courant 1925, 201).

It seems likely that Klein formed while in Munich the "genial idea" of becoming, in Courant's words:

> the pathfinder for the Mathematics and Mathematical Physics of the future.... In the depths of his soul,...he found the intuitive formulation of geometrical connections congenial. Klein was the most painstaking and effective apostle of Riemann's spirit... If mathematics can

build higher today on Riemann's foundations in tranquil clarity, this is thanks to Klein's special service. (Courant 1925, 202)

The connection of (periodic) trigonometric functions with the cylinder and of doubly periodic *elliptic functions* with the torus is obvious. It was apparently around 1880 that Klein first recognized analogous connections between other *automorphic functions* and the hyperbolic plane. We will return to this idea, which deserves a much more thorough historical study than we have had time to make, in the next section.

It was also at about this time that Klein, after taking over the editorship of the *Mathematische Annalen* with his friend A. Mayer, began to have assistants, Gierster and Dyck being the first two. Both wrote doctoral theses under Klein's guidance in the years 1879-81, as did Hurwitz. Dyck became coeditor of the M.A. in 1886, soon after becoming a professor in Munich, and remained in that position until 1919 when Einstein replaced him. (Hilbert had become coeditor in 1902). Dyck was also director of the Technische Hochschule during 1900-1906 and 1919-25, receiving the title of *von* Dyck (the German pre-1918 equivalent of being knighted) for his leadership.

Hurwitz went on to have an even more distinguished mathematical career; his collected works are available in two volumes. His early work shows clearly the influence of Klein's ideas, and his *Funktionentheorie* (1929), coauthored by Courant, also reflects Klein's ideas about geometric function theory, as seen in retrospect.

9. Klein and Poincaré

For a decade after writing the E.P., Klein's brilliance seemed unrivaled. By 1882 he had achieved leadership in geometry, with Gaston Darboux (1842-1917) as his closest rival. Lie's reputation was not yet comparable. Indeed, as we have seen, Klein was in 1882 in some sense Lie's patron, a status that was later to rankle Lie. Klein was also gaining in reputation as an analyst, through his contributions (and those of his students Hurwitz and Dyck) to Riemann's 'geometric function theory'. Indeed, in 1882, Teubner had already published Klein's first monograph on this subject.

Then suddenly, still in his early thirties, Klein became outshone by the incredibly original and versatile French mathematician Henri Poincaré (1854-1912). Although only five years younger than Klein, Poincaré was nearly ten years his mathematical junior. Poincaré had taken time off from his studies to help his physician-father during the Franco-Prussian war;

furthermore, it took longer to complete a doctorate in France than in Germany. As a result, Poincaré did not receive his Ph.D. from the University of Paris until 1879—ten years after Klein's degree was awarded.

Then, after two years in Caen, Poincaré returned to Paris. There he quickly published a series of notes on what are today called automorphic functions,[14] soon expanding on these notes in 1882-84 in a celebrated series of papers published in the first five volumes of *Acta Mathematica*. Morris Kline, in his informative discussion of automorphic functions (Kline 1972, 726-29), reports that Kronecker had tried to dissuade Mittag-Leffler from publishing Poincaré's first paper for fear that "this immature and obscure article would kill the journal" (p. 728).

In Poincaré's own words, the story of his first major breakthrough was as follows:

> For fifteen days I strove to prove that there could not be any functions like those I have since called Fuchsian functions. . . . One evening, contrary to my custom, I drank black coffee and could not sleep. Ideas rose in crowds; I felt them collide until pairs interlocked, so to speak, making a stable combination. By the next morning I had established the existence of a class of Fuchsian functions, those which come from the hypergeometric series: I had only to write out the results, which took but a few hours. (Poincaré 1907-8, 647-48)

Early on, Poincaré also published (in M.A. 19: 553-64) a synopsis of his ideas on the subject as of December 1881. This contains frequent references to "fuchsian" and "kleinian" functions, and it was followed by two short notes by Klein (M.A. 19: 565-68; 20: 49-51) on "functions that reproduce themselves under linear transformations," to which Klein would give their current name of "automorphic functions" in 1890. Of special historical interest are Klein's editorial comments (M.A. 19: 564) on Poincaré's synopsis. He questioned the relevance of the work of Fuchs; Poincaré later explained this relevance.

Curiously, Poincaré's "kleinian functions" refer to the parabolic case to which Klein had contributed little; this was a historical accident. Poincaré had been influenced by Fuchs's papers on solutions of (homogeneous) linear differential equations in the complex domain before he became aware of Klein's work, so he referred to "fuchsian functions" for the richer hyperbolic case to the half-plane. Klein complained of the inappropriateness, so Poincaré corrected his omission by later giving Klein's name to a much less novel class of functions.

It is clear that, at the time, Klein was having serious health problems and that he suffered a real breakdown. Young (1928, vii-viii) attributes the breakdown at least partially to "the antagonism he experienced at Leipzig" (at the hands of colleagues jealous of his rapid academic promotion). Young (1928, vii) and Row (1985, 288) both attribute the breakdown to overwork, aggravating an asthmatic condition. Young also reports that, while suffering from asthma, Klein proved the theorem that he himself "prized highest among his mathematical discoveries, known as the *'Grenzkreistheorem'* in the theory of automorphic functions."

