On the Dynamics of the Electron

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Introduction.

It seems at first sight that the aberration of light and the related optical and electrical phenomena will provide us a means of determining the absolute motion of the Earth, or rather its motion, not in relation to the other stars, but in relation to the ether. Fresnel had already tried it, but he recognized soon that the motion of the earth does not alter the laws of refraction and reflection. Similar experiments, like that of a telescope filled with water and all those which take into consideration only terms of first order in respect to aberration, give no other but negative results; soon an explanation was discovered; but Michelson, having imagined an experiment where the terms depending on the square of the aberration became sensitive, failed as well.

It seems that this impossibility of demonstrating an experimental evidence for absolute motion of the Earth is a general law of nature; we are naturally led to admit this law, which we will call the *Postulate of Relativity* and admit it without restriction. This postulate, which is up to now in accord with experiments, may be either confirmed or disproved later by more precise experiments, it is in any case interesting to see which consequences follow from it.

An explanation was proposed by LORENTZ and FITZGERALD, who introduced the hypothesis of a contraction undergone by all bodies into the direction of the motion of earth and proportional to the square of aberration; this contraction, which we will call *LORENTZ contraction*, would give an account of the experiment of MICHELSON and all those which were carried out up to now. The hypothesis would become insufficient, however, if one were to assume the postulate of relativity in all its generality.

LORENTZ sought to supplement and modify it in order to put it in perfect agreement with this postulate. He succeeded in doing so in his article entitled *Electromagnetic phenomena in a system moving with any velocity smaller than that of light* (*Proceedings* de l'Académie d'Amsterdam, May 27, 1904).

The importance of the question determined me to take it up again; the results which I obtained are in agreement with those of LORENTZ on all important points; I was only led to modify and supplement them in some points of detail; one will further see the differences which are of secondary importance.

The idea of LORENTZ can be summarized as follows: if we can bring the whole system to a common translation, without modification of any of the apparent phenomena, it is because the equations of the electromagnetic medium are not altered by certain transformations, which we will call *Lorentz transformation*; two systems, one motionless, the other in translation, thus become exact images of one another.

Langevin[1] had sought to modify the idea of Lorentz; for both authors the moving electron takes the shape of a flattened ellipsoid, but for Lorentz two of the axes of the ellipsoid remain constant, while for Langevin on the contrary it is the volume of the ellipsoid which remains constant. Besides, both scientists showed hat these two hypothesis are in agreement with the experiments of Kaufmann, as well as the original hypothesis of Abraham (undeformable spherical electron).

The advantage of the theory of Langevin is that it uses only electromagnetic forces and binding forces; but it is incompatible with the postulate of relativity; this is what Lorentz had shown, this is what I find again in another way by relying upon the principles of group theory.

It is thus necessary to return from here to the theory of Lorentz; but if one wants to preserve it and avoid intolerable contradictions, it is necessary to suppose a special force which explains at the same time the contraction and the constancy of two of the axes. I sought to determine this force, I found that *it can be compared to a constant external pressure, acting on the deformable and compressible electron, and whose work is proportional to the variations of the volume of the electron.*

So if the inertia of matter is exclusively of electromagnetic origin, as it is generally admitted since the experiment of Kaufmann, and except that constant pressure from which I come to speak, all forces are of electromagnetic origin, the postulate of relativity can be established in any rigour. It is what I show by a very simple calculation founded on the principle of least action.

But this is not all. LORENTZ, in the quoted work, considered it to be necessary to supplement his hypothesis so that the postulate remains when there are other forces as the electromagnetic forces. According to him, all the forces, whatever is their origin, are affected by the LORENTZ transformation (and consequently by a translation) in the same way as the electromagnetic forces.

It was important to examine this assumption more closely and in particular to seek which modifications it would oblige us to bring to the laws of gravitation.

It is found at first sight, that we are forced to suppose that the propagation of gravitation is not instantaneous, but happens with the speed of light. One could believe that this is a sufficient reason to reject the hypothesis, as LAPLACE has shown that this cannot be so. But actually, this propagation effect is mainly compensated by a different cause, so that there is no more contradiction between the proposed law and the astronomical observations.

Is it possible to find a law, which satisfies the condition imposed by LORENTZ, and which at the same time is reduced to the law of Newton when the speeds of the stars are rather small, so that one can neglect their squares (as well as the product of acceleration and distance) in respect to the square speed of light?

To this question, as it further will be seen, one must answer in the affirmative.

Is the law thus amended compatible with the astronomical observations?

At first sight it seems that it is the case, but this question can be decided only by a thorough discussion.

But even accepting that the discussion turns to the advantage of a new hypothesis, what should we conclude? If the propagation of attraction happens with the speed of light, it cannot be by a fortuitous coincidence, it must be due to a function of the ether; and then it will be necessary to seek to penetrate the nature of this function, and to relate it to the other functions of the fluid.

We cannot be satisfied with simply juxtaposed formulas which would agree only by a lucky stroke; it is necessary that these formulas are so to speak able to be penetrated mutually. Our mind will not be satisfied before it believes to see the reason of this agreement, at the point where it has the illusion that it could have predicted it.

But the question can still be seen form another point of view, which could be better understood by analogy. Let us suppose an astronomer before Copernicus who reflects on the system of Ptolemy; he will notice that for all planets one of the two circles, epicycle or deferent, is traversed in the same time. This cannot be by chance, there is thus between all planets a mysterious binding.

But Copernicus, by simply changing the axes of coordinates regarded as fixed, destroyed this appearance; each planet does not describe any more than only one circle and the durations of the revolutions become independent (until Kepler restores between them the binding which was believed to be destroyed).

Here it is possible that there is something analogue; if we admit the postulate of relativity, we would find in the law of gravitation and the electromagnetic laws a common number which would be the speed of light; and we would still find it in all the other forces of any origin, which could be explained only in two manners:

Either there would be nothing in the world which is not of electromagnetic origin.

Or this part which would be, so to speak, common to all the physical phenomena, would be only apparent, something which would be due to our methods of measurement. How do we perform our measurements? By transportation, one on the other, of objects regarded as invariable solids, one will answer immediately; but this is not true any more in the current theory, if the LORENTZ contraction is admitted. In this theory, two equal lengths are, by definition, two lengths for which light takes the same time to traverse.

Perhaps it would be enough to give up this definition, so that the theory of LORENTZ is as completely rejected as it was the system of PTOLEMY by the intervention of COPERNICUS. If that happens one day, it will not prove that the effort made by LORENTZ was useless; because PTOLEMY, no matter what we think about him, was not useless for COPERNICUS.

Also I did not hesitate to publish these few partial results, although in this moment even the whole theory seems to be endangered by the discovery of magnetocathodic rays.

§ 1. — LORENTZ transformation

LORENTZ had adopted a particular system of units, so as to eliminate the factors 4π in the formulas. I'll do the same, plus I choose the units of length and time so that the speed of light is equal to 1. Under these conditions the fundamental formulas become (by calling f, g, h the electric displacement, α, β, γ the magnetic force, F, G and H the vector potential, φ the scalar potential, ρ the electric density, ξ, η, ζ the electron velocity, *u*, *v*, *w* the current):

$$(1)\quad \begin{cases} u = \frac{df}{dt} + \rho\xi = \frac{d\gamma}{dy} - \frac{d\beta}{dz}, \quad \alpha = \frac{dH}{dy} - \frac{dG}{dz}, \; f = -\frac{dF}{dt} - \frac{d\psi}{dx}, \\ \frac{d\alpha}{dt} = \frac{dg}{dz} - \frac{dh}{dy}, \quad \frac{d\rho}{dt} + \sum \frac{d\rho\xi}{dx} = 0, \quad \sum \frac{df}{dx} = \rho, \quad \frac{d\psi}{dt} + \sum \frac{dF}{dx} = 0, \\ \square = \Delta - \frac{d^2}{dt^2} = \sum \frac{d^2}{dx^2} - \frac{d^2}{dt^2}, \quad \square\psi = -\rho, \quad \square F = -\rho\xi. \end{cases}$$

A material element of volume *dx dy dz* suffers a mechanical force whose components X *dx dy dz*, Y *dz dx dy*, Z *dx dy dz* are deduced from the formula:

$$(2)\qquad X = \rho f + \rho(\eta\gamma - \zeta\beta)$$

These equations are capable of a remarkable transformation discovered by LORENTZ and which owes its interest from the fact, that it explains why no experience is suited to show us the absolute motion of the universe. Let:

$$(3)\qquad x' = kl(x + \epsilon t), \; t' = kl(t + \epsilon x), \; y' = ly, \; z' = lz$$

l and ε are two arbitrary constants, and

$$k = \frac{1}{\sqrt{1-\epsilon^2}}.$$

If we now set:

$$\square' = \sum \frac{d^2}{dx'^2} - \frac{d^2}{dt'^2},$$

it follows:

$$\Box' = \Box l^{-2}.$$

Consider a sphere entrained with the electron in a uniform translational motion, and

$$(x - \xi t)^2 + (y - \eta t)^2 + (z - \zeta t)^2 = r^2,$$

is the equation of that moving sphere whose volume is $\frac{4}{3}\pi r^2$.

