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Knots and physics: Old wine in new bottles[a)]

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The history of the interplay between physics and mathematics in the theory of knots is briefly reviewed. In particular, Gauss' original definition of the linking number in the context of electromagnetism is presented, along with analytical, algebraical, and geometrical derivations. In a modern context, the linking number appears in the first-order term in the perturbation expansion of a Wilson loop in Chern–Simons quantum field theory. New knot invariants, the Vassiliev numbers, arise in higher-order terms of the expansion, and can be written in a form which shows them to be generalizations of the linking number.

I. INTRODUCTION

The theory of knots is a fascinating branch of topology, often pleasing and surprising us with new and unexpected connections between algebra, geometry, and analysis. It attracts the nonspecialist because its objects can be plastically represented and imagined. Its central question—when can two knots be continuously deformed into each other?—is simple to formulate, yet continues to challenge successive generations of mathematicians. The roots of the subject go back to questions which arose in physical contexts. Recent decades have seen a strong revival of interest, as new links to physics, and also to such diverse fields as theoretical chemistry and molecular biology, have come to light.[1]

The first recorded result involving knot theory was found by Carl Friedrich Gauss. It appeared in the posthumous edition of his unpublished works, in a section devoted to remarks on electrodynamics.[2] The breadth of Gauss' interests is phenomenal; he made epochal contributions to number theory, analysis (Gauss' divergence theorem), statistics (the Gaussian distribution, least squares), astronomy, geodesy (non-Euclidean geometry), and physics—theoretical physicists still use Gaussian units, and the strength of the magnetic field is measured in gauss. His longest and most productive collaboration was with the physicist Wilhelm Weber—they worked on the theory and phenomenology of magnetism. Next to the Göttingen Observatory (also built according to Gauss' specifications) he constructed a laboratory for the study of magnetism made entirely of wood—even the joinings avoided the use of metal nails. Together with the explorer Alexander von Humboldt he laid out plans for a world-wide network of measuring stations to determine the terrestial magnetic field. In the course of his travels in Northern Russia, von Humboldt made the first measurements of the field declination.[3]

The fundamental problem of knot theory is illustrated in Fig. 1. The question is—are the two knots shown equivalent (in the sense that the one can be continuously deformed into the other), or not? The answer is not immediately apparent. The first person to attempt a systematic listing of knots, P. G. Tait, thought they were distinct, and included both in his table of knots with ten crossings.[4] It took almost a hundred years until Perko proved that they are actually equivalent.[5]

In the attempt to find a systematic way of answering such questions an ever-growing list of knot invariants has been developed: these are quantities associated with a given knot which are unaffected by continuous deformations. The initial hope was to find some invariant which could distinguish knots of distinct equivalence classes. All the invariants found to date indeed have the property that distinct values of the invariant signify inequivalent knots. However, the converse is not true: Inequivalent knots may yield the same value for the invariant. Mathematicians are now involved in the search for a *class* of invariants, for which distinct knots would generate distinct values for at least some invariants in the class. The status of this search will be further discussed below.

II. GAUSS' DISCOVERY: THE LINKING NUMBER

Gauss' note on knot theory appears in Volume V of the complete works. It reads: ''*Of the* Geometria Situs, *which was foreseen by Leibnitz, and into which only a pair of geometers (Euler and Vandermonde) were granted a bare glimpse, we know and possess today, a century and a half later, little more than nothing.*

A principal task at the interface of Geometria Situs *and* Geometria Magnitudinis *will be to determine the linking of two closed or infinite lines.*

Let x, y, z be the coordinates of a given point on the first line, x', y', z' *those of a point on the second, and*

$$I=\int\int[(x-x')^2+(y-y')^2+(z-z')^2]^{-3/2}[(x-x')\times(dydz'-dzdy')+(y-y')(dzdx'-dxdz')+(z-z')(dxdy'-dydx')],$$

then the integral taken over both lines is $4\pi n$, *where n is the linking number. This value is mutual, i.e. it is unaltered when the two lines are interchanged.*

January 22, 1833''

Characteristically, Gauss provides no hint of how he arrived at his result. However, the editor of the *Works*, through careful study of the context in which the remark appears, was able to deduce the method he probably used. I shall discuss this method in Sec. V.

