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Physics, Computation, and Why Biology Looks so Different

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The biological world is a physical system whose properties and behaviors seem entirely foreign to physics. The origins of this discrepancy lie in the very high information content in biological systems (the large amount of dynamically broken symmetry) and the evolutionary value placed on predicting the future (computation) in an environment which is inhomogeneous in time and in space. Within this context, "free will" can be described as a useful predictive myth.

1. Introduction

Why does the world of biology look so different? As physicists, we know that the workings of biology are to be explained by the known laws of quantum mechanics and statistical physics. The existence of the sun and the nature of the planet earth are important as the substrate for biology, but there are no fundamental aspects of cosmology or particle physics necessary to address the mysteries of biology. The essence of biology is fundamentally properties of molecular physics in non-equilibrium circumstances and on a large scale. By and large, quantum mechanics is not relevant (Hopfield, 1986). Of course, the quantum mechanics of chemical bonds is essential, but while the making and breaking of these bonds by the enzyme catalysts in cells is of paramount importance, it is the rates of these processes as expressed in a network of chemical reactions and not the quantum-mechanical details that matter. The essential specificity of biochemical reactions chiefly involve molecules that are so large, and binding forces that are so weak, that classical descriptions are entirely adequate. For example, the forces which hold double-helical DNA together are of quantum mechanical origin, but can be adequately modeled as effective forces acting between classical atom masses. (Contrary to the expectations of a long history of ill-prepared physicists approaching biology, there is absolutely no indication that quantum mechanics plays any significant role in biology.) So why does biology look so different to a physicist?

2. Broken Symmetry and Complexity

The first important point to note is that the micro- and macro-structures of the plants and animals which make up biology are a consequence of a massive amount of broken symmetry (Anderson, 1972; Palmer, 1982). Broken symmetry, originally a part of phase transition lore in condensed matter physics, has been slowly making its way into the rest of physics. Even the laws of elementary particle physics, which would have been believed in 1960 to be unique, are now thought of as containing elements of broken symmetry. But in most of physics, broken symmetries are few in number.

Geology is a physical system with much broken symmetry. The fact that a particular mountain is located at a specific place and not elsewhere represents broken symmetry, as does the fact that the mountain is chiefly granite rather than sandstone. The details of a particular rock—its mineral micro-composition—represent the consequence of a long and detailed evolutionary process filled with arbitrary choices. But while the mineralogy of a particular rock

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is complex, it is simple compared to that of an equivalent size piece of biological matter. This complexity can be specified by describing the set of instructions necessary to make an equivalent piece of rock. Such instructions might state the following: break crystals of quartz, feldspar, and alumina into small pieces, mix together, and heat at 1200°C for 500 000 years. A specific procedure of this kind will generate a piece of rock in every way equivalent to a particular geological specimen, although not identical to it. They are equivalent in the sense of there being no significant macroscopic consequences of their micro-differences, an equivalence like that of two different members of a thermodynamic ensemble. In a language appropriate for describing crystal mixtures and heat treatments for producing rocks, the program to make typical rocks need be no more than 100 bits. If instead of describing a rock the size of a cat, you were required to specify how to make a cat from its chemical components, the problem appears impossible. But since there are less than 1 000 000 000 bits of information in the cat genome, the description of a cat must be shorter than this in an appropriate chemical language. A genome containing 1 000 000 bits is large enough to describe a bacterium, but not a cat. So within crude limits, we know how much information is required to specify a cat in an unknown language.

When we say that biology is a complex system, it is really this immense amount of information necessary to specify the significant state of biological matter (compared to an equivalent mass of geological matter) that is being referred to. This difference is on the scale of thousands to millions, and is fundamentally involved in the apparent difference between biological systems and physical systems.