The transformation will change it into an ellipsoid, and it is easy to find the equation. It is easily deduced because of equations (3):

$$(3^{\text{bis}}) \quad x = \frac{k}{l}(x' - \epsilon t'),\ t = \frac{k}{l}(t' - \epsilon x'),\ y = \frac{y'}{l},\ z = \frac{z'}{l}.$$

The equation of the ellipsoid becomes:

$$k^2(x' - \epsilon t' - \xi t' + \epsilon \xi x')^2 + (y' - \eta k t' + \eta k \epsilon x')^2 + (z' - \zeta k t' + \zeta k \epsilon x')^2 = l^2 r^2.$$

This ellipsoid moves in uniform motion; for $t' = 0$, it reduces to

$$k^2 x'^2 (1 + \xi \epsilon)^2 + (y' + \eta k \epsilon x')^2 + (z' + \zeta k \epsilon x')^2 = l^2 r^2,$$

and has the volume:

$$\frac{4}{3}\pi r^3 \frac{l^3}{k(1 + \xi \epsilon)}.$$

If we want that the charge of an electron is not altered by the transformation, and when we call ρ' the new electrical density, it follows:

(4) $$\rho' = \frac{k}{l^3}(\rho + \epsilon\rho\xi).$$

Those are the new velocities ξ', η', ζ '; we must have:

$$\xi' = \frac{dx'}{dt'} = \frac{d(x+\epsilon t)}{d(t+\epsilon x)} = \frac{\xi+\epsilon}{1+\epsilon\xi},$$

$$\eta' = \frac{dy'}{dt'} = \frac{dy}{kd(t+\epsilon x)} = \frac{\eta}{k(1+\epsilon\xi)}, \quad \zeta' = \frac{\zeta}{k(1+\epsilon\xi)},$$

where:

4bis $$\rho'\xi' = \frac{k}{l^3}(\rho\xi + \epsilon\rho), \quad \rho'\eta' = \frac{1}{l^3}\rho\eta, \quad \rho'\zeta' = \frac{1}{l^3}\rho\zeta$$

Here I should mention for the first time a discrepancy with LORENTZ.

LORENTZ poses (with different notations) (loco citato, page 813, formulas 7 and 8):

$$\rho' = \frac{1}{kl^3}\rho, \quad \xi' = k^2(\xi+\epsilon), \quad \eta' = k\eta, \quad \zeta' = k\zeta.$$

We thus find the formulas:

$$\rho'\xi' = \frac{k}{l^3}(\rho\xi + \epsilon\rho), \quad \rho'\eta' = \frac{1}{l^3}\rho\eta, \quad \rho'\zeta' = \frac{1}{l^3}\rho\zeta;$$

but the value of ρ' differs.

It is important to note that formulas (4) and (4^{bis}) satisfy the continuity condition

$$\frac{d\rho'}{dt'} + \sum \frac{d\rho'\xi'}{dx'} = 0.$$

Indeed, let λ be an undetermined quantity and D the functional determinant

(5) $$t + \lambda\rho,\ x + \lambda\rho\xi,\ x + \lambda\rho\eta,\ z + \lambda\rho\zeta$$

with respect to *t, x, y, z*. We will have:

$$D = D_0 + D_1\lambda + D_2\lambda^2 + D_3\lambda^3 + D_4\lambda^4$$

$$\text{with } D_0 = 1,\ D_1 = \frac{d\rho}{dt} + \sum \frac{d\rho\xi}{dx} = 0.$$

Let $\lambda' = l^2\lambda$, we see that the four functions

5^{bis} $$t' + \lambda'\rho',\ x' + \lambda'\rho'\xi',\ y' + \lambda'\rho'\eta',\ z' + \lambda'\rho'\zeta'$$

are related to the functions (5) by the same linear relations as the old variables to the new variables. Then, if we denote by D' the functional determinant of the functions (5^{bis}) in relation to the new variables, we have:

$$D' = D,\ D' = D'_0 + D'_1\lambda' + \ldots + D'_4\lambda'^4,$$

where:

$$D'_0 = D_0 = 1,\ D'_1 = l^{-2}D_1 = 0 = \frac{d\rho'}{dt'} + \sum \frac{d\rho'\xi'}{dx'}.\ \text{C. Q. F.}$$

D

With the hypothesis of LORENTZ, this condition is not satisfied, since ρ' has not the same value.

We will define the new potentials, vector and scalar, in order to satisfy the conditions

$$\Box'\psi' = -\rho', \quad \Box' F' = -\rho'\xi'. \tag{6}$$

Then we obtain from this:

$$\psi' = \frac{k}{l}(\psi + \epsilon F),\ F' = \frac{k}{l}(F + \epsilon\psi),\ G' = \frac{1}{l}G,\ H' = \frac{1}{l}H. \tag{7}$$

These formulas differ significantly from those of LORENTZ, but the difference is ultimately due to the definitions.

We will choose the new electric and magnetic fields so as to satisfy the equations:

$$f' = -\frac{dF'}{dt'} - \frac{d\psi'}{dx'}, \quad \alpha' = \frac{dH'}{dy'} - \frac{dG'}{dz'}. \tag{8}$$

It is easy to see that:

$$\frac{d}{dt'} = \frac{k}{l}\left(\frac{d}{dt} - \epsilon\frac{d}{dx}\right), \quad \frac{d}{dx'} = \frac{k}{l}\left(\frac{d}{dx} - \epsilon\frac{d}{dt}\right), \quad \frac{d}{dy'} = \frac{1}{l}\frac{d}{dy}, \quad \frac{d}{dz'} = \frac{1}{l}\frac{d}{dz}$$

and we conclude:

$$(9) \qquad \begin{cases} f' = \dfrac{1}{l^2}f, g' = \dfrac{k}{l^2}(g + \epsilon\gamma), & h' = \dfrac{k}{l^2}(h - \epsilon\beta), \\ \alpha' = \dfrac{1}{l^2}\alpha, \beta' = \dfrac{k}{l^2}(\beta - \epsilon h), & \gamma' = \dfrac{k}{l^2}(\gamma + \epsilon g). \end{cases}$$

These formulas are identical to those of LORENTZ.

Our transformation does not alter the equations (I). Indeed, the continuity condition, and the equations (6) and (8), already provided us with some of the equations (I) (except the accentuation of letters).

Equations (6) close to the continuity condition give:

$$(10) \qquad \frac{d\psi'}{dt'} + \sum \frac{dF'}{dx'} = 0.$$

It remains to establish that:

$$\frac{df'}{dt'} + \rho'\xi' = \frac{d\gamma'}{dy'} - \frac{d\beta'}{dz'}, \quad \frac{dz'}{dt'} = \frac{dg'}{dz'} - \frac{dh'}{dy'}, \quad \sum \frac{df'}{dx'} = \rho'$$

and it is easy to see that these are necessary consequences of equations (6), (8) and (10).