We first of all ask ourselves: What is this result doing in the volume on mathematical physics, in the middle of the section on electromagnetism? I present here an answer to this question which can easily be understood by a modern undergraduate physics student.

Consider the integral of a magnetic field **B** around a closed circuit *c*. Stokes' theorem relates the value of this integral to curl **B**: If **B** is irrotational the field is conservative and the

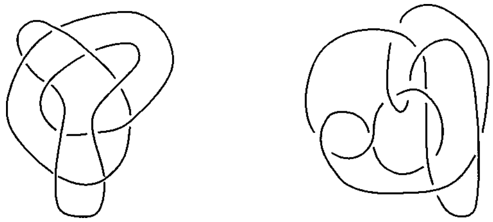

Fig. 1. A pair of knots.

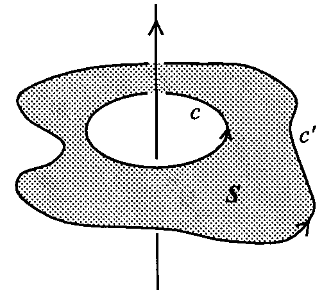

Fig. 3. Topological invariance of the linking number.

integral vanishes. Stokes's theorem was not available to Gauss in this form in 1833, although we today consider it to be a variant of Gauss' own divergence theorem.[6] curl **B** is controlled by Maxwell's equation: For the case of steady currents we have

$$\oint_c \mathbf{B}\cdot d\mathbf{x} = \int_S \text{curl } \mathbf{B}\cdot d\mathbf{S} = 4\pi \int_S \mathbf{j}\cdot d\mathbf{S} = 4\pi I, \tag{1}$$

where S is any oriented surface spanning the closed curve c, $\mathbf{j}$ the electric current density, and I the electric current flowing through the surface S. This result is of course Ampère's law. We generally present it in a first course on electrodynamics for the case of a simple circular curve around an infinite straight wire, see Fig. 2(a). If a current is flowing in the wire it must constitute part of a closed circuit, so a more realistic representation is achieved if we close the ends of the wire as in Fig. 2(b). Mathematicians would say that we have thereby identified the points at infinity and compactified the Euclidean space $\mathbb{R}^3$ to the three-sphere S^3, a common strategy in knot theory, where knots are defined as embeddings of the circle S^1 in S^3.

The situation depicted in Fig. 2(b) can be continuously deformed into the configuration depicted in Fig. 2(c), which is known as the Hopf link. A link is the disjoint union of two or more knots. Obviously the two components in Fig. 2 are linked, and the number which Gauss calls n attempts to measure the degree of linking. For the Hopf link the linking number is 1. For the configuration shown in Fig. 2(d), where the current-carrying wire winds n times around the circuit c,

$$\oint_c \mathbf{B}\cdot d\mathbf{x} = 4\pi n I, \tag{2}$$

and the linking number is n.

We see that the linking number is related to the integral of the magnetic field, but how can we recover the integral expression of Gauss? The answer is given by the law of Biot–Savart:

$$\mathbf{B}(\mathbf{x}) = I \oint_{c'} \frac{d\mathbf{x}'\times(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}. \tag{3}$$

It follows that

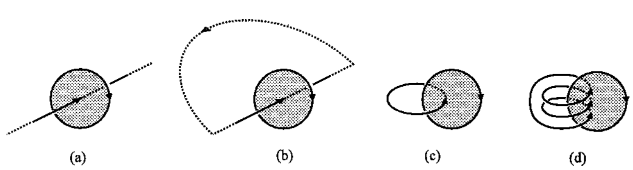

Fig. 2. Current-carrying wires.

$$\oint_c \mathbf{B}\cdot d\mathbf{x} = I \oint_c \oint_{c'} \frac{d\mathbf{x}\cdot d\mathbf{x}'\times(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}. \tag{4}$$

Inserting this into Eq. (2) yields

$$n = \frac{1}{4\pi} \oint_c \oint_{c'} \frac{(\mathbf{x}-\mathbf{x}')\cdot d\mathbf{x}\times d\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}, \tag{5}$$

where we have used the cyclic property of the triple product in the numerator to rearrange the order of the factors. With the notations $\mathbf{x}=(x,y,z)$, $\mathbf{x}'=(x',y',z')$ and $d\mathbf{x}=(dx,dy,dz)$, $d\mathbf{x}'=(dx',dy',dz')$, this is just the integral in Gauss' note.