Whatever the cause and nature of his problems, these years marked a turning point in Klein's career; he never regained the remarkable research activity of his earlier years. However, his GMA lists sixteen theses written at Leipzig under his direction, including those of Hurwitz (1881), Fine and Fiedler (1885), and his nephew Fricke (1886).

10. Years of Transition

Klein's first major act after recovering from his breakdown was to publish his famous book *Das Ikosaeder* (Klein 1884), which explained to mathematicians at large some of the connections that he had discovered between algebra, geometry (Euclidean and non-Euclidean), and analysis. Two years later, he left Leipzig to join H. A. Schwarz (1843-1921) in Göttingen.

In Göttingen, Klein continued to work on discontinuous groups. Especially, he constructed various new automorphic functions by multiple Schwarz reflections in the edges of regular circular polygons ("fundamental regions") and their images. In collaboration with his nephew Robert Fricke (1861-1930), he wrote up his ideas in classic treatises on the elliptic modular function and automorphic functions. The first of these is still studied today because of its applications to algebraic number theory.

By 1890, Lie's profound results on continuous groups and Klein's continuing applications of discontinuous groups had stimulated Fano to translate the E.P. into Italian. A French translation (by Padé) and an English translation by Klein's American student M. Haskell soon followed. Klein's foreword to the English translation states that his E.P.

> had but a limited circulation at first.... But...the general development of mathematics has taken, in the meanwhile, the direction corresponding precisely to these views, and particularly since *Lie* has begun the publication...of his *Theorie der Transformationsgruppen* (vol. I,

1888, vol. II, 1890) it seems proper to give a wider circulation to my Programme.

Klein's foreword to the German republication of the E.P. (M.A. 43 [1893]: 63-100) goes further, stating his desire "to include the collected applications of the theory of manifolds... not only to geometry, but also to mechanics and mathematical physics," and to work in "much material... which has been added in the intervening 20 years, naturally Lie's theory of continuous groups in particular, but also geometric connections that are implicit in the theory of automorphic functions." However, he concluded that this was simply too big a task.

American Influence. Klein's beneficial influence on American mathematicians seems to have begun in 1883-85, when F. N. Cole from Harvard and H. B. Fine from Princeton came to Leipzig (Archibald 1938, 100, 167). "There [Fine] attended lectures and seminars of Klein, Mayer, Fr. Schur, Carl Neumann, and W. Wundt (philosophy)." He also "wrote a thesis on a topic approved by Klein but suggested by Study, later one of Fine's closest friends."[15] After his return, Fine went on to lead the development of Princeton into one of the world's greatest mathematical centers. As regards Cole, Archibald states:

> After two years under Klein at Leipzig, Cole spent the next three years at Harvard, where his career as an undergraduate had been so brilliant. Aglow with enthusiasm, he gave courses in modern higher algebra, and in the theory of functions of a complex variable, geometrically treated, as in Klein's famous course of lectures at Leipzig in 1881-82. He was the first to open up modern mathematics to Prof. Osgood as a student, who characterized the lectures as "truly inspiring." Another student, M. Bôcher, as well as nearly all members of the Department, Profs. J. M. Peirce, B. O. Peirce and W. E. Byerly attended his lectures. He received the doctor's degree from Harvard on a topic suggested by Klein. . . .

A professor at Columbia for 31 years, Cole served as secretary to the American Mathematical Society from 1896 to 1920.[16]

Before 1892 Klein had attracted a steady stream of American graduate students to Göttigen. These included M. W. Haskell, M. Bôcher, H. S. White, H. D. Thompson, and E. B. van Vleck, three of whom later became presidents of the American Mathematical Society. In 1893, Klein was chosen by the German government to head a delegation sent to the International Congress held in Chicago in conjunction with the World's Fair. One senses Klein's influence in the choice of those who presented papers. These included Weber, Hurwitz, Study, Meyer, Netto, Max Noether, Pringsheim, Fricke, Minkowski, and Hilbert (see *Bulletin of the American Mathematical Society* [1893]: 15-20).

Klein was made honorary president of the congress and gave a special series of lectures after it (Klein 1893b). These give a very readable account of Klein's views about many of the topics we have been discussing. To quote from W. F. Osgood's foreword to the 1911 edition: "His instinct for that which is vital in mathematics is sure, and the light with which his treatment illumines the problems here considered may well serve as a guide for the youth who is approaching the study of the problems of a later day."

11. Klein as a Leader

Meanwhile, in 1892, Schwartz left Göttingen to become the successor of Weierstrass in Berlin. From then on, Klein's gift for leadership increasingly dominated his activities. As Courant has written:

> When Schwarz went to Berlin in 1892, giving Klein a free hand in Göttingen, there began a new period of activity, in which his organizational involvement became more and more prominent.... The word *organize* meant for Klein not ruling by power: it was a symbol of deep insight and understanding. (Courant 1925, 207)

From that time until the outbreak of World War I, Klein was extraordinarily influential.