We must now compare the force before and after transformation.

Let X, Y, Z be the force before, and X', Y', Z' the force after transformation, both related to unit volume. In order for X' to satisfy the same equations as before the transformation, we must have:

$$X' = \rho' f' + \rho'(\eta'\gamma' - \zeta'\beta'),$$
$$Y' = \rho' g' + \rho'(\zeta'\alpha' - \xi'\gamma'),$$
$$X' = \rho' h' + \rho'(\xi'\beta' - \eta'\alpha'),$$

or, replacing all quantities by their values (4), (4^{bis}) and (9) and taking into account equations (2):

$$(11) \qquad \begin{cases} X' = \dfrac{k}{l^5}(X + \epsilon \sum X\xi), \\ Y' = \dfrac{1}{l^5} Y, \\ Z' = \dfrac{1}{l^5} Z. \end{cases}$$

If we represent the components of the force X_1, Y_1, Z_1, not per unit volume, but per unit of electric charge of the electron, and X'_1, Y'_1, Z'_1 are the same quantities after the transformation, we would have:

$$X_1 = f + \eta\gamma - \zeta\beta,\ X'_1 = f' + \eta'\gamma' - \zeta'\beta',\ X = \rho X_{1,}\ X' = \rho' X'_1$$

and we would have the equations

$$(11^{\text{bis}})\qquad \begin{cases} X_1' = \dfrac{k}{l^5}\dfrac{\rho}{\rho'}(X_1 + \epsilon \sum X_1 \xi), \\ Y_1' = \dfrac{1}{l^5}\dfrac{\rho}{\rho'} Y_1, \\ Z_1' = \dfrac{1}{l^5}\dfrac{\rho}{\rho'} Z_1. \end{cases}$$

LORENTZ found [with different notation, page 813, formula (10)]:

$$(11^{\text{ter}})\qquad \begin{cases} X_1 = l^2 X_1' - l^2 \epsilon(\eta' g' + \zeta' h'), \\ Y_1 = \dfrac{l^2}{k} Y_1' + \dfrac{l^2 \epsilon}{k} \xi' g', \\ Z_1 = \dfrac{l^2}{k} Z_1' + \dfrac{l^2 \epsilon}{k} \xi' h'. \end{cases}$$

Before going further, it is important to investigate the cause of this significant discrepancy. It is obvious that the formulas for ξ', η', ζ' are not the same, while the formulas for the electric and magnetic fields are the same.

If the inertia of electrons is exclusively of electromagnetic origin, if in addition they are subject only to forces of electromagnetic origin, the equilibrium condition requires that we have inside the electrons:

$$X = Y = Z = 0.$$

But in virtue of equations (11) those relations are equivalent to

$$X' = Y' = Z' = 0.$$

The equilibrium conditions of the electrons are not altered by the transformation.

Unfortunately, a hypothesis as simple as that is unacceptable. If, indeed, we assume $\xi = \eta = \zeta = 0$, the conditions $X = Y = Z = 0$ entrain $f = g = h = 0$, and consequently $\sum \frac{df}{dx} = 0$, *i.e.* ρ = 0. We arrive at similar results in the most general case. We must therefore admit that there are, in addition to electromagnetic forces, either other forces or bindings. It is necessary to search for conditions which must satisfy these forces or bindings, so that the equilibrium of the electron is not disturbed by the transformation. This will be the subject of a later paragraph.

§ 2. — Principle of least action

We know how Lorentz deduced his equations from the principle of least action. I will return to this question, even though I have nothing substantial to add to the analysis of Lorentz, because I prefer to present it in a slightly different form which will be useful for my purpose. I will pose:

$$(1) \qquad J = \int dt\, d\tau \left[\frac{\sum f^2}{2} + \frac{\sum \alpha^2}{2} - \sum F u \right],$$

assuming that *f*, α, F, *u*, etc.. are subject to the following conditions and the ones deduced by symmetry:

$$(2) \qquad \sum \frac{df}{dx} = \rho, \quad \alpha = \frac{dH}{dy} - \frac{dG}{dz}, \quad u = \frac{df}{dt} + \rho\xi.$$

Regarding the integral J, it must be extended:

I° in relation to the volume element dτ = *dx dy dz* over the whole space;

2° in relation to time t, over the interval between the limits $t = t_0$, $t = t_1$.

According to the principle of least action, the integral J must be a minimum, if one sets the various quantities which appear in:

1° the conditions (2);

2° the condition that the state of the system is determined by both limiting times $t = t_0$, $t = t_1$.

This last condition allows us to transform our integral by partial integration with respect to time. If we have indeed an integral of the form

$$\int dt\, d\tau A \frac{dB\delta C}{dt},$$

where C is a quantity that defines the system state and its variation δC, it will be equal to (by partial integration with respect to time):

$$\int d\tau \mid AB\delta C \mid \begin{matrix} t = t_1 \\ t = t_0 \end{matrix} - \int dt\, d\tau \frac{dA}{dt} dB\delta C.$$

Since the system state is determined by both limiting times, it is δC = 0 for $t = t_0$, $t = t_1$, so the first integral which is related to these two periods is zero, and the 2nd one remains.

We can also integrate by parts with respect to x, y or z, we have indeed

$$\int A \frac{dB}{dx} dx\, dy\, dz\, dt = \int AB\, dy\, dz\, dt - \int B \frac{dA}{dx} dx\, dy\, dz\, dt.$$

Our integrations are extended to infinity, it must be $x = \pm\infty$ in the first integral on the right-hand side; so, since we always assume that all our functions vanish at infinity, this integral will be zero and it follows

$$\int A\frac{dB}{dx}d\tau\, dt = -\int B\frac{dA}{dx}d\tau\, dt.$$

If the system is supposed to be subject to bindings, the binding conditions should be connected to the conditions imposed on the various quantities appearing in the integral J.

Let us first give to F, G, H the increasements δF, δG, δH; where:

$$\delta\alpha = \frac{d\delta H}{dy} - \frac{d\delta G}{dz}.$$

We should have

$$\delta J = \int dt\, d\tau \left[\sum \alpha\left(\frac{d\delta H}{dy} - \frac{d\delta G}{dz}\right) - \sum u\delta F\right] = 0,$$

or, integrating by parts,

$$\delta J = \int dt\, d\tau \left[\sum \left(\delta G\frac{d\alpha}{dz} - \delta H\frac{d\alpha}{dy}\right) - \sum u\delta F\right] =$$

$$= -\int dt\, d\tau \sum \delta F\left(u - \frac{d\gamma}{dy} + \frac{d\beta}{dz}\right) = 0$$

whence, by setting the arbitrary coefficient δF equal to zero,

(3) $$u = \frac{d\gamma}{dy} - \frac{d\beta}{dz}.$$

This relationship gives us (by partial integration):

$$\int \sum F u d\tau = \int \sum F \left(\frac{d\gamma}{dy} - \frac{d\beta}{dz} \right) d\tau = \int \sum \left(\beta \frac{dF}{dz} - \gamma \frac{dF}{dy} \right) d\tau =$$

$$= \int \sum \alpha \left(\frac{dH}{dy} - \frac{dG}{dz} \right) d\tau,$$

or

$$\int \sum F u d\tau = \int \sum \alpha^2 d\tau,$$

hence finally:

(4) $$J = \int dt \, d\tau \left(\frac{\sum f^2}{2} - \frac{\sum \alpha^2}{2} \right).$$

Now, thanks to equation (3), δJ is independent from δF and thus δα; let us vary now the other variables

It follows, by returning to expression (1) of J,

$$\delta J = \int dt \, d\tau \left(\sum f \delta f - \sum F \delta u \right).$$

But *f*, *g*, *h* are first subject to conditions (2), so that

$$(5) \qquad \sum \frac{d\delta f}{dx} = \delta\rho$$

and for convenience we write:

$$(6) \quad \delta J = \int dt\, d\tau \left[\sum f df - \sum F\delta u - \psi\left(\sum \frac{d\delta f}{dx} - \delta\rho\right)\right].$$

The principles of variation calculus tells us that we must do the calculation as if ψ is an arbitrary function, as if δJ is represented by (6), and as if the changes were no longer subject to the condition (5).