It is easy to see that the number n is unaffected by continuous deformations of either of the loops. Consider, e.g., the situation depicted in Fig. 3. We wish to compare the results obtained for the original circuit c and the deformed circuit c'. Consider an oriented surface, such as that denoted in Fig. 3 by S, which is bounded by the curves c and c'. Applying Stokes's theorem, and paying attention to the orientation of the boundary of S, yields

$$\int_S \text{curl } \mathbf{B}\cdot d\mathbf{S} = \int_{\partial S} \mathbf{B}\cdot d\mathbf{x} - \oint_c \mathbf{B}\cdot d\mathbf{x} + \oint_{c'} \mathbf{B}\cdot d\mathbf{x}' = 0, \tag{6}$$

since curl $\mathbf{B}=4\pi\mathbf{j}=0$ throughout S. Hence,

$$n = \frac{1}{4\pi I} \oint_c \mathbf{B}\cdot d\mathbf{x} = \frac{1}{4\pi I} \oint_{c'} \mathbf{B}\cdot d\mathbf{x}'. \tag{7}$$

III. THE ANALYTICAL APPROACH

For a general configuration the task of evaluating Gauss' integral seems formidable; for a certain simple configuration of the loops in space Spivak[6] writes: ''*You may easily convince yourself that evaluating the linking number n by the above integral is hopeless...*''. However, since we have just ascertained that the value of n is unaffected by a continuous deformation of the link components we may use this fact in order to find a configuration for which the integral becomes manageable.

How to arrive at such a configuration is indicated in Fig. 4. We consider the projection of the three-dimensional link onto a two-dimensional plane, where the projection is chosen in such a way that the image of the link in the plane exhibits at most a finite number of isolated double points. This intuitively plausible procedure can be rigorously justified.[7] Choose a circle centered on each crossing whose radius is so small that no other crossing falls within it, and imagine a cylinder in three-space with this circle as its base. Now deform the link in three-space in such a way that the two

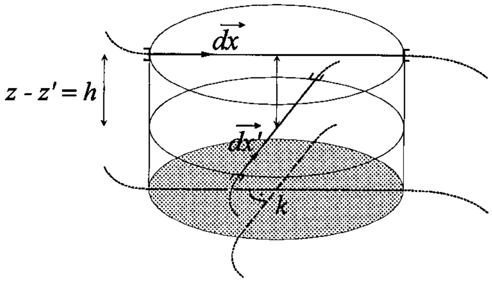

Fig. 4. The flat knot limit.

strands intersecting each of these cylinders are horizontal, and perpendicular to each other, at least within the cylinder. Furthermore, parametrise the curves with parameters s and s' in such a way that the segments enter the cylinder at the points corresponding to $s = s' = -1$, and leave it at the points $s = s' = +1$. We also require that $|\dot{\mathbf{x}}(s)| = |\dot{\mathbf{x}}'(s')| = 1$. When the segments are oriented as in Fig. 4, the parametrisation within the crossing region is

$$\begin{aligned} &\mathbf{x} = (s,0,z), \quad \mathbf{x}' = (0,s',z'), \\ &\dot{\mathbf{x}}(s) = (1,0,0), \quad \dot{\mathbf{x}}'(s') = (0,1,0), \\ &\mathbf{dx} = \dot{\mathbf{x}}(s)ds, \quad \mathbf{dx}' = \dot{\mathbf{x}}'(s')ds'. \end{aligned} \tag{8}$$

The contribution to the integral (5) from the region involving the kth crossing then becomes

$$n(k) = \frac{1}{4\pi} \int_{-1}^{1} ds \int_{-1}^{1} ds' \frac{h}{\sqrt{h^2 + s^2 + s'^2}^3} \tag{9}$$

$$= \frac{1}{\pi} \arctan\left(\frac{1}{h\sqrt{2+h^2}} \right), \tag{10}$$

where $h = z - z'$.