His influence on American mathematics continued. Two more Americans (F. S. Woods and V. Snyder) wrote Ph.D. theses under Klein's at least nominal direction. Fine, Bôcher, White, van Vleck, Woods, and Snyder were all active for many years in the American Mathematical Society, and Klein's influence on Haskell, Bôcher, and van Vleck in particular was considerable.

In addition, Klein was the key architect and organizer of: (i) a major expansion in the importance of Göttingen as a center of mathematical activity, (ii) the publication of the *Enzyklopädie der Mathematischen Wissenschaften* (EMW), and (iii) various 'reforms' in the style, standards, and substance of German mathematical education.

Klein and Göttingen. During these years, Klein was busy rebuilding Göttingen, the work place of Gauss and Riemann, into a preeminent world center of mathematics.[17] One of his first activities was to organize an international commission to fund a monument to Gauss and the physicist W. Weber, who had collaborated in constructing an early telegraph. (It was to honor the dedication of this monument that Hilbert, at Klein's invitation, wrote his famous *Grundlagen der Geometrie* [see §12].)

In 1895 Klein invited Hilbert to Göttingen, and Hilbert accepted with alacrity. Although Klein continued to give masterful advanced expository lectures, Hilbert was soon attracting the lion's share of doctoral candidates (see §12). Indeed, by 1900 Klein had become primarily a policy-maker and elder statesman, although barely 50. In this role, Klein obtained governmental and industrial support from an Institute of Applied Mathematics, with Prandtl and Runge as early faculty members. Sommerfeld, at one time his assistant and later coauthor with Klein of *Die Kreisel* ("The Top"), was another link of Klein with applied mathematics and physics.[18]

Klein and Education. Already in his *Antrittsrede* (see §4), Klein had expressed his concern about separation into humanistic and scientific education, stating that: "Mathematics and those fields connected with it are relegated to the natural sciences, and rightly so.... On the other hand, its conceptual content belongs to neither of the two categories" (Rowe 1985, 135). Klein's later involvement with German educational policy-making is described in (Pyenson 1983); in fact, Klein is the main subject of two of its chapters.

Klein was a universalist who believed strongly in integrating *pure* with *applied* mathematics, in the importance of both *logic* and *intuition* in geometry, and in the importance of having high-school teachers who understood and appreciated higher mathematics. Especially widely read by high-school teachers were his 1895 lectures on *Famous Problems in Elementary Geometry*, written for this purpose and translated into English, French, and Italian. (See R. C. Archibald, *American Mathematical Monthly* 21 [1914]: 247-59, where various slips were carefully corrected.)

In 1908, Klein became the president of the International Mathematical Teaching Commission. In this capacity he worked closely with the American David Eugene Smith and the Swiss Henri Fehr to improve mathematics education throughout the western world. As with other cooperative enterprises, this one was ended by World War I.

Klein and the Encyclopedia. One of Klein's most important legacies to the mathematical world was the EMW, to which we referred above. We have Young's first-hand description of Klein's rationale for his project (Young 1928, xiii):

> One day in the '90's the concept of the Enzyklopädie was formulated by Klein in the presence of the writer: the progress of mathematics, he said, using a favourite metaphor, was like the erection of a great tower; sometimes the growth in height is evident, sometimes it remains apparently stationary; those are the periods of general revision, when the advance, though invisible from the outside, is still real, consisting in underpinning and strengthening. And he suggested that such was the then period. What we want, he concluded, is a general view of the state of the edifice as it exists at present.

Klein himself edited the volume on mechanics.

Almost 35 years elapsed from the time Klein conceived his plan for an encyclopedia, to be published in French as well as German, to its completion in the late 1920s with articles surveying advances of the preceding two decades. From around 1905 to at least 1935, it truly lived up to its name. Not surprisingly, Fano wrote for it an article on "Continuous Groups and Geometry," which summarizes developments stemming from the E.P. Since the contents of this article cover somewhat the same topics as our §§1-7, we shall defer its discussion until we take up Élie Cartan's 1912 revision of it in §13 below.

12. Klein and Hilbert

The forty years that Klein spent at Göttingen transformed it into an almost legendary center of pure and applied mathematical research. One of the key figures in this transformation was David Hilbert (1862-1943), a mathematical genius who may have owed more to Klein than he cared to realize.[19] As we have stated, Klein brought Hilbert to Göttingen in 1895 to replace Heinrich Weber. Having studied with Lindemann and Hurwitz, Hilbert was in some sense Klein's "academic grandson"; moreover, he had gone to Paris on Klein's advice (in 1886), at that time arguably the world's greatest center of mathematical research.

Klein's example may also have stimulated Hilbert to broaden his research interests after coming to Göttingen, Hilbert having previously devoted his mathematical genius almost exclusively to invariant theory and algebraic number theory (Weyl 1944, 635). Soon after arriving in Göttingen, he showed in a letter to Klein that the Laguerre-Cayley-Klein projective metric defined by them in general ellipsoids had an analogue in arbitrary convex bodies. A few years later, he vindicated the "Dirichlet Principle" of Riemann, which had been discredited by Weierstrass.