3. Stability and Evolution

Broken symmetry situations which persist for appreciable times must be stable against perturbations. The most elementary broken symmetry events are those of static stability. Statistical mechanics or thermodynamics can provide this stability, as in the case of the direction of magnetization of a ferromagnet. In the case of macroscopic systems, the laws of classical mechanics may suffice, as in the case of a block lying on the table with a particular side up. More complex cases of broken symmetry occur in open systems, in which the flow of energy through the system provides the stability to a dynamical system. Examples of this range from the great red spot of Jupiter to the mundane "flip-flop" storage circuit in digital computers. The stability of biological systems is generally a case of dynamical broken symmetry.

The ability of systems of finite size to maintain a dynamically stable broken symmetry situation is always limited. Fluctuations will inevitably cause a finite lifetime of a particular broken symmetry solution. In biology, different aspects of the broken symmetry have quite different time scales. The choice of left-handed amino acids as the building blocks of proteins must have occurred over three billion years ago. The races of man, on the other hand, are believed to have diverged more recently than 100 000 years ago. Identifiable new strains of a flu virus are created every year. The finite stability of biological dynamical systems creates the diversity seen in biology by continuing to generate new molecules, structures and species.

The notion of evolution is also unique to biology. To be sure, rocks also evolve. Granite, under high temperature and pressure, will evolve to marble. But biological evolution is fundamentally different, owing to its much greater complexity. The progression from granite to marble is a change between simple forms of low information content. It is not necessary to seed the formation of marble with a pre-existing piece. The available space of possibilities is so small that the random fluctuations of crystal growth can spontaneously generate marble nuclei in a short time. The evolutionary progression (under appropriate physical circumstances) is essentially inevitable. Even when there are competing forms of crystal growth, the number of more or less equivalently stable crystal structures is generally a few at most. What makes biology chiefly different is that the "crystal" whose structure is essential is a one-dimensional strand of nucleic acid. All sequences of DNA are similarly chemically stable, and there are about 10^{10^8} such sequences possible in a billion-base piece of DNA. Of course, only a tiny fraction of these turn out to be biologically viable, but even so there are a huge number of possible species. The age of the earth has allowed the exploration of a negligible fraction of available species space. This is to be contrasted with geology, where the space of stable crystal forms is by comparison minuscule and fully explored.

Both biology and geology can replicate by replicating the information contained in a structure (DNA or a crystal form). For example, a crystal can be broken into two pieces and each used to seed the growth of a crystal equivalent to the starting one. DNA replication has much in common with this elementary process. Both examples involve duplicating the information contained in the broken symmetry description (see previous section) by templating on a physical structure, in one case a DNA strand, in the other a crystal surface. In the case of geology, this is not really necessary—viable crystal seeds are easy to produce by fluctuations. In biology, this process is essential, for a random sequence of DNA generated through fluctuations does not make a viable organism. Biology has thus evolved in a fashion which is qualitatively different from the evolution of a mineral. Information replication has been essential. With a system evolving through replication, non-trivial new forms are created by events which generate less than faithful replication of the genomic information.

Crystal growth far from equilibrium often results in a multiplicity of crystal forms which grow in a dynamic competition for "resources" (material to add) and may also continue to compete after growth by exchanging material. Given time and favorable kinetics, the most stable form may be capable of capturing all the available material in this dynamic competition. Biological competitions between species are different in that the whole notion of "equilibrium" and most stable or "best" is ill-defined in biology. If there is a "terrain" on which the dynamics is evolving downhill, it must also be one with an extremely rich local minimum structure. In most complicated dynamical systems, it has not been possible to isolate a Lyapunov function which is being optimized by the dynamical system.

4. Behavior

The term "behavior" in physics concerns the response of a system to a change in its environment. When we say water behaves as a liquid, we are really stating its ability to conform to the shape of a container, to shape itself into spherical droplets, to flow downhill, etc, in ways common to other liquids. Such behaviors are all consequences of the tendency of a near-equilibrium system to minimize its free energy. For such a system, the notion of behavior is most commonly a manifestation of Le Chatelier's principle.