Now deform the link further in such a way that the strands everywhere approach the projection plane, in the limit lying in this plane, except in the vicinity of the crossing points, where the overlying strand lies directly above, but infinitesimally close to, the underlying strand. This of course still leaves the value of the linking number invariant. In Ref. 8 we refer to this procedure as *taking the flat knot limit*. In Eq. (10) it corresponds to the limit $h \to 0$, which yields

$$n(k) = \tfrac{1}{2}\,\mathrm{sgn}(h) = \tfrac{1}{2}\epsilon(k). \tag{11}$$

The crossing number $\epsilon(k) = +1$ if the segment $\mathbf{dx}$ lies above $\mathbf{dx}'$ in the original three-dimensional configuration (we call this case an *overcrossing*), and $\epsilon(k) = -1$ in the case of an *undercrossing*. We further note that the integrand vanishes outside the crossing regions, for in this case the denominator is nonvanishing but the numerator vanishes, since it involves the triple product of three vectors in a plane. Hence the integral over the complete link is

$$n = \sum_k n(k) = \frac{1}{2} \sum_k \epsilon(k), \tag{12}$$

where the summation is over all the crossings which involve strands belonging to distinct link components.

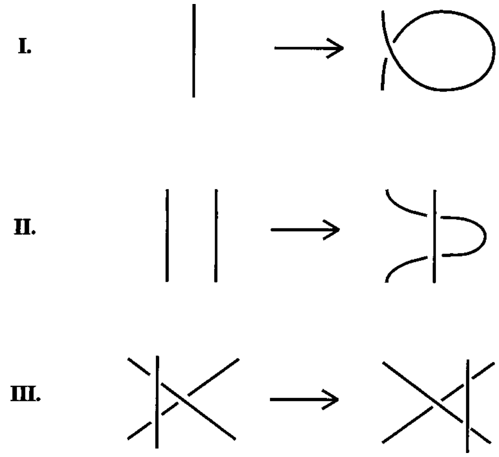

Fig. 5. The Reidemeister moves.

IV. THE ALGEBRAICAL APPROACH

Up to now I have attempted to motivate and illustrate the concept of the linking number by referring to the relatively simple configurations shown in the figures. Expression (12) is, however, completely general. It is valid for links with an arbitrary number of components, where each component may itself be knotted in some more or less complicated way. This is the expression for the linking number used in the *algebraical*, or *combinatorial*, approach to knot theory. In this approach the central objects are the *link diagrams*, which are the plane projections of the links supplied with the necessary crossing information; the convention is $\epsilon(k) = +1$ at the kth crossing if the upper strand must be rotated *counterclockwise* to get to the direction of the lower strand, and $\epsilon(k) = -1$ if the rotation is *clockwise*.

Link diagrams are used extensively in knot theory. A fundamental result due to Reidemeister is that two links are equivalent if and only if their associated diagrams can be made to coincide by a finite succession of the moves shown in Fig. 5. Of course, continuous deformations in the plane which do not involve crossings are also allowed.

It is intuitively clear that these moves do not affect the topological nature of the link; the deep part of Reidemeister's result is that these moves are sufficient. To be complete, the moves listed here must be distinguished according to the orientations of the various segments, and complemented by their obvious counterparts involving alternative choices for the overcrossings and undercrossings.

It is an elementary exercise to check that expression (12) for the linking number is invariant under each of the Reidemeister moves. This provides a *combinatorial* proof of the fact that the linking number is a link invariant, which was already established above (for a restricted case) by an *analytical* method.