By 1898, Hilbert had largely taken over the supervision of Ph.D. theses at Göttingen, of which no fewer than 60 were written under his direction between then and 1916. This was also the year in which Klein's plans for the Gauss-Weber Denkmal matured, and he invited Hilbert to be one of the two speakers to celebrate the great occasion. Hilbert chose to speak on the foundations of Euclidean geometry. His lecture notes of the previous winter on the subject, hastily polished, became the first edition of his famous *Grundlagen der Geometrie.* This book, now in its tenth edition, concluded with a study of 'constructability with ruler and compass' that is closely related in theme (though not in style) to Klein's beautiful exposition of the same subject in his "Ausgewählten Kapiteln..." of 1895.

Hilbert did not refer to this at all, an omission almost amounting to a discourtesy to a senior colleague. However, since a major stimulus for Klein's lectures had been the simplifications by Hilbert, Gordan, and Hurwitz of Lindemann's original (1882) proof of the transcendence of π, this was perhaps only fair, although a reference to Klein's brilliant booklet would have been gracious.

The purely *formal* approach of Hilbert's *Grundlagen* contrasts sharply with Klein's emphasis on the *intuitive* visualization of geometric ideas, and it is interesting to recall what Klein had to say about Hilbert's *Grundlagen* in his 1908 *Elementarmathematik vom Höhere Standpunkt aus*, vol. 2. After a brief review (pp. 130-59) of the E.P. and some of his later ideas (cf. §5), Klein discusses other approaches to geometry. Among these, his book takes up last the "modern theory of geometric axioms," observing that (p. 185):

> In it, we determine what parts of geometry can be set up without using certain axioms, and whether or not, by assuming the opposite of a given axiom, we can also secure a system free from contradiction, that is, a so-called 'pseudo-geometry.'
>
> As the most important work belonging here, I should mention Hilbert's *Grundlagen der Geometrie*. Its chief aim, as compared with earlier investigations, is to establish, in the manner indicated, the significance of the axioms of continuity.

It does seem curious that finally, near the end of his life, Hilbert should have apparently forsaken his extreme formalism and written (with S. Cohn-Vossen) a book entitled *Anschauliche Geometrie* ("Intuitive Geometry"). (For a fuller account of the *Grundlagen der Geometrie*, its background and influence, see [Birkhoff and Bennett, forthcoming].)

Hilbert's social and scientific personalities were very different from those of the dignified and highly intuitive Klein, and there is little doubt about Hilbert's restiveness as regards Klein's regal manner. Thus in a letter to his future wife, Courant wrote in 1907-8 that "Hilbert now rebels everywhere against Klein's assumed dictatorship" (Reid 1976, 19). Likewise, Ostrowski (coeditor of vol. 1 of Klein's GMA) wrote one of us in 1980: "As to Hilbert I do not think that you will find any reference to the Erlanger Programm. As a matter of fact, Hilbert did not think very much of it."

13. Study and Élie Cartan

Three more major mathematicians whose work reflects the influence of the E.P. are Eduard Study (1862-1930), Élie Cartan (1869-1951), and Hermann Weyl (1885-1955). Although they had related interests, they had very different backgrounds and tastes. We shall discuss next the influence of the E.P. on Study and Cartan, taking up its influence on Weyl in §14.

Eduard Study. As a geometer, Study was more influential than either Killing or Engel. Blaschke dedicated the first volume (1921) of his famous *Vorlesungen über Differentialgeometrie* to Study, and Study's Ph.D. students included not only Fine but also J. L. Coolidge, the author of several widely read books, whose *History of Geometrical Methods* (Coolidge 1940) is a standard reference. From the chapter "Higher Space Elements" in this treatise, we quote the following passage:

> The connecting thread in [this chapter] is the idea of treating directly as a space element some figure previously treated as a locus. The idea of doing this was dominant in geometrical circles, especially in the schools of Klein and Study, at the end of the nineteenth century.... It shades off imperceptibly into the theory of geometrical transformations. (Coolidge 1940, book 2, chap. 6)

This idea, obviously generalizing Plücker's "line geometry," can be used in the spirit of the E.P. to construct many "global" representations of continuous groups as transformation groups of manifolds.

Study began his career at the University of Munich, where he wrote his doctoral thesis *Ueber die Massbestimmungen Extensiver Grössen* in

1885. It was concerned with metric magnitudes in Grassmannian geometry, while his *Habilitationsschrift* dealt with the geometry of the conic section. Although Klein had been at Munich some years earlier, and Klein's student Dyck was to go there a year later, the only reference to Klein in either paper is the statement: "One sees that the geometric interest of this formulation of the problem has the closest connections with the researches of Riemann, Helmholtz, and Klein."