Strongly non-equilibrium physical systems have responses which are more difficult to analyze. Occasionally, they can be described by ideas such as extremal entropy production, but in general they are not unified by a simple principle. They can display complex dynamics, as in the case of turbulence. Because biological systems have much richer physical structure, they exhibit correspondingly more complex behaviors.

When the environment changes with time, it may be possible to divide the variables of a dynamic system into fast variables (with response times much faster than the timescale of environmental change), and slow variables. In such a case, an adiabatic separation can be made. The slow variables will adapt (i.e. change slowly) to changes in the environment, and the fast variables will simply see the slow variables as changing parameters. The motion of such a physical system can be termed "adaptive".

The environment in which a cell or organism finds itself in biology is normally complicated, fluctuating with significant correlation patterns both in space and in time. The organisms which can best compete in this environment will not merely have fast variables appropriately chosen for growth and reproduction in an average environment. In addition, use will be made of the adaptation of slow variables. But because these variables can only change slowly, the organism which is able to initiate adaptation in advance of an environmental fluctuation by a prediction of the future environment from the recent past and present is at a strong competitive advantage. Such predictions are useful even in a spatially homogeneous environment. For example, a yeast cell deprived of energy sources forms spores, which have a very low metabolic rate, and which generate new yeast cells when a rich nutrient broth is provided. Such behavior is very well adapted to survival in an environment where periods of plentiful food can be followed by long periods of deprivation. Forming spores itself takes an hour, and is a useful response to environment only because the correlations of nutrient circumstances have a long correlation time. The act of sporulation as a behavior can be thought of as a prediction by the organism that the deprivation will last a long time. The organism has, through evolution, learned about the nature of the correlation time in its environment. And an organism which "understands" its environment in this fashion has a major competitive advantage over one which does not.

Bacterial adaptation can be seen at the biochemical level. Bacteria raised in the presence of a single sugar make proteins which transport that sugar across the cell membrane, and make very little protein for transporting other sugars. When another kind of sugar is added, the bacterium begins to generate more protein for transporting the new sugar. The natural environment tends to be stable for a long time, followed by environmental change. The behavior is then appropriately predictive, representing the idea that any given environment tends to persist. If the sugar environment fluctuated very rapidly, this behavior would have no value.

Such prediction becomes immensely more important when the environment is spatially non-uniform on a scale larger than the size of the organism. In such a case, if the organism is able to move it can induce environmental changes by its own motions. The organism which develops movement patterns which take it into more favorable environmental circumstances has a great advantage. Such behaviors are identifiable even at the level of bacteria, which can sense and swim up concentration gradients of nutrients.

Crudely put, one who can predict the future from the present and make advantageous choices of action on the basis of that prediction will generally win in the game of evolution. Much of the history of evolution can be read as the evolution of systems to make environmental measurements, make predictions, and generate appropriate actions. This pattern has the essential aspects of a computational system, where the inputs are from environmental measurements, the outputs are the signals (chemical or electrical) which modulate the behavior, and the computation represents an appropriate generation of outputs in response to environmental signals.

The relationship between sensory inputs and behavioral outputs (or the signals which drive them) is the essential mystery of what appears to us as observers to be motivated biological behavior. The sensory input is a form of symbol, and the signals driving muscles or turning on genes can also be described as symbols. The behavioral computation done by the organism is to generate symbolic outputs appropriate to the environmental symbolic inputs. This is an example of computation in the sense that the term is generally understood in computer science. Indeed, the history of biology can be described as the evolution of symbol-manipulating systems.

5. Brain and Computation

The human brain is the most mysterious and complex of these biological computational systems. To understand its computations, we first describe a view of digital computation which moves away from purely logical descriptions, and can deal with the physical systems behind the mathematics. Our understanding of biological computation and its origins must come through studying the relation between computation and its underlying hardware, not computation as a logical structure.