We have in addition:

$$\delta u = \frac{d\delta f}{dt} + \delta\rho\xi,$$

whence, after partial integration,

$$(7) \quad \delta J = \int dt\, d\tau \sum \delta f\left(f + \frac{dF}{dt} + \frac{d\psi}{dx}\right) + \int dt\, d\tau\left(\psi\delta\rho - \sum F\delta\rho\xi\right).$$

If we assume at first that the electrons do not undergo a variation, δρ = δρξ = 0 and the second integral is zero. Because δJ must vanish, we should have:

$$(8) \qquad f + \frac{dF}{dt} + \frac{d\psi}{dx} = 0$$

It remains in the general case:

$$\delta J = \int dt\, d\tau \left(\psi \delta\rho - \sum F \delta\rho \xi\right). \tag{9}$$

It remains to determine the forces acting on the electrons. To do this we must suppose that a supplementary force -Xdτ, -Ydτ, -Zdτ applies to each element of an electron, and write that this force is in equilibrium with the forces of electromagnetic origin. Let U, V, W be components of the displacement of the element dτ of the electron, where the displacement is counted from an arbitrary initial position. Let δU, δV, δW be the variations of this displacement; the virtual work corresponding to the supplementary force is:

$$-\int \sum X \delta U\, d\tau,$$

so that the equilibrium condition about which we have spoken can be written:

$$\delta J = -\int \sum X \delta U\, d\tau\, dt. \tag{10}$$

It's about the transformation of δJ. To begin the search for the continuity equation, we express how the charge of an electron is preserved by the variation.

Let x_0, y_0, z_0 be the initial position of an electron. Its current position is:

$$x = x_0 + U, \quad y = y_0 + V, \quad z = z_0 + W.$$

We also introduce an auxiliary variable ε, which produces changes in our various functions, so that for any function A we have:

$$\delta A = \delta\epsilon \frac{dA}{d\epsilon}$$

It is indeed convenient to switch from the notation of variation calculus to that of ordinary calculus, or vice versa.

Our functions should be regarded: 1° as dependent on five variables *x, y, z, t*, ε, so that we can remain at the same place when ε and *t* vary alone: we then indicate their derivatives by the ordinary *d*; 2° as dependent on five variables x_0, y_0, z_0, *t*, ε so that we may always follow a single electron when *t* and ε vary alone, then we denote their derivatives by ∂. We will have then:

$$(11) \qquad \xi = \frac{\partial U}{\partial t} = \frac{dU}{dt} + \xi\frac{dU}{dx} + \eta\frac{dU}{dy} + \zeta\frac{dU}{dz} = \frac{\partial x}{\partial t}.$$

Denote now by Δ the functional determinant of *x, y, z* with respect to x_0, y_0, z_0:

$$\Delta = \frac{\partial(x,\ y,\ z)}{\partial(x_0,\ y_0,\ z_0)}.$$

If ε, x_0, y_0, z_0 remain constant, we give to *t* an increasement ∂*t*; to x, y, z the increasements ∂x_0, ∂y_0, ∂z_0 will result; and to Δ the increasement ∂Δ, and there will be:

$$\partial x = \xi \partial t, \quad \partial y = \eta \partial t, \quad \partial z = \zeta \partial t,$$

$$\Delta + \partial \Delta = \frac{\partial (x + \partial x,\ y + \partial y,\ z + \partial z)}{\partial (x_0,\ y_0,\ z_0)}$$

hence

$$1 + \frac{\partial \Delta}{\Delta} = \frac{\partial (x + \partial x,\ y + \partial y,\ z + \partial z)}{\partial (x,\ y,\ z)} = \frac{\partial (x + \xi \partial t,\ y + \eta \partial t,\ z + \zeta \partial t)}{\partial (x,\ y,\ z)}$$

We deduce:

$$\frac{1}{\Delta}\frac{\partial \Delta}{\partial t} = \frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz} \tag{12}$$

The mass of each electron is invariable, we have:

$$\frac{\partial \rho \Delta}{\partial t} = 0 \tag{13}$$

where:

$$\frac{\partial \rho}{\partial t} + \sum \rho \frac{d\xi}{dx} = 0, \quad \frac{\partial \rho}{\partial t} = \frac{d\rho}{dt} + \sum \xi \frac{d\rho}{dx}, \quad \frac{d\rho}{dt} + \sum \frac{d\rho\xi}{dx} = 0.$$

These are the different forms of the continuity equation with respect of variable *t*. We find similar forms with respect to the variable ε. Either:

$$\delta U = \frac{\partial U}{\partial \epsilon}\delta\epsilon, \quad \delta V = \frac{\partial V}{\partial \epsilon}\delta\epsilon, \quad \delta W = \frac{\partial W}{\partial \epsilon}\delta\epsilon;$$

it follows:

(11bis) $$\delta U = \frac{dU}{d\epsilon}\delta\epsilon + \delta U\frac{dU}{dx} + \delta V = \frac{dU}{dy} + \delta W\frac{dU}{dz},$$

(12bis) $$\frac{1}{\Delta}\frac{\partial\Delta}{\partial\epsilon} = \sum\frac{dU}{d\epsilon}, \quad \frac{\partial\rho\Delta}{\partial\epsilon} = 0,$$

(13bis) $$\delta\epsilon\frac{\partial\rho}{\partial\epsilon} + \sum\rho\frac{d\rho U}{dx} = 0, \quad \frac{\partial\rho}{\partial\epsilon} = \frac{d\rho}{d\epsilon} + \sum\frac{\delta U}{\delta\epsilon}\frac{d\rho}{dx}, \quad \delta\rho + \frac{d\rho\delta U}{dx} = 0.$$

Note the difference between the definition of $\delta U = \frac{dU}{d\epsilon}\delta\epsilon$ and that of $\delta\rho = \frac{d\rho}{d\epsilon}\delta\epsilon$, we note that it is this definition of δU that suits to formula (10).

This equation will allow us to transform the first term of (9); we find in fact:

$$\int dt\, d\tau\, \psi\delta\rho = -\int dt\, d\tau\, \psi\sum\frac{d\rho\delta U}{dx},$$

or, by partial integration,

(14bis) $$\int dt\, d\tau\, \psi\delta\rho = \int dt\, d\tau\sum\rho\frac{d\psi}{dx}\delta U.$$

Let us propose now to determine

$$\delta(\rho\xi) = \frac{d(\rho\xi)}{d\epsilon}\delta\epsilon.$$

Note that ρΔ does not depend on x_0, y_0, z_0; indeed, if we consider an electron whose initial position is a rectangular parallelepiped whose edges are dx_0, dy_0, dz_0, the charge of this element is

$$\rho\Delta dx_0\ dy_0\ dz_0$$

and this charge should remain constant, then:

$$\frac{\partial\rho\Delta}{\partial t} = \frac{\partial\rho\Delta}{\partial\epsilon} = 0. \tag{15}$$

We deduce:

$$\frac{\partial^2\rho\Delta U}{\partial t\,\partial\epsilon} = \frac{\partial}{\partial\epsilon}\left(\rho\Delta\frac{\partial U}{\partial t}\right) = \frac{\partial}{\partial t}\left(\rho\Delta\frac{\partial U}{\partial\epsilon}\right). \tag{16}$$

Now we know that for any function A, we have by the continuity equation,

$$\frac{1}{\Delta}\frac{\partial A\Delta}{\partial t} = \frac{dA}{dt} + \sum\frac{dA\xi}{dx}$$

and also

$$\frac{1}{\Delta}\frac{\partial A\Delta}{\partial\epsilon} = \frac{dA}{d\epsilon} + \sum\frac{dA\frac{\partial U}{\partial\epsilon}}{dx}$$