More relevant to the E.P. were Study's investigations on the three-dimensional Euclidean group and its subgroups (M.A. 39 [1891]: 444-566; 60 [1905]: 321-77). Its most general one-parameter subgroup is the group of helical or "screw" motions, previously studied by Klein and Lie. Stemming from these investigations, and hence indirectly from the Theorem of Chasles, was Study's major work *Geometrie der Dynamen* (1903). Concerned with the connected component of the Euclidean group and its one-parameter subgroups, this deals with the geometry of the "space" of force systems and rigid displacements, and its philosophy is akin to that of Plücker's "line geometry." It can be regarded as a sequel to Sir Robert Ball's *Geometry of Screws*, whose 1871 edition had excited Klein and Clifford, and whose third edition would appear in 1911, but Study's book says little about any earlier work.[20]

Study's beautiful researches on the Problem of Apollonius (to construct the circles tangent to three given circles) were also, according to his first paper on the subject (M.A. 49 [1897]: 497-542), inspired by Klein's use of inversions in circles to generate symmetrical patterns. Also related to the E.P. was Study's original but obscure and rambling *Methoden zur Theorie der Ternaerien Formen* (Teubner 1889), dedicated to "my dear friend Friedrich Engel." In this book, Study applied Lie's concept of an *infinitesimal* transformation to invariants and covariants, mentioning (p. 143) the problem of determining "all types of r-parameter subgroups" of the full linear group. He also distinguished "integral" invariants, and "algebraic" (as well as "integral algebraic") invariants from general invariants.

Élie Cartan. Perhaps the greatest geometer of the twentieth century, Élie Cartan's thesis (1894) was purely algebraic. In it, he determined all simple complex Lie algebras, thus completing and making precise the earlier results of Killing. Presumably inspired to undertake this task by Poincaré (cf. §8), Cartan describes his advances over Killing as follows:

> Unfortunately Killing's research lacks rigor, particularly concerning groups which are not simple; it makes constant use of a theorem which is not proved in its generality; I show in this work an example where the theorem is not true, and when the occasion arises, a number of other errors of lesser importance. (Cartan 1952, partie I, 139)

Cartan's brilliant thesis was only the beginning of an outstanding career, which reached its climax when he was in his fifties and sixties, between World War I and World War II. His first major geometric effort was his translation and extension of Fano's article "Continuous Groups and Geometry" (ESM, vol. 3, 5; Cartan 1952, vol. 3, 2), which was concerned with developments stemming from the E.P. Written in 1912, Cartan's article notably amplified Fano's original by including the extension of the E.P. to space-time suggested by Einstein's then new theory of special relativity, and by summarizing Study's important but involved contributions.

Lorentz Group. Much as the E.P. notably extended the invariant theory of Cayley and Sylvester by pointing out "the possibility of constructing other than projective invariants," so a celebrated paper by Minkowski (written at Göttingen) identified the main contribution of Einstein's theory of special relativity as the replacement of the Galileo-Newton group by the "Lorentz group," actually first identified as a group by Poincaré. As Klein immediately recognized, this suggested that the E.P. might be applicable not only to geometry but to physics as well. Following a 1910 paper by Bateman, Cartan would later go further and describe the role of the conformal group, extended to space-time, in electromagnetic theory and special relativity.

Eight years later, in the preface to the second volume of his classic *Projective Geometry* (Veblen and Young 1917), Veblen would state:

> We have in mind two principles for the classification of any theorem of geometry: (a) the axiomatic basis. . . from which it can be derived. . . ; and (b) the group to which it belongs in a given space.
>
> The two principles of classification, (a) and (b), give rise to a double sequence of geometries, most of which are of consequence in present-day mathematics.[21]

Chapter 3 of that volume was devoted to applying Klein's classification scheme (b).

General Relativity. Unfortunately for the E.P., Einstein's "general relativity" theory fits much less neatly into Klein's classification scheme than his "special theory." See (Bell 1940, chap. 20, esp. 443-49), where a quotation from Veblen's chapter 3 is contrasted with one made by Veblen ten years later, and one made in 1939 by J. H. C. Whitehead.[22]

However, the properties of Einstein's curved space-time are expressed by local *differential invariants*, which can be classified as 'conformal', 'metric', 'affine', 'projective', etc., very much in the spirit of the E.P.

J. A. Schouten (1926) spelled out this connection. Cartan's invited address on "Lie Groups and Geometry" at the Oslo congress (1936) and J. H. C. Whitehead's biography of Cartan ("Obituary Notices of the Fellows of the Royal Society" 8, 1952) give perhaps the most authoritative opinions on the subject.

In the 1920s and 1930s, Cartan also showed his consummate and creative command of old and new mathematics by his major contributions to the "globalization" of Lie's purely local theory of Lie groups. The resulting global theory has made both Klein's and Lie's expositions technically obsolete.

Between 1927 and 1935, Cartan published what Chern and Chevalley call "his most important work in Riemannian geometry. . .the theory of symmetric spaces" (*Bulletin of the American Mathematical Society* 58 [1952]: 244). These are Riemannian manifolds in which ds^2 is invariant under reflection in any point. In retrospect, Cartan's tortuous path (via the parallelism of Levi-Civita) to the recognition of this simple idea seems amazing.[23] Even more amazing is the failure of Klein, after applying reflections in lines in the hyperbolic plane in so many ways, to identify "symmetric Riemann spaces" at all!