The operation of a real digital computer for batch-mode computation can be described as follows. A computer has N storage registers, each storing a single binary bit. The logical state of the machine at a particular time is specified by a binary vector 1001011000.. of N bits. This binary state changes into a new state each clock cycle. The transition map, describing which state follows which, is implicitly built into the machine by its design. Thus, the machine can be described as a dynamical system which changes its discrete state in discrete time (Fig. 1).

The user of the machine has no control over the dynamics, the state transition flow map. His program, data, and a standard initialization procedure describe the starting state of the machine. The computation is carried out by the motion of the dynamical system. In batch-mode computation, the answer represents a stable point of the discrete dynamical system, where the state space motion comes to a halt. The amount of computation which is done in this process depends on the complexity of the flow map. If it is very simple [Fig. 1(a) (i)], methods can be found to locate the terminal point without following each step along the

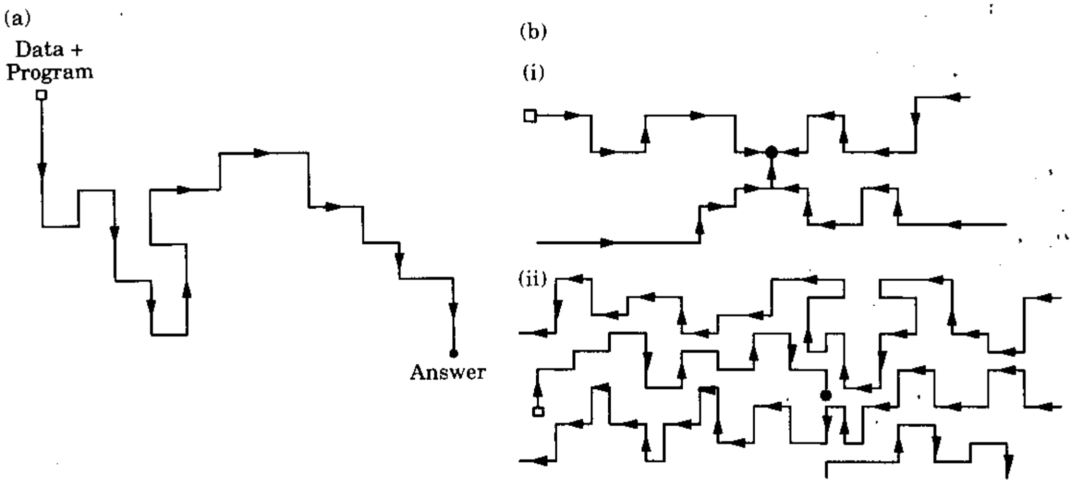

FIG. 1. The state space motion (flow) of a digital computer. In batch mode computation (left), the path goes from an initial state representing the program and data to a final stable state representing the answer. The amount of computation done depends on the complexity of the flow field. A trivial computation is shown on the upper right, and a hard one on the lower right.

pathway. If, on the other hand, the dynamics is very rich [Fig. 1(a) (ii)], it will be essential to follow each step in order to find the answer, and the amount of computation done is then much larger.

The electrical and chemical activity of a set of nerve cells also form a dynamical system, one which moves in continuous time and with continuous state variables. But batch-mode computation can still be described in exactly the same fashion as in the digital case, as a motion to a stable attractor. (The problem of extending computation to systems with time-varying inputs is the same for digital and analog systems.) The only additional complication is the necessity of restoration. In the analog system, noise and imprecision in manufacture lead to errors in the desired trajectory. It is essential that the system recover from such fluctuations back toward the correct path in order that the computation reach the correct answer. This process is called restoration, and is unnecessary in a digital system, where operation can be made essentially perfect. The restoration process is represented by a flow pattern which locally focuses motion back onto pathways. Most of the time, states which are close to each other must lead to later states which are also close to each other. This point limits the complexity of appropriate state space motions for an analog computer. It will not be possible to use the rich complexities of chaotic dynamics in a profound fashion in biological computation (Fig. 2).