V. THE GEOMETRICAL APPROACH

I will consider here one more method of establishing Gauss' result, presumably that used by Gauss himself. In the course of his investigations of magnetostatic phenomena, Gauss noticed that the magnetic field at a point $\mathbf{x}$ induced by a steady current in a loop c' can be calculated with the help

Fig. 6. Solid angles.

of a *magnetostatic potential*, which is equal to the solid angle $\Omega(\mathbf{x})$ subtended by a surface S' spanning the loop c' as seen from $\mathbf{x}$. The formula is

$$\mathbf{B}(\mathbf{x}) = I\boldsymbol{\nabla}\Omega(\mathbf{x}), \tag{13}$$

where

$$\Omega(\mathbf{x}) = \int \frac{(\mathbf{x}'-\mathbf{x})\cdot\mathbf{dS}'}{|\mathbf{x}'-\mathbf{x}|^3}, \tag{14}$$

and I is the electric current flowing in the circuit. The sign convention for the solid angle is that it is positive when the projection of the outward normal to the oriented surface onto the direction of the line-of-sight is positive, and negative when this projection is negative. The formula is most easily established by starting from the Biot–Savart law and using standard vector identities.[9] Inserting this into Eq. (2) yields

$$n = \frac{1}{4\pi I}\oint_c \mathbf{B}\cdot\mathbf{d}x = \frac{1}{4\pi}\oint_c (\boldsymbol{\nabla}\Omega)\cdot\mathbf{dx}$$
$$= \frac{1}{4\pi}[\Omega(\mathbf{x}^+) - \Omega(\mathbf{x}^-)], \tag{15}$$

where $\mathbf{x}^+$ and $\mathbf{x}^-$ are the points on c directly above and below the surface S' for the simple case depicted in Fig. 2(c).

You can easily convince yourself that $\Omega(\mathbf{x}^+) - \Omega(\mathbf{x}^-) = 4\pi$ for an arbitrary surface S'. For example, if S' is planar, this just means that when you are standing directly in front of an extended wall you see it covering a solid angle of 2π, and when you are directly behind the wall the solid angle is -2π. For a sphere with an opening, as in Fig. 6, you see a solid angle of $\Omega = 4\pi - \Omega'$ from a point inside the sphere and directly opposite the opening [Fig. 6(a)], and a solid angle of $-\Omega'$ from a point just outside the sphere [in any direction which does not point to the opening your line-of-sight meets the sphere *twice*, and the corresponding contributions to the solid angle have opposite signs and cancel, Fig. 6(b)].

The geometrical method thus again yields a linking number $+1$ for the Hopf link shown in Fig. 2(c) and n for the link in Fig. 2(d). We may say that we get a contribution of $+1$ to the linking number each time one link component goes though an oriented surface spanned by the other link component in the direction of the outward normal, and a contribution of -1 each time it goes through in the opposite direction. We easily see with this method that for the so-called Whitehead link shown in Fig. 7(a) the linking number is zero! This illustrates the point mentioned in Sec. I, that a single link invariant is not sufficient to classify links. It is certainly clear that the components of the Whitehead link really *are* linked, in the sense that they cannot be separated without cutting at least one of the components. But the linking number does not distinguish the Whitehead link from the trivial link shown in Fig. 7(b).

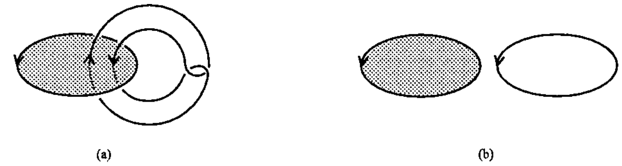

Fig. 7. The Whitehead link.

The geometrical method can also be extended to more general cases than those illustrated in the figures. To do this we would use a fundamental result of Seifert, which tells us that an arbitrary knot can be spanned by a connected, oriented surface.[7] However, for such cases the method quickly loses its intuitive appeal, and modern mathematical texts invoke more general concepts to describe the linking number, such as the *degree* of a mapping.[10] Despite these developments, tom Dieck[11] tells us, ''*Even after all these decades there is still nothing in topology to match (Gauss') result for elegance and insight...*''.