We thus have:

$$(17) \quad \frac{1}{\Delta}\frac{\partial}{\partial\epsilon}\left(\rho\Delta\frac{\partial U}{\partial t}\right) = \frac{d\rho\frac{\partial U}{\partial t}}{d\epsilon} + \frac{d\left(\rho\frac{\partial U}{\partial t}\frac{\partial U}{\partial\epsilon}\right)}{dx} + \frac{d\left(\rho\frac{\partial U}{\partial t}\frac{\partial V}{\partial\epsilon}\right)}{dy} + \frac{d\left(\rho\frac{\partial U}{\partial t}\frac{\partial W}{\partial\epsilon}\right)}{dz},$$

$$(17^{\text{bis}}) \quad \frac{1}{\Delta}\frac{\partial}{\partial t}\left(\rho\Delta\frac{\partial U}{\partial\epsilon}\right) = \frac{d\rho\frac{\partial U}{\partial\epsilon}}{dt} + \frac{d\left(\rho\frac{\partial U}{\partial t}\frac{\partial U}{\partial\epsilon}\right)}{dx} + \frac{d\left(\rho\frac{\partial V}{\partial t}\frac{\partial U}{\partial\epsilon}\right)}{dy} + \frac{d\left(\rho\frac{\partial W}{\partial t}\frac{\partial U}{\partial\epsilon}\right)}{dz}.$$

The right-hand sides of (17) and (17^{bis}) must be equal, and if one remembers that

$$\frac{\partial U}{\partial t} = \xi, \quad \frac{\partial U}{\partial\epsilon}\delta\epsilon = \delta U, \quad \frac{d\rho\xi}{d\epsilon}\delta\epsilon = \delta\rho\xi$$

we get:

$$(18) \quad \delta\rho\xi + \frac{d(\rho\xi\delta U)}{dx} + \frac{d(\rho\xi\delta V)}{dy} + \frac{d(\rho\xi\delta W)}{dz} = \frac{d(\rho\delta U)}{dt} + \frac{d(\rho\xi\delta U)}{dx} + \frac{d(\rho\eta\delta U)}{dy} + \frac{d(\rho\zeta\delta U)}{dz}$$

Transforming now the second term of (9); we get:

$$\int dt\, d\tau \sum F\delta\rho\xi$$

$$= \int dt\, d\tau \left[\sum F\frac{d(\rho\delta U)}{dt} + \sum F\frac{d(\rho\eta\delta U)}{dy} + \sum F\frac{d(\rho\zeta\delta U)}{dz} - \sum F\frac{d(\rho\xi\delta V)}{dy} - \sum F\frac{d(\rho\xi\delta W)}{dz}\right].$$

The right-hand side becomes by partial integration:

$$\int dt\, d\tau \left[-\sum \rho\delta U\frac{dF}{dt} - \sum \rho\eta\delta U\frac{dF}{dy} - \sum \rho\zeta\delta U\frac{dF}{dz} + \sum \rho\xi\delta V\frac{dF}{dy} + \sum \rho\xi\delta W\frac{dF}{dz}\right].$$

Now note, that:

$$\sum \rho\xi\delta V \frac{dF}{dy} = \sum \rho\zeta\delta U \frac{dH}{dx}, \quad \sum \rho\xi\delta W \frac{dF}{dz} = \sum \rho\eta\delta U \frac{dG}{dx}$$

If, indeed, we develop Σ on the two sides of these relations, they become identities; and remember that

$$\frac{dH}{dx} - \frac{dF}{dz} = -\beta, \quad \frac{dG}{dx} - \frac{dF}{dy} = \gamma$$

the right-hand side in question will become:

$$\int dt\, d\tau \left[-\sum \rho\delta U \frac{dF}{dt} - \sum \rho\gamma\eta\delta U - \sum \rho\beta\zeta\delta U \right],$$

so that finally:

$$\delta J = \int dt\, d\tau \sum \rho\delta U \left(\frac{d\psi}{dx} + \frac{dF}{dt} + \beta\zeta - \gamma\eta \right) = \int dt\, d\tau \sum \rho\delta U \left(-f + \beta\zeta - \gamma\eta \right).$$

Equating the coefficient of δU on both sides of (10) we get:

$$X = f - \beta\zeta + \gamma\eta$$

This is equation (2) of the preceding §.

§ 3. — The Lorentz transformation and the principle of least action

Let us see if the principle of least action gives us the reason for the success of the Lorentz transformation. We must look at the transformation of the integral:

$$J=\int dt\,d\tau\left(\frac{\sum f^2}{2}-\frac{\sum\alpha^2}{2}\right)$$

(formula 4 of § 2).

We first find

$$dt'\ d\tau'=l^4dt\ d\tau$$

because *x'*, *y'*, *z'*, *t'* are related to *x, y, z, t* by linear relations whose determinant is equal to l^4; then we have:

$$(1)\quad\begin{cases} l^4\sum f'^2=f^2+k^2(g^2+h^2)+k^2\epsilon^2(\beta^2+\gamma^2)+2k^2\epsilon(g\gamma-h\beta)\\ l^4\sum\alpha'^2=\alpha^2+k^2(\beta^2+\gamma^2)+k^2\epsilon^2(g^2+h^2)+2k^2\epsilon(g\gamma-h\beta)\end{cases}$$

(formula 9 of § 1), hence:

$$l^4\left(\sum f'^2-\sum\alpha'^2\right)=\sum f^2-\sum\alpha^2$$

so that if we set

$$J'=\int dt'\ d\tau'\left(\frac{\sum f'^2}{2}-\frac{\sum\alpha'^2}{2}\right),$$

we get:

$$J'=J.$$

However, to justify this equality, the integration limits have to be the same; so far we have assumed that *t* varies from t_0 to t_1, and *x, y, z* from ∞ to + ∞. On this account the integration limits would be affected by the LORENTZ transformation, but nothing prevents us from assuming $t_0 =- \infty$, $t_1 = + \infty$; with those conditions the limits are the same for J and J'.

We then compare the following two equations analogues to equation (10) of § 2:

$$(2) \qquad \begin{cases} \delta J = -\int \sum X \delta U \, d\tau \, dt \\ \delta J' = -\int \sum X' \delta U' \, d\tau' \, dt' . \end{cases}$$

For this, we must first compare δU with δU.

Consider an electron whose initial coordinates are x_0, y_0, z_0; its coordinates at the instant t are

$$x = x_0 + U, \quad y = y_0 + V, \quad z = z_0 + W.$$

If one considers the electron after the corresponding LORENTZ transformation, it will have as coordinates

$$x' = kl\,(x + \epsilon t)\,, \quad y' = ly, \quad z' = lz$$

where

$$x' = x_0 + U', \quad y' = y_0 + V', \quad z' = z_0 + W'$$

but it will only attain these coordinates at the instant

$$t' = kl\,(t + \epsilon x)\,.$$

If we subject our variables to the variations δU, δV, δW, and when we give at the same time t an increasement δt, the coordinates x, y, z

will experience a total increasement

$$\delta x = \delta U + \xi \delta t, \quad \delta y = \delta V + \eta \delta t, \quad \delta z = \delta W + \zeta \delta t.$$

We will also have:

$$\delta x' = \delta U' + \xi' \delta t', \quad \delta y' = \delta V' + \eta' \delta t', \quad \delta z' = \delta W' + \zeta' \delta t',$$

and in virtue of the LORENTZ transformation:

$$\delta x' = kl\left(\delta x + \epsilon \delta t\right), \quad \delta y' = l\delta y, \quad \delta z' = l\delta z, \quad \delta t' = kl\left(\delta t + \epsilon \delta x\right).$$

hence, assuming $\delta t = 0$, the relations:

$$\begin{cases} \delta x' = \delta U' + \xi' \delta t' & = kl\delta U, \\ \delta y' = \delta V' + \eta' \delta t' & = l\delta V, \\ \delta t' = kl\epsilon \delta U. \end{cases}$$

Note that

$$\xi' = \frac{\xi + \epsilon}{1 + \xi \epsilon}, \quad \eta' = \frac{\eta}{k\left(1 + \xi \epsilon\right)}$$