The theoretical view of neurobiology in greatest use today represents neurodynamics as a set of first-order dynamical equations. A neuron i is generally taken as an input–output device, with output V_i given in terms of input $\boldsymbol{u}_i$ by a function such as

$$V_i = 1/(1 + \exp - \boldsymbol{u}_i)$$

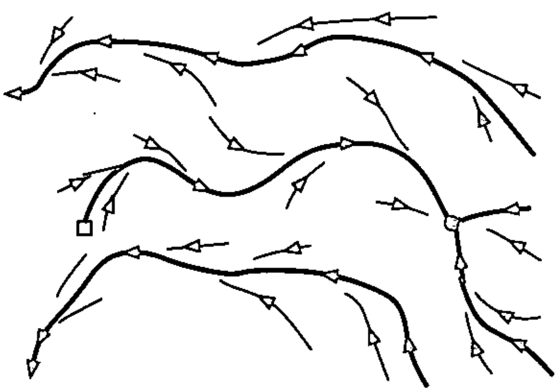

FIG. 2. The state space flow field for a batch mode analog computational system. The flow must be focussed onto paths to restore effects of errors, but is otherwise similar to those in Fig. 1.

The vector $\boldsymbol{u}$ (or V) represents the state vector of activity of the system. The influence of the synapse (connection) from cell j to cell i (if any) is represented by a connection strength matrix **T**. The equation of motion of the activity vector is often taken to be

$$\mathrm{d}\boldsymbol{u}_i/\mathrm{d}t = -\boldsymbol{u}_i/t_{\text{fast}} + \Sigma \mathrm{T}_{ij} \cdot V_j + I_i. \quad (1)$$

This equation generates computation through producing a dynamics similar to those illustrated above (Hopfield, 1984).

However, the system is also adaptive. The connection strengths themselves change with time, though typically on a slower timescale. The general structure of the change with time is often represented by an equation in the style of

$$\mathrm{dT}_{ij}/\mathrm{d}t = -\mathrm{T}_{ij}/t_{\text{slow}} + \mathrm{del} \cdot V_i \cdot V_j, \quad (2)$$

and involves the activity state of the neurons. These two equations are the essence of an adaptive computing system. (It is, of course, an oversimplification to represent the two timescales as completely non-overlapping). While these equations are a mere parody of the complexities of neurobiology, they contain enough of the general neurobiological themes that these equations are capable of powerful computation. There are many successful applications of such equations to real-world problems. Learning systems (Sejnowski & Rosenberg, 1987; LeCun *et al.*, 1989) have generally emphasized the computations done by an adaptive process such as eqn (2), [using eqn (1) only in a computationally trivial fashion as in Fig 1(a) (i)]. Optimization approaches have emphasized the computation done by eqn (1), and replace eqn (2) by a set of connections provided by design (Hopfield & Tank, 1985; Takefui & Lee, 1989).

6. Free Will

Most physicists, when asked whether they have free will, respond "yes" with little hesitation. Asked the same question about a Cray, they equally quickly respond in the negative. This instant intuitive differentiation between the biological and non-biological computers points to the most profound area in which biology truly "looks different". Can the properties which we associate with the human mind be equivalent to those of a very large digital computer, or is there fundamentally something irreducibly different about the operation of the biological brain?

What do we mean by "free will"? The idea that "I am responsible for my actions" is somehow central, but is woefully imprecise, since it involves an undefined "I". This statement does, however, emphasize one major aspect, the unimportance of noise. Even in the simple case of associative memory, the continuous dynamical system represented by the biological computing equations of motion will sometimes be in a regime of delicate balance, where a little noise will be the determining factor in the choice between two memories. This noise might be thermal, from external perturbations, or even the result of quantum fluctuations. Actions which are determined by noise are not what we mean by free will, for the entire idea of responsibility is lost in actions resulting from noise. (We do not care whether noise influences the underlying microstates of neurobiology, just as it is present in the fluctuating voltages on individual gates in a digital computer, as long as it does not determine macroscopic decisions.)