VI. FURTHER DEVELOPMENTS IN KNOT THEORY

After Gauss' remark concerning the linking number, and another remark he made on winding numbers and crossing information, some further results concerning knots were found by his student Listing. But the systematic study of knots began with the investigations of the Scottish mathematician P. G. Tait.[4] Tait was interested in Helmholz's findings concerning persistent vortex rings in incompressible fluids, and communicated these results to W. G. Thomson (the later Lord Kelvin). Thomson was thereby inspired to his theory of ''vortex atoms,'' in which the atoms of the various elements correspond to differently knotted vortex rings in the ether. Put this way the notion sounds to us naive, but Tait's words ''*There is, of course, an infinite number of possible modes of vibration for every vortex...*'' seem almost uncannily prophetic when we remember that Schrödinger finally succeeded in calculating the frequencies of atomic radiations using a method devised for finding the frequencies of the normal modes of a vibrating string. We may also be reminded of modern string theories of the elementary particles. In any case, Tait was motivated to embark on a systematic cataloging of the knots with up to ten crossings, generating in the process the famous series of ''Tait conjectures,'' some of which have just been proven in the last decade. The modern theory of magnetic flux tubes in perfectly conducting incompressible fluids may be considered an extension of Helmholtz's work on vortex rings.[12]

After Tait's work many important developments in knot theory were made in the first half of this century by mathematicians such as Seifert and Alexander. The next significant conjunction of physics and knot theory occurred in 1985, when the physicist V. Jones realized the relevance of his results concerning von Neumann algebras to the theory of knots, and introduced an important invariant called the *Jones polynomial*.[13] This turned out to be only the first in a whole series of new invariants. In the course of this work intimate relationships between knot theory and soluble models in statistical mechanics were uncovered. In this context the methods of the theory of *quantum groups* played an important role. This whole development is often referred to as the *Jones revolution*.[1]

It turns out, however, that even with the help of all these new invariants we are not in a position to solve the original problem of classifying knots and links. The latest twist in our story involves the work of the Russian mathematician Vassiliev.[14] Vassiliev proposed a class of new invariants which he called invariants of finite type. He conjectured that this class might be powerful enough to solve the classification problem.

A surprising link between quantum field theory and knot theory was discovered in a separate line of development. In the early 1970's theoretical physicists became interested in nonperturbative effects in quantum field theories, especially in their topological aspects, which are relevant to studies of solitons, integral quantum numbers, tunneling effects in Yang–Mills theory, and anomalies.[15] Topological effects result, e.g., when a Chern–Simons term is present in the Lagrangian of the field theory under consideration. Such field theories were studied, beginning in 1980, by Jackiw[16] and others.[17]

Chern–Simons forms originally arose in mathematics in connection with studies of invariant polynomials and de Rham cohomology classes.[18] A gauge field theory in three dimensions which has a Chern–Simons form for its Lagrangian is an example of a *topological field theory*, since its action is metric independent. The observables of a gauge theory are quantities which physicists call ''Wilson loops,'' closely related to what mathematicians call ''holonomies.'' In a topological field theory the vacuum expectation value of such a Wilson loop cannot depend on any metric properties of the loop, in contrast, e.g., to Yang–Mills theory, where this expectation value depends on the area of the loop in a way which is connected with the *confinement* phenomena of the strong interactions.[19] The expectation value in a topological field theory can thus only depend on how the loop is knotted, in other words it must be a knot invariant. Following up on suggestions made by Polyakov[20] and Atiyah,[21] Witten[22] proved in 1989 that in the SU(2) Chern–Simons theory the expectation value of a given loop is related to the Jones polynomial of that loop. It was later shown that using other gauge groups yields many of the other new knot invariants.[23]

In a series of papers by Witten's erstwhile student Bar-Natan,[24] the topologists Birman and Lin,[25] and the field theorists Guadagnini *et al.*,[26] it was proven that the individual terms in the perturbation expansion of the vacuum expectation value in powers of the inverse coupling constant correspond to the Vassiliev numbers. Bar-Natan and others have already proven that the Vassiliev invariants indeed separate a large class of knots.[27]

Chern–Simons theory and knot solutions continue as active research areas in theoretical physics. I refer here only to applications involving quantum gravity,[28] cosmology,[29] and finite-energy solitons.[30]

In the final section of this paper I shall make some of these last remarks more explicit, indicate how the connection between knot theory and quantum field theory came to light, and sketch some recent work which shows that at least some of the Vassiliev invariants may be cast in the form of ''generalized linking numbers.''