It follows, by replacing $\delta t'$ by its value

$$kl\left(1 + \xi \epsilon\right) \delta U = \delta U' \left(1 + \xi \epsilon\right) + \left(\xi + \epsilon\right) kl\epsilon \delta U,$$

$$l\left(1 + \xi \epsilon\right) \delta V = \delta V' \left(1 + \xi \epsilon\right) + \eta l\epsilon \delta U.$$

If we recall the definition of k, we draw from this:

$$\delta U = \frac{k}{l}\delta U' + \frac{k\epsilon}{l}\xi\delta U',$$

$$\delta V = \frac{1}{l}\delta V' + \frac{k\epsilon}{l}\eta\delta U',$$

and also

$$\delta W = \frac{1}{l}\delta W' + \frac{k\epsilon}{l}\zeta\delta U';$$

hence

$$(3)\quad \sum X\delta U = \frac{1}{l}\left(kX\delta U' + Y\delta V' + Z\delta W'\right) + \frac{k\epsilon}{l}\delta U'\sum X\xi.$$

Now, in virtue of equations (2) we must have:

$$\int\sum X'\delta U'dt'\ d\tau' = \int\sum X\delta U\ dt\ d\tau = \frac{1}{l^4}\sum X\delta U\ dt'\ d\tau'.$$

By replacing ΣXδU by its value (3) and by identifying, it follows:

$$X' = \frac{k}{l^5}X + \frac{k\epsilon}{l^5}\sum X\xi, \quad Y' = \frac{1}{l^5}Y, \quad Z' = \frac{1}{l^5}Z.$$

These are the equations (11) of § 1. The principle of least action leads us to the same result as the analysis of § 1.

If we turn to formulas (1), we see that $\Sigma f^2 - \Sigma\alpha^2$ is not affected by the LORENTZ transformation, except one constant factor; it is not the case with expression $\Sigma f^2 + \Sigma\alpha^2$ which represents the energy. If we confine ourselves to the case where ε is sufficiently small, so that the square can be neglected so that $k = 1$, and if we also assume $l = 1$, we find:

$$\sum f'^2 = \sum f^2 + 2\epsilon\left(g\gamma - h\beta\right),$$

$$\sum \alpha'^2 = \sum \alpha^2 + 2\epsilon\left(g\gamma - h\beta\right),$$

or by addition

$$\sum f'^2 + \sum \alpha'^2 = \sum f^2 + \sum \alpha^2 + 4\epsilon\left(g\gamma - h\beta\right).$$

§ 4. — The LORENTZ group

It is important to note that the LORENTZ transformations form a group.

Indeed, if we set:

$$x' = kl(x + \epsilon t), \quad y' = ly, \quad z' = lz, \quad t' = kl(t + \epsilon x),$$

and in addition

$$x'' = k'l'(x' + \epsilon' t'), \quad y'' = l'y', \quad z'' = l'z', \quad t'' = k'l'(t' + \epsilon' x'),$$

with

$$k^{-2} = 1 - \epsilon^2, \quad k'^{-2} = 1 - \epsilon'^2$$

it follows:

$$x'' = k''l''(x + \epsilon''t), \quad y'' = l''y, \quad z'' = l''z, \quad t'' = k''l''(t + \epsilon''x),$$

with

$$\epsilon'' = \frac{\epsilon + \epsilon'}{1 + \epsilon\epsilon'}, \quad l'' = ll', \quad k'' = kk'(1 + \epsilon\epsilon') = \frac{1}{\sqrt{1 - \epsilon''^2}}.$$

If we take for *l* the value 1, and we suppose ε infinitely small,

$$x' = x + \delta x, \quad y' = y + \delta y, \quad z' = z + \delta z, \quad t' = t + \delta t$$

it follows:

$$\delta x = \epsilon t, \quad \delta y = \delta z = 0, \quad \delta t = \epsilon x.$$

This is the infinitesimal generator of the transformation group, which I call the transformation T_1, and which can be written in LIE's notation:

$$t\frac{d\varphi}{dx} + x\frac{d\varphi}{dt} = T_1.$$

If we assume ε = 0 and *l* = 1 + δ*l*, we find instead

$$\delta x = x\delta l, \quad \delta y = y\delta l, \quad \delta z = z\delta l, \quad \delta t = t\delta l$$

and we would have another infinitesimal transformation t_0 of the group (assuming that l and ε are regarded as independent variables) and we would have with LIE's notation:

$$T_0 = x\frac{d\varphi}{dx} + y\frac{d\varphi}{dy} + z\frac{d\varphi}{dz} + t\frac{d\varphi}{dt}.$$

But we could give the *y*- or *z*-axes the special role, which we gave the *x*-axis; thus we have two further infinitesimal transformations:

$$T_2 = t\frac{d\varphi}{dy} + y\frac{d\varphi}{dt},$$

$$T_3 = t\frac{d\varphi}{dz} + z\frac{d\varphi}{dt},$$

which also would not alter the equations of LORENTZ.

We can form combinations devised by LIE, such as

$$[T_1, T_2] = x\frac{d\varphi}{dy} - y\frac{d\varphi}{dx},$$

but it is easy to see that this transformation is equivalent to a coordinate change, the axes are rotating a very small angle around the *z*-axis. We should not be surprised if such a change does not alter the form of the equations of LORENTZ, obviously independent of the choice of axes.

We are thus led to consider a continuous group which we call the *LORENTZ group* and which admit as infinitesimal transformations:

1° the transformation T_0 which is permutable with all others;

2° the three transformations T_1, T_2, T_3;

3° the three rotations $[T_1, T_2]$, $[T_2, T_3]$, $[T_3, T_1]$.

Any transformation of this group can always be decomposed into a transformation of the form:

$$x' = lx, \quad y' = ly, \quad z' = lz, \quad t' = lt$$

and a linear transformation which does not change the quadratic form

$$x^2 + y^2 + z^2 - t^2$$

We can still generate our group in another way. Any transformation of the group may be regarded as a transformation of the form:

$$(1) \qquad x' = kl(x + \epsilon t), \quad y' = ly, \quad z' = lz, \quad t' = kl(t + \epsilon x)$$

preceded and followed by a suitable rotation.

But for our purposes, we should consider only a part of the transformations of this group; we must assume that *l* is a function of ε, and it is a question of choosing this function in such a way that this part of the group that I call P still forms a group.

Let's rotate the system 180° around the *y*-axis, we should find a transformation that will still belong to P. But this amounts to a sign change of *x*, *x'*, *z* and *z'*; we find:

$$(2) \qquad x' = kl(x - \epsilon t), \quad y' = ly, \quad z' = lz, \quad t' = kl(t - \epsilon x)$$

So l does not change when we change ε into -ε.

On the other hand, if P is a group, then the inverse substitution of (1)

$$(3) \quad x' = \frac{k}{l}(x - \epsilon t), \quad y' = \frac{y}{l}, \quad z' = \frac{z}{l}, \quad t' = \frac{k}{l}(t - \epsilon x),$$

must also belong to P; it will therefore be identical with (2), that is to say that

$$l = \frac{1}{l}.$$

We must therefore have $l = 1$.

§ 5. — Langevin waves

Langevin has put the formulas that define the electromagnetic field produced by the motion of a single electron in a particularly elegant form.