If, however, the influence of noise is not to be thought central, then the operation of neurobiology is to be understood through deterministic equations of motion of the general type described earlier. We must think of free will in the context of deterministic equations of motion. These equations can be simulated with arbitrary accuracy on a digital machine. So we must search for a meaning to free will in a context where, in principle, the phenomenon must also be present in a suitably programmed digital computer. The point is that biology is not different from physics, it only seems different. (Delicate circumstances surrounding chaotic systems can result in long-term simulation results which are sensitive to digital noise. In such circumstances, however, noise in the neural system will also result in major influences, which is not then a case relevant to free will.)

Useful definitions tend to come from operational circumstances rather than from philosophy. "Did you strangle the infuriating student of your own free will, Professor Hopfield?" In a court of law, a jury will hand down completely different verdicts, and my treatment chosen between a short stay in a mental hospital and twenty years in prison based on whether the answer to this question is "yes" or "no". Free will is not merely an academic issue.

There are at least two forms of processing of information by the brain. One involves conscious processing, and is roughly characterized by being logical, sequential, and aware. The other is non-conscious, parallel, multitasking. When you are searching for a name you cannot remember while continuing the conversation, or driving over a very familiar route, or placing your foot for the next step, you are engaged in non-conscious activities.

Free will, as usually considered, concerns conscious behavior. It involves the realm of goals and values, and procedures for achieving goals. A possible action is consciously evaluated in terms of its contribution to goals and how well it serves values. Rational and conscious man then makes the choice which best serves these ends, goals and values.

What is the difference between these two forms of generating actions? In both cases, the action is completely determined in advance by the information already in the human machine. Knowledge, previous experience, emotional state, the present neural state, and current sensory information completely determine the computational trajectory of the physical system (in the case not dominated by noise).

Within the legal system, the reason for the importance of the notion of free will is chiefly as a means of differentiating these two different modes of brain action. If we accept the idea that a major role of the legal sentence or punishment is to protect society through behavior modification, then different treatments of aberrant behavior are appropriate to different brain computations, even though those computations led to the same behavior. Similarly, when a digital machine makes an error, we seek to discover why it made the error before replacing components. We do not have a single repair strategy for a given output error independent of other diagnosis.

Neither in conscious nor unconscious information processing does the term "choice" seem appropriate. We would never say that while a planet is moving along its trajectory that it chooses to follow a Keppler orbit. The term choice seems completely out of place when describing the behavior of a deterministic physical system. Yet we insistently use the term choice in describing human actions. What do we mean by choice?

Suppose I am following a car on the highway, and see it approaching an intersection. At the moment, I have no knowledge of whether the car will continue straight ahead or turn right. I observe it to turn right, and say that the occupant of the car chose to turn right. If you later tell me that the car was driven by a computer, I tend to retract my statement, and say instead that the car control system was programmed to turn right. If it was driven by a person following the instructions to get to a dinner party, I will say that she chose to turn right. If it was driven by someone going home after work, by the same route taken every day and who was thinking of tomorrow's sailing, I might well also admit that no choice was made, that the driver was really "on autopilot". Yet the driver following instructions to the dinner party was also performing a predetermined pattern of action. Choice seems more a statement about my knowledge of possibilities rather than a description of what is taking place in the car.

The myth of rational man and free will contains the following elements. At any moment, a person has a set of available actions. The person has also a set of goals, attitudes, values and knowledge (which to a great extent is shared between individuals). There is an agent, "you" or "I", in our brains which freely selects the action to be taken on the basis of our particular goals, attitudes, and knowledge. I call this a myth because, like most myths, it is culturally passed down from generation to generation, is blatantly untrue, but nevertheless does have value to the believer.

This myth is untrue in that the action is determined, so the notion of "free choice" is meaningless. The set of available actions is not truly available at all, since the action to be taken is in fact determined. And no decision-making central agent, no localized arbiter of decisions, has been found in the brain.