14. Klein and Hermann Weyl

Hermann Weyl began his career in the Göttingen that Klein had built up, and Hilbert was his thesis advisor. Hence he was in some sense an academic great-grandson of Klein, whom he must have known. Weyl seems never to have been overawed by Hilbert; thus his 1908 thesis stated unequivocally (M.A. 66 [1909]: 273): "As will be shown below, the applicability of Hilbert's method is by no means limited to the continuous kernels treated by Hilbert. . . but also leads to interesting consequences in certain more general cases." Beginning in 1917, Weyl outdistanced Hilbert (again) in mathematicizing Einstein's then new general theory of relativity. In papers and in his famous book *Raum. Zeit. Materie* (later translated into

English as *Space, Time, Matter*), he introduced *local* generalizations of affine, projective, and conformal geometry that are related to their global counterparts as Riemannian geometry is related to Euclidean, as Schouten (1926) was later to explain. Still under 40, he then attacked Hilbert's "formalist" logic of mathematics in the early 1920s supporting the conflicting "intuitionist" logic of L. E. J. Brouwer.

Weyl showed more respect for Klein, to whom he dedicated the first (1912) edition of his *Die Idee der Reimannschen Fläche*.[24] This was because, as he stated in the preface to its 1955 edition, "Klein had been the first to develop the freer conception of a Riemann surface,. . . thereby he endowed Riemann's basic ideas with their full power."

Then, in the middle 1920s, Weyl was the spark plug of the famous Peter-Weyl theory of group representations. He used a very concrete theory of *group-invariant* measure on *compact Lie groups*, the existence of which permits one to extend to compact Lie groups the result of E. H. Moore and Maschke: that every group of linear transformations having a finite group-invariant measure is equivalent to a group of orthogonal transformations.[25] In his *Gruppentheorie und Quantenmechanik* (translated into English by H. P. Robertson), Weyl later applied the analogous result for the orthogonal group to the then new quantum mechanics.[26]

By an irony of fate, very little of Weyl's deep and influential research work was done at Göttingen, the source of much of his inspiration. It was only three years after he finally accepted a professorship there that Hitler seized power in Germany. In the next year, Weyl emigrated to the new Institute of Advanced Study in Princeton, where he spent his last twenty years, creative and versatile to the end. His later books, *The Classical Groups* (1939) and *Symmetry* (1952), show his spiritual affinity with Klein, and it seems fitting to conclude our review with two quotations from the former. First, Weyl remarks:

> This is not the place for repeating the string of elementary definitions and propositions concerning groups which fill the first pages of every treatise on group theory. Following Klein's "Erlanger Program" (1872), we prefer to describe in general terms the significance of groups for the idea of *relativity*, in particular in geometry. (1939, 14)

Later, he makes a more specific evaluation:

> The dictatorial regime of the projective idea in geometry was first broken by the German astronomer and geometer Möbius, but the

classical document of the democratic platform in geometry, establishing the group of transformations as the ruling principle in any kind of geometry, and yielding equal rights of independent consideration to each and every such group, is F. Klein's Erlanger Programm. (1939, 28)

We hope that the great influence of this classical document, through its extensions and interpretations by Klein himself, by Lie, by Weyl, and by many other mathematicians, has been clarified by our review.

Notes

1. In discussing this last phase of Klein's career, Rowe (1985, 278) has referred to him as the "doyen of German mathematics for nearly three decades."

2. Weierstrass himself always viewed geometry *metrically*.

3. For the significance of W-curves, Klein refers his reader to ##13-20 and #34 of the article of Scheffers (EMW, vol. 3, D4).

4. In the second of these (Klein 1873), submitted in June 1872, Klein describes briefly how "the different methods of geometry can be characterized by an associated transformation group." This paper and the E.P. were the first publications by Klein or Lie that used the phrase *transformation group*. The word *invariant* is conspicuous by its absence, although invariant theory had been studied in Germany (without explicit mention of the word *group*) for at least a decade.

5. See (Birkhoff and Bennett, forthcoming) for references to this work of Klein and Lüroth, as well as its influence on later theories.

6. A glance at the index of names in the relevant volumes of Lie's *Collected Papers* (LGA) makes Lie's scientific indebtedness to Klein obvious.

7. Lie's *local* theory (LGA, vol. 5) was finally extended into a rigorous *global* theory by G. D. Mostow (*Annals of Mathematics* 52 [1950]: 606-36). There Mostow showed that the two-dimensional manifolds are the plane, cylinder, torus, sphere, projective homogeneous plane, Möbius strip, and Klein bottle.

8. Picard had used the simplicity of the (Lie) projective group on two variables to prove the impossibility of solving $u'' + p(x)u' + q(x)u = 0$ by quadratures—a theorem that Lie wished he had discovered himself.

9. See also (Hawkins 1982, §3).

10. For a very human account of Lie's jealousy of Klein and their final reconciliation, which reproduces a moving letter from Frau Klein, see (Young 1928, xviii-xix).

11. Wedekind's summary mentions his use of a result in the E.P.; Klein entitled his paper "Binary Forms Invariant under Linear Transformations."

12. By this, Klein meant that the subgroup of the group of the icosahedron leaving each point invariant is of order 2, as contrasted with 5 for the vertices of the icosahedron and 3 for the dodecahedron.

13. As late as 1886, Brioschi (M.A. 26 [1886]: 108) would refer to the "icosahedral" hyper geometric equation:

$$x(x - 1)v'' + (7x - 4v')/6 + 11v/3^2 4^2 5^2 = 0.$$

14. See, for example, L. R. Ford, *Automorphic Functions* (New York: McGraw-Hill, 1929).

15. Another close friend of Fine was Woodrow Wilson, president of Princeton University, and later of the United States. Further information on H. B. Fine can be found in William Aspray's paper in this volume.