Let us remember the equations

$$(1) \qquad \Box\psi = -\rho, \quad \Box F = -\rho\xi.$$

We know we can integrate by the retarded potentials and we have:

$$(2) \qquad \psi = \frac{1}{4\pi}\int\frac{\rho_1 d\tau}{r}, \quad F = \frac{1}{4\pi}\int\frac{\rho_1\xi_1 d\tau_1}{r}.$$

In these formulas we have:

$$d\tau_1 = dx_1 dy_1 dz_1, \quad r^2 = (x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2,$$

whereas ρ_1 and ξ_1 are the values of ρ and ξ at the point x_1, y_1, z_1 and the instant

$$t_1 = t - r$$

x_0, y_0, z_0 being coordinates of a molecule of the electron at the instant t;

$$x_1 = x_0 + U,\ y_1 = y_0 + V,\ z_1 = z_0 + W$$

being its coordinates at the instant t_1;

U, V, W are functions of x_0, y_0, z_0, so that we can write:

$$dx_1 = dx_0 + \frac{dU}{dx_0}dx_0 + \frac{dU}{dy_0}dy_0 + \frac{dU}{dz_0}dz_0 + \xi_1 dt_1;$$

and if we assume t to be constant, as well as x, y and z:

$$dt_1 = +\sum \frac{x - x_1}{r} dx_1.$$

We can therefore write:

$$dx_1\left(1 + \xi_1\frac{x_1 - x}{r}\right) + dy_1\xi_1\frac{y_1 - y}{r} + dz_1\xi_1\frac{z_1 - z}{r} = dx_0\left(1 + \frac{dU}{dx_0}\right) + dy_0\frac{dU}{dy_0} + dz_0\frac{dU}{dz_0}$$

so that the other two equations can deduced by circular permutation.

We therefore have:

$$(3)\quad d\tau_1 \mid 1 + \xi_1\frac{x_1 - x}{r},\quad \xi_1\frac{y_1 - y}{r},\quad \xi_1\frac{z_1 - z}{r} \mid = d\tau_0 \mid 1 + \frac{dU}{dx_0},\quad \frac{dU}{dy_0},\quad \frac{dU}{dz_0} \mid,$$

we set

$$d\tau_0 = dx_0 dy_0 dz_0$$

Consider the determinants that appear in both sides of (3) and at the begin of the first part; if we seek to develop, we see that the terms of the 2^d and 3^{rd} degree from ξ_1, η_1, ζ_1 disappear and that the determinant is equal to

$$1 + \xi_1 \frac{x_1 - x}{r} + \eta_1 \frac{y_1 - y}{r} + \zeta_1 \frac{z_1 - z}{r} = 1 + \omega,$$

ω designates the radial component of the velocity ξ_1, η_1, ζ_1, that is to say, the component directed along the radius vector indicating from point x, y, t to point x_1, y_1, z_1.

In order to obtain the second determinant, I look at the coordinates of different molecules of the electron at instant t', which is the same for all molecules, but in such a way that for the molecule considered we have $t_1 = t'_1$. The coordinates of a molecule will then be:

$$x'_1 = x_0 + U',\ y'_1 = y_0 + V',\ z'_1 = z_0 + W'$$

U', V', W' is what become of U, V, W, when we replace t_1 by t'_1; since t'_1 is the same for all molecules, we have:

$$dx'_1 = dx_0 \left(1 + \frac{dU'}{dx_0}\right) + dy_0 \frac{dU'}{dy_0} + dz_0 \frac{dU'}{dz_0}$$

and therefore

$$d\tau'_1 = d\tau_0 \mid 1 + \frac{dU'}{dx_0}, \quad \frac{dU'}{dy_0}, \quad \frac{dU'}{dz_0} \mid$$

by setting

$$d\tau_1' = dx_1' dy_1' dz_1'.$$

But the element of electric charge is

$$d\mu_1 = \rho_1 d\tau_1'$$

and moreover *for the molecule considered*, we have $t_1 = t'_1$, and therefore $\frac{dU'}{dx_0} = \frac{dU}{dx_0}$ etc..; we can write:

$$d\mu_1 = \rho_1 d\tau_0 \mid 1 + \frac{dU}{dx_0}, \quad \frac{dU}{dy_0}, \quad \frac{dU}{dz_0} |$$

so that equation (3) becomes:

$$\rho_1 d\tau_1 (1 + \omega) = d\mu_1$$

and equations (2):

$$\psi = \frac{1}{4\pi} \int \frac{d\mu_1}{r(1+\omega)}, \quad F = \frac{1}{4\pi} \int \frac{\xi_1 d\mu_1}{r(1+\omega)}.$$

If we are dealing with a single electron, our integrals are reduced to a single element, provided we consider only the points *x*, *y*, *x* which are sufficiently remote so that *r* and ω have substantially the same value for all points of the electron. The potentials ψ, F, G, H depend on the position of the electron and also its velocity, because not only ξ_1, η_1, ζ_1 show up in the numerator of F, G, H, but the radial component ω shows up in the denominator. It is of course its position and its velocity at the instant t_1.

The partial derivatives of φ, F, G, H with respect to *t*, *x*, *y*, *z* (and therefore the electric and magnetic fields) will also depend on its acceleration. Moreover, they depend *linearly*, since the acceleration in these derivatives is introduced as a result of a single differentiation.

LANGEVIN was thus led to distinguish the electric and magnetic field terms which do not depend on the acceleration (this is what he calls the velocity wave) and those that are proportional to acceleration (that is what he calls the acceleration wave).

The calculation of these two waves is facilitated by the LORENTZ transformation. Indeed, we can apply this transformation to the system, so that the velocity of the single electron under consideration becomes zero. We will use for the *x*-axis the direction of the velocity before the transformation, so that, at the instant t_1,

$$\eta_1 = \zeta_1 = 0$$

and we will take ε = -ξ, so that

$$\xi_1' = \eta_1' = \zeta_1' = 0$$

We can therefore reduce the computation of the two waves to the case where the electron velocity is zero. Let's start with the velocity wave, we first note that this wave is the same as if the electron motion was uniform.

If the electron velocity is zero, then:

$$\omega = 0, \quad F = G = H = 0, \quad \psi = \frac{\mu_1}{4\pi r},$$

μ_1 is the electrical charge of the electron. The speed was reduced to zero by the LORENTZ transformation, we have now:

$$F' = G' = H' = 0, \quad \psi' = \frac{\mu_1}{4\pi r'},$$

r' is the distance from point x', y', z' at point x'_1, y'_1, z'_1, and therefore:

$$\alpha' = \beta' = \gamma' = 0,$$

$$f' = \frac{\mu_1 (x' - x'_1)}{4\pi r'^3} \quad g' = \frac{\mu_1 (y' - y'_1)}{4\pi r'^3}, \quad h' = \frac{\mu_1 (z' - z'_1)}{4\pi r'^3}.$$

Now let us carry out the reverse LORENTZ transformation to find the true field corresponding to the velocity ε, 0, 0. We find, with reference to equations (9) and (3) of § 1:

$$(4) \quad \begin{cases} \alpha = 0, \quad \beta = \epsilon h, \quad \gamma = -\epsilon g, \\ f = \frac{\mu_1 k l^3}{4\pi r'^3}(x + \epsilon t - x_1 - \epsilon t_1), \quad g = \frac{\mu_1 k l^3}{4\pi r'^3}(y - y_1), \quad h = \frac{\mu_1 k l^3}{4\pi r'^3}(z - z_1). \end{cases}$$

We see that the magnetic field is perpendicular to the x-axis (direction of velocity) and the electric field, and the electric field is directed to the point:

$$(5) \qquad x_1 + \epsilon(t_1 - t), \ y_1, \ z_1.$$

If the electron continues to move in a rectilinear and uniform way with the velocity it had at the instant t_1, that is to say, with the velocity $-\varepsilon$, 0, 0, the point (5) would be the one occupied at the instant t.

Taking the acceleration wave, we can, through the LORENTZ transformation, reduce its determination to the case of zero velocity. This is the case if we imagine an electron whose oscillation amplitude is very small, but very fast, so that the displacements and velocities are much smaller, but the accelerations are finished. We thus come back to the field that has been studied in the famous work by HERTZ entitled *Die Kräfte elektrischer Schwingungen nach der Maxwell'schen Theorie*, and that for a point at great distance. In these conditions:

I° Both electric and magnetic fields are equal.

2° They are perpendicular to each other.

3° They are perpendicular to the normal of the spherical wave, that is to say to the sphere whose center is the point x_1, y_1, z_1.

I say that these three properties will remain, even when the velocity is not zero, and for this it is enough to show that they are not altered by the LORENTZ transformation.