VII. CHERN–SIMONS THEORY AND VASSILIEV INVARIANTS

The Chern–Simons theory is a gauge field theory, where the gauge fields $A^a_\mu(\mathbf{x})$ are defined on the space S^3. The index μ runs from 1 to 3, corresponding to the three dimensions of S^3, and the index a is a group index which runs from 1 to dim G, where G is the gauge group. The action is

$$S_{\mathrm{CS}}=\frac{k}{8\pi}\int d^3x\,\epsilon^{\mu\nu\lambda}\Big[A^a_\mu(\mathbf{x})\partial_\nu A^a_\lambda(\mathbf{x}) - \tfrac{1}{3}f_{abc}A^a_\mu(\mathbf{x})A^b_\nu(\mathbf{x})A^c_\lambda(\mathbf{x})\Big], \tag{16}$$

where $\epsilon^{\mu\nu\lambda}$ is the totally antisymmetric Levi–Cevita tensor, f_{abc} are the structure constants of the gauge group, and repeated indices are summed. The Wilson loop associated with a knot K in S^3 is

$$W(K)=\mathrm{tr}\,P\left[\exp\left(i\oint_K A^a_\mu(\mathbf{x})T_a dx^\mu\right)\right], \tag{17}$$

where the T_a are the generators of the gauge group, and P denotes path ordering of the terms in the expansion of the exponential. For a link with n components, $L=\{K_1,\dots,K_n\}$, the Wilson loop is $W(L)=W(K_1)\dots W(K_n)$.

The vacuum expectation value of the Wilson loop is

$$\langle W(K)\rangle=\int \mathscr{D}A\{W(K)\}\exp(iS_{\mathrm{CS}}[A]), \tag{18}$$

where $\int\mathscr{D}A$ indicates a Feynman integral over the gauge fields. The propagator function in quantum field theory is $D^{ab}_{\mu\nu}(\mathbf{x},\mathbf{x}')=\langle A^a_\mu(\mathbf{x})A^b_\nu(\mathbf{x}')\rangle$, for the Chern–Simons theory it works out, in the Landau gauge, to

$$D^{ab}_{\mu\nu}(\mathbf{x},\mathbf{x}')=\frac{i}{k}\,\delta^{ab}\epsilon_{\mu\nu\lambda}\,\frac{(\mathbf{x}-\mathbf{x}')^\lambda}{|\mathbf{x}-\mathbf{x}'|^3}. \tag{19}$$

When the expectation value (18) is expanded in terms of the inverse coupling constant, the first-order term is

$$\oint dx^\mu \oint dx'^\nu\, D^{ab}_{\mu\nu}(\mathbf{x},\mathbf{x}')=\delta^{ab}\int\frac{(\mathbf{x}-\mathbf{x}')\cdot\mathbf{dx}\times\mathbf{dx}'}{|\mathbf{x}-\mathbf{x}'|^3}. \tag{20}$$

Comparing this expression to Eq. (5) we see that it is nothing other than our old friend the linking number. Polyakov[20] was the first to notice the occurrence of the linking number in this context, thus drawing attention to a possible connection between Chern–Simons theory and knot theory. Since we are here considering a single knot, i.e., a one-component link, one may legitimately ask what is being linked here. The answer involves the regularisation procedure used to make sense of the Feynman integrals. In analogy to the familiar *point-splitting method*, one regularises the terms in the Chern–Simons theory by using *framed knots*.[22] That is, the one-dimensional knot is first thickened to a two-dimensional ribbon, and at the end of the calculation the width of the ribbon is set to zero. The linking number in Eq. (20) actually describes how the edges of the ribbon are linked when the ribbon is twisted, see Fig. 8.