16. "For a quarter of a century no one could think of the American Mathematical Society apart from the personality of Professor Cole" (Archibald 1938, 101).

17. Many references to Klein's role in Göttingen may be found in (Reid 1970, 1976); see also chapter 11 of (Klein 1893b).

18. Klein had shown his belief in the importance of applied mathematics in 1875, by leaving a full professorship at Erlagen for a position at the Technische Hochschule in Munich and referring to it as a great advance ("einem grossen Sprung") (Young 1928, vii). There he held a seminar on pure and applied mathematics with von Linde (Pyenson 1979, 55-61).

19. For Klein's role in suggesting Hilbert's Fifth Problem, see (Birkhoff and Bennett, forthcoming).

20. See (Ziegler 1985) and (EMW, vol. 3, parts 1, 2).

21. Principle (a) clearly refers to the method used by Hilbert in his *Grundlagen der Geometrie*.

22. Actually, Whitehead did not lose interest in the E.P. as quickly as Bell suggests; see (*Annals of Mathematics* 33 [1932]: 681-87).

23. For Cartan's mature exposition of the theory of symmetric spaces, see Proceedings, International Congress of Mathematicians, Zurich, 1932, 152-61.

24. A few years earlier, Paul Koebe (also in Göttingen) had finally solved rigorously the "uniformization problem" that had eluded both Klein and Poincaré. Koebe's "Primenden," one of his major technical tools, seem related to the ideas of Klein's "Grenzkreistheorem" (see §9).

25. The Peter-Weyl theory surely helped to inspire Haar's 1933 theory of invariant *Lebesgue* measure on compact topological groups. It was von Neumann's subsequent application of Haar measure to solve the Klein-Hilbert Fifth Problem for compact groups, and Pontrjagin's parallel solution for Abelian groups, that paved the way for its complete solution.

26. One of us adapted Weyl's title to the chapter on group theory and fluid mechanics in his *Hydrodynamics* (Birkhoff, Princeton University Press, 1950). In this the notion of 'self-similar solution' (exploited earlier by Sedov and others) was generalized to arbitrary groups. The ultimate inspiration for this chapter was Klein's group-theoretic interpretation of special relativity in his (EdM), as an extension of the E.P.

References

[EdM] Klein, Felix. 1926-27. *Entwicklung der Mathematik in 19ten Jahrhundert*. Berlin: Springer. Reprinted New York: Chelsea, 1950.

[EMW] *Enzyklopädie der Mathematischen Wissenschaften*, esp. vol. 3.1.

[ESM] *Encyclopédie des Sciences Mathématiques*, esp. vol. 3.

[GMA] Klein, Felix. 1921-23. *Gesammelte Mathematische Abhandlungen*. 3 vols. Berlin: Springer.

[LGA] Lie, Sophus. 1922-37. *Gesammelte Abhandlungen*. 6 vols. Ed. F. Engel and P. Heegard. Berlin: Springer.

Archibald, R. C. 1938. *A semi-centennial History of the American Mathematical Society; 1888-1938*. Providence, R.I.: American Mathematical Society, vol. 1.

Bell, E. T. 1940. *The Development of Mathematics*. New York: McGraw-Hill.

Birkhoff, G. 1973. *Source Book in Classical Analysis*. Cambridge, Mass.: Harvard University Press.

Birkhoff, G., and Bennett, M. K. Hilbert's "Grundlagen der Geometrie." *Rendiconti del Circolo Mathematico di Palermo*. Forthcoming.

Cartan, E. 1952. *Oeuvres complètes*. 12 vols. Paris: Gauthier-Villars.

Clebsch, A., and Lindemann, F. 1876-91. *Vorlesungen uber Geometrie*, bearb. und hrsg. von F. Lindemann. Leipzig: Teubner.

Clifford, W. K. 1873. Preliminary Sketch of Biquaternions. *Proceedings of the London Mathematical Society* 4: 381-95.

Coolidge, J. L. 1940. *A History of Geometrical Methods*. Oxford: Clarendon Press.

Courant, R. 1925. Felix Klein. *Jahresbericht der Deutschen Mathematiker Vereiningung* 34: 197-213.