A major evolutionary function of a nervous system is to provide an appropriate connection between present actions and future events. Our nervous system has embedded within it both implicit and explicit descriptions of the likely future consequences of our actions. When Pavlov trained dogs to salivate at the sound of a bell by ringing a bell and following that ringing with food, he was embedding an implicit representation of a prediction of the future. Our conscious and unconscious reasoning processes, in conjunction with factual knowledge, are very powerful predictors of the future course which will follow from a present state of affairs.

Man is a social animal, and it is therefore particularly important that an individual human be able to predict the likely actions of others. Knowledge as to how to do that can be inherited, learned from direct experience, or acquired through culture. The myth of rational man and free will is useful in that it provides considerable ability to predict the future actions of other individuals. If I reason from the hypotheses of this myth, and presume that other humans have roughly the same value system that I do, I will make very useful predictions about the future. This myth is not at all unique in its ability to transmit the relevant social information. An equally powerful myth can be constructed in which the reason an individual gives for taking an action is always "God wills it". This myth will contain examples of the kind of things which God wills and does not will. If God consists only of a set of examples, the knowledge is not very compact, and the prediction ability will be limited. So the description of God should also contain some guiding principles which will help figure out what God is likely to will in novel situations.

Physics has also known many myths in its day. Caloric is now regarded as a mythical substance, but its invention at the time allowed some understandings and predictions about heat flow and temperature to be made. Phlogiston similarly described some of the aspects of combustion. The rigid ether of Maxwell encapsulated some truths about electromagnetic waves. Successful myths in science provide a compact encapsulation of a large body of empirical knowledge, which would otherwise have to be described in detail. Indeed, scientific truth seems merely myth with few blatant errors. In everyday social interactions and in physics alike, the worthwhile myths are ones which provide this compact distillation of empirical knowledge, and thus can be readily transmitted to others.

The myth of free will and rational man is thus one primitive representation of the science of human behavior. Free will is not a problem for brain physics, but rather a social phenomenon. "Free", "choice", and "decide" do not seem to be a part of the equations or symbols of physics at all, but instead, a matter of the ambiguous language which we use to describe that physics to ourselves and to others. Notions of "voluntary movement" or "free choice" refer to different modes of (deterministic) information processing in the nervous system, not to a new physics.

7. Discussion

Why does biology look so different? As a physical system, it is merely another example of dynamical broken symmetry. But what sets it apart so much from other such systems is its complexity—its meaningful information content. Meaningful content, as distinct from noise entropy, can be distinguished by the fact that a change in a meaningful bit will have an effect on macroscopic behavior of a system. In addition, the meaningful bits describing the macroscopic broken symmetry are represented at the DNA level by a single long string of nucleic acids. Changing one of them can easily generate a macroscopic effect on behavior or viability. Physics is unaccustomed to looking at systems whose macroscopic dynamic properties are so influenced by single events at a molecular level, and with such massive quantities of significant broken symmetry information.

The possibility of evolution leading to a selection among information systems necessitates the transfer of a great deal of information when a structure is replicating. If this takes place through chemical templating, as it does both when a fragment of a crystal structure is used to grow a new one and when DNA is replicated, then the dimension of the informational structure must be one or two dimensional. The larger three-dimensional structure of an organism thus reflects the information carried by a tiny fraction of its matter.

The second point which sets it apart is the selection, through evolutionary pressures, of a computational system as its stable structure. It is difficult to ascertain what went on in the earliest era of the creation of biology, when the amount of dynamical information per cubic micron went from the few bits typical of physical systems to the thousands of bits essential to an elementary biological system (Eigen & Schuster, 1977; 1978*a*, *b*). But once a situation was established that spontaneous fluctuations could no longer generate adequately competitive forms, then the evolution of ever richer computational systems, better able to predict and to learn from the environment, was an inevitable consequence of the competition between organisms in a fluctuating but somewhat predictable environment.

Finally, higher mental function seems most completely different from physics. But when we examine carefully what is meant by an issue such as "free will", we find again that neurobiology and mind are not separate from physics, but are merely surrounded by different predictive myths.

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