Witten considered the Chern–Simons theory with the gauge group SU(2), and found that in this case

$$\langle W(K)\rangle\approx J_K(q), \tag{21}$$

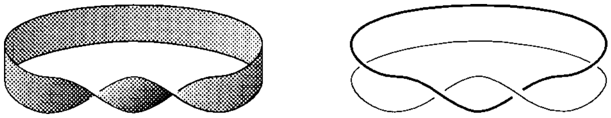

Fig. 8. A framed knot.

where $J_K(q)$ is the Jones polynomial associated to the knot K, and $q = \exp(-2\pi i/k)$.[23] To make the connection indicated in this relation precise would involve field-theoretic technicalities beyond the scope of this article. For details see Ref. 23 and references therein. A theorem of Birman and Lin[25] concerning the Jones polynomial then implies that the separate terms in the perturbation series for $\langle W(K)\rangle$ are Vassiliev invariants.

The standard technique involving Feynman diagrams may be used to calculate the terms of the perturbation series. In performing this calculation for the second-order term we encountered Feynman integrals which, in analogy to the integral for the linking number in Sec. III, could be evaluated by use of the *flat knot limit*. For the relevant second-order Vassiliev invariant (also known as the total twist[31]) we found by this method the following expression:[8]

$$v_2(K) = \frac{1}{4} \sum_{j_1 > j_2 > j_3 > j_4} [\epsilon(j_1, j_3)\epsilon(j_2, j_4) - \epsilon_\alpha(j_1, j_3)\epsilon_\alpha(j_2, j_4)]. \quad (22)$$

The notation used in this expression is explained below. First, choose for the oriented knot diagram an arbitrary but fixed point, the *basepoint* (it turns out that all final results are independent of this arbitrary choice). Now, starting from this point, traverse the knot in the direction dictated by its orientation. Each time a crossing is encountered assign it a number, e.g., the number 1 for the first crossing. On traversing the entire knot each crossing will be encountered exactly twice, so each crossing is characterized by two numbers. Now define the *crossing function*: $\epsilon(j_1, j_2) = +1$ if the numbers j_1, j_2 are associated with the same crossing, and this crossing is an *overcrossing*; $\epsilon(j_1, j_2) = -1$ if it is an *undercrossing*; $\epsilon(j_1, j_2) = 0$ if j_1, j_2 are associated with different crossings. The indices j_i obviously run from 1 to $2m$, where m is the number of crossings. This takes care of the first term in Eq. (22).

The second term in Eq. (22) involves the concept of the *standard ascending diagram* K_α associated with a knot diagram K. Start from the basepoint of the original diagram K and traverse it again in the direction of its orientation, but this time, each time you encounter a crossing *for the first time*, change it into an undercrossing, irrespective of whether it was originally an overcrossing or an undercrossing. The resulting diagram is the *standard ascending diagram* K_α. It turns out that for any knot K, the diagram K_α is always the knot diagram associated with the *trivial knot*, which is just an unknotted loop. This is illustrated for the trefoil knot in Fig. 9. Finally, $\epsilon_\alpha(j_1, j_2)$ is the crossing function for the standard ascending diagram K_α.

Comparing Eq. (22) with Eq. (12), we see in what sense the Vassiliev invariant $v_2(K)$ may be considered a generalisation of the linking number, the first link invariant of all.

We have also calculated the third-order terms in the expansion of the expectation values of the Wilson loops in the

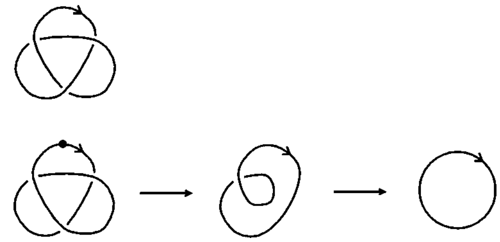

Fig. 9. The standard ascending diagram for the trefoil knot.

Chern–Simons theory, and found similar expressions for the Vassiliev invariant $v_3(K)$,[32] as well as for invariants associated with multicomponent links.[33]

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