Engel, F. 1900. Sophus Lie. *Jahresbericht der Deutschen Mathematiker Vereiningung* 8: 30-46.
Engel, F. and Faber, K. 1932. *Die Liesche Theorie der Partiellen Differentialgleichungen Erster Ordnung*. Leipzig: Teubner.
Gray, Jeremy J. 1980. Olinde Rodrigues' Paper of 1840 on Transformation Groups. *Archive for History of Exact Sciences* 21: 375-85.
Hawkins, Thomas. 1981. Non-Euclidean Geometry and Weierstrassian Mathematics.... *Historia Mathematica* 7: 289-342.
——. 1981. Wilhelm Killing and the Structure of Lie Algebras. *Archive for History of Exact Sciences* 26: 127-92.
——. 1984. The *Erlanger Programm* of Felix Klein.... *Historia Mathematica* 11: 442-70.
Hilbert, David. 1899. *Grundlagen der Geometrie*. 1st ed. Liepzig: Teubner.
Hopf, Heinz. 1926. Zum Clifford-Kleinschen Raumproblem. *Mathematische Annalen* 95: 313-39.
Jordan, Camille. 1869. Mémoire sur les groupes de mouvements. *Annali di Matematica* (2) 2: 167-213 and 322-451.
——. 1870. *Traité des substitutions*. Paris: Gauthier-Villars.
Killing, Wilhelm. 1886. *Zur Theorie der Lie'schen Transformationsgruppen*. Braunschweig.
——. 1891. Uber die Clifford-Kleinsche Raumformen. *Mathematische Annalen* 39: 257-78.
Klein, Felix. 1872. *Vergleichende Bertrachtungen über neuere geometrische Forschungen Erlangen*. Reprinted with additional footnotes in *Mathematische Annalen* 43 (1893): 63-100 and in GMA, vol. 1, 460-97.
——. 1873. Uber die sogenannte nicht-Euklidische Geometrie. *Mathematische Annalen* 6: 112-45.
——. 1884. *Vorlesungen über das Ikosaeder*.... Leipzig: Teubner. (English translation by G. G. Morrice, 1888; cf. also F. N. Cole's review in *American Journal of Mathematics* 9 [1886]: 45-61.
——. 1892. *Einleitung in die geometrischen Funktionentheorie*. Lectures given in 1880-81, Leipzig, and reproduced in Göttingen a decade later.
——. 1893a. *Einleitung in die Höhere Geometrie*. 2 vols. Lithographed lecture notes by Fr. Schilling.
——. 1893b. The Evanston Colloquim "Lectures on Mathematics." Republished by the American Mathematical Society in 1911.
——. 1895. *Vorträge über ausgewählte Fragen der Elementargeometrie*. Göttingen. Translated as *Famous Problems of Elementary Geometry* by W. W. Beman and D. E. Smith in 1900; 2d ed. reprinted by Stechert, 1936.
——. 1908. *Elementarmathematik vom Höheren Standpunkt aus*. Vol. 2. Leipzig: Teubner.
Kline, Morris. 1972. *Mathematical Thought from Ancient to Modern Times*. New York: Oxford University Press.
Lie, Sophus. 1880. Theorie der Transformationsgruppen I. *Mathematische Annalen* 16: 441-528.
Lie, S., and Engel, F., 1888-93. *Theorie der Transformationsgruppen*. 3 vols. Leipzig: Teubner. Reprinted in 1930-36.
Lie, S., and Scheffers, G. 1896. *Vorlesungen über continuierliche Gruppen*. Leipzig; Teubner.
Noether, Max. 1900. Sophus Lie. *Mathematische Annalen* 53: 1-41.
Plücker, Julius. 1868-69. *Neue Geometrie des Raumes*..., hrsg. von Felix Klein. Leipzig: Teubner.
Poincaré, Henri. 1907-8. *Science et Méthode*. English translation published by the Science Press in 1921 with the title *Foundations of Science*.
——. 1916. *Oeuvres complètes*. 12 vols.; vol. 11 most relevant. Paris: Gauthier-Villars.
Pyenson, L. 1979. Mathematics, Education, and the Göttingen Approach to Physical Reality, 1890-1914. *Europa* 2(2): 91-127.
——. 1983. *Neohumanism and the Persistence of Pure Mathematics in Wilhelmian Germany*. Philadelphia: American Philosophical Society.
Reid, Constance. 1970. *Hilbert*. Berlin: Springer.
——. 1976. *Courant in Göttingen and New York*. Berlin: Springer.

Rowe, D. E. 1983. A Forgotten Chapter in the History of Felix Klein's *Erlanger Programm*. *Historia Mathematica* 10: 448-57.

——. 1984. Felix Klein's *Erlanger Antrittsrede*. . . . *Historia Mathematica* 12: 123-41.

——. 1985. Essay Review. *Historia Mathematica* 12: 278-91.

Schouten, J. A. 1926. Erlanger Programm and Uebertragungslehre. *Rendiconti del Circolo Mathematico di Palermo* 50: 142-69.

Study, E. 1903. *Geometrie der Dynamen*. Leipzig: Teubner.

Veblen, O., and Young, J. W. 1917. *Projective Geometry*. Vol. 2. (Actually written by Veblen.) Boston: Ginn.

Weyl, Hermann. 1912. *Die Idee der Riemannschen Fläche*. Leipzig: Teubner. (English translation of the 2d ed. by Gerald Mac Lane, Addison-Wesley, 1955.)

——. 1939. *The Classical Groups*. Princeton, N.J.: Princeton University Press.

——. 1944. David Hilbert and His Mathematical Work. *Bulletin of the American Mathematical Society* 50: 612-44.

Young, W. H. 1928. Christian Felix Klein: 1849-1925. *Proceedings of the Royal Society of London* A121: i-xix.

Ziegler, R. 1985. *Die Geschichte der Geomestrischen Mechanik in 19. Jahrhundert*. Stuttgart: Fr. Steiner Verlag Wiesbaden GMBH.