Indeed, let A be the intensity common to both fields, let

$$(x - x_1) = r\lambda, \quad (y - y_1) = r\mu, \quad (z - z_1) = r\nu, \quad \lambda^2 + \mu^2 + \nu^2 = 1.$$

These properties expressed through the equalities

$$\left\{\begin{array}{l} A^2 = \sum f^2 = \sum \alpha^2, \quad \sum f\alpha = 0, \quad \sum f\,(x - x_1) = 0, \quad \sum \alpha\,(x - x_1) = 0 \\ \sum f\lambda = 0, \quad \sum \alpha\lambda = 0; \end{array}\right.$$

which means again that

$$\begin{array}{ccc} \frac{b}{A}, & \frac{g}{A}, & \frac{h}{A} \\ \frac{\alpha}{A}, & \frac{\beta}{A}, & \frac{\gamma}{A} \\ \lambda, & \mu, & \nu \end{array}$$

are the direction cosines of three rectangular directions, and we deduce the relations:

$$f = \beta\nu - \gamma\mu, \quad \alpha = h\mu - g\nu,$$

or

(6) $fr = \beta\,(z - z_1) - \gamma\,(y - y_1) \quad \alpha r = h\,(y - y_1) - g\,(z - z_1)\,,$

with the equations that we can deduce by symmetry.

If we take the equations (3) of § 1, we find:

$$
(7) \quad \begin{cases} x' - x'_1 = kl\left[(x - x_1) + \epsilon\,(t - t_1)\right] = kl\left[(x - x_1) + \epsilon r\right], \\ y' - y'_1 = l\,(y - y_1)\,, \\ z' - z'_1 = l\,(z - z_1)\,. \end{cases}
$$

We found above in § 3:

$$
l^4\left(\sum f'^2 - \sum \alpha'^2\right) = \sum f^2 - \sum \alpha^2.
$$

So

$$
\sum f^2 = \sum \alpha^2 \text{ entrain } \sum f'^2 - \sum \alpha'^2.
$$

On the other hand, from equations (9) of § 1, we get:

$$
l^4 \sum f'\alpha' = \sum f\alpha,
$$

This shows that

$$
\sum f\alpha = 0 \text{ entrain } \sum f'\alpha' = 0.
$$

I say now that

$$(8) \qquad \sum f'\left(x'-x'_1\right)=0, \quad \sum \alpha'\left(x'-x'_1\right)=0.$$

Indeed, by virtue of equations (7) (and equations 9, § 1) the first parts of equations (8) are written respectively:

$$\frac{k}{l}\sum f\left(x-x_1\right)+\frac{k\epsilon}{l}\left[fr+\gamma\left(y-y_1\right)-\beta\left(z-z_1\right)\right],$$

$$\frac{k}{l}\sum \alpha\left(x-x_1\right)+\frac{k\epsilon}{l}\left[\alpha r-h\left(y-y_1\right)-g\left(z-z_1\right)\right].$$

They then vanish in virtue of equations $\sum f\left(x-x_1\right)=\sum \alpha\left(x-x_1\right)=0$ and in virtue of equations (6). Yet this is precisely what was demonstrated.

We can also achieve the same result by considerations of homogeneity.

Indeed, ψ, F, G, H are functions of $x-x_1,\ y-y_1,\ z-z_1,\ \xi_1=\dfrac{dx_1}{dt_1},\ \eta_1=\dfrac{dy_1}{dt_1},\ \zeta_1=\dfrac{dz_1}{dt_1}$ being homogeneous of degree -1 with respect to $x, y, z, t, x_1, y_1, z_1, t_1$ and their differentials.

So the derivatives of ψ, F, G, H with respect to *x, y, z, t* (and hence also the two fields *f, g, h*; α, β, γ) will be homogeneous of degree -2 with respect to the same quantities, if we remember also that the relation

$$t-t_1=r=\sqrt{\sum\left(x-x_1\right)^2}$$

is homogeneous with respect to these quantities.

But these derivatives depend on these fields of x - x_1, the velocities $\frac{dx_1}{dt_1}$, and the accelerations $\frac{d^2x_1}{dt_1^2}$; they consist of a term independent of accelerations (velocity wave) and a term linear in respect to accelerations (acceleration waves). But $\frac{dx_1}{dt_1}$ is homogeneous of degree 0 and $\frac{d^2x_1}{dt_1^2}$ is homogeneous of degree -1; hence it follows that the velocity wave is homogeneous of degree -2 with respect to $x - x_1$, $y - y_1$, $z - z_1$, and the acceleration wave is homogeneous of degree -1. So in a very distant point an acceleration wave is predominant and can therefore be regarded as being assimilated to the total wave. In addition, the law of homogeneity shows that the acceleration wave is similar to itself at a distance and at any point. It is therefore, at any point, similar to the total wave at a remote point. But in a distant point the disturbance can propagate as plane waves, so that the two fields should be equal, mutually perpendicular and perpendicular to the direction of propagation.

I shall refer for more details to a work by LANGEVIN in the *Journal de Physique* (Year 1905).

§ 6. — Contraction of electrons

Suppose a single electron in rectilinear and uniform motion. From what we have seen, we can, through the LORENTZ transformation, reduce the study of the field determined by the electron to the case where the electron is motionless; the LORENTZ transformation replaces the real electron in motion by an ideal electron without motion.

Let α, β, γ, *f*, *g*, *h* be the real field; let α', β', γ', *f'*, *g'*, *h'* be the field after the LORENTZ transformation, so the ideal field α', *f'* corresponds to the case where the electron is motionless; we have:

$$\alpha' = \beta' = \gamma' = 0, \quad f' = -\frac{d\psi'}{dx'}, \quad g' = -\frac{d\psi'}{dy'}, \quad h' = -\frac{d\psi'}{dz'};$$

and the actual field (in virtue of the formulas 9 of § 1):

$$(1) \qquad \begin{cases} \alpha = 0, \quad \beta = \epsilon h, \quad \gamma = -\epsilon g \\ f = l^2 f', \quad g = kl^2 g', \; h = kl^2 h'. \end{cases}$$

We now determine the total energy due to the motion of the electron, the corresponding action, and the electromagnetic momentum, in order to calculate the electromagnetic mass of the electron. For a distant point, it suffices to consider the electron as reduced to a single point; we are thus brought back to the formulas (4) of the preceding § which generally can be appropriate. But here they do not suffice, because the energy is mainly located in the ether parts nearest to the electron.

On this subject we can make several hypotheses.

According to that of ABRAHAM, the electrons are spherical and not deformable.

So when we apply the LORENTZ transformation when the real electron is spherical, the electron becomes a perfect ellipsoid. The equation of this ellipsoid is based on § 1:

$$k^2(x' - \epsilon t' - \xi t' + \epsilon\xi x')^2 + (y' - \eta k t' + \eta k \epsilon x')^2$$
$$+(z' - \zeta k t' + \zeta k \epsilon x')^2 = l^2 r^2$$

But here we have:

$$\xi + \epsilon = \eta = \zeta = 0, \quad 1 + \epsilon\xi = 1 - \epsilon^2 = \frac{1}{k^2},$$

so that the equation of the ellipsoid becomes:

$$\frac{x'^2}{k^2} + y'^2 + z'^2 = l^2 r^2.$$

If the radius of the real electron is *r*, the axes of the ideal electron would therefore be:

$$klr, \; lr, \; lr.$$

In LORENTZ's hypothesis, however, the moving electrons are deformed, so that the real electron would become a ellipsoid, while the ideal electron is still always a perfect sphere of radius *r*; the axes of the real electron will then be:

$$\frac{r}{lk}, \quad \frac{r}{l}, \quad \frac{r}{l}.$$

We denote by

$$A = \frac{1}{2} \int f^2 d\tau$$

the *longitudinal electric energy*; by

$$B = \frac{1}{2} \int \left(g^2 + h^2\right) d\tau$$

the *transverse electric energy*; by

$$C = \frac{1}{2} \int \left(\beta^2 + \gamma^2\right) d\tau$$

the *transverse magnetic energy*. There is no longitudinal magnetic energy, since α = α' = 0. We denote by A', B', C' the corresponding quantities in the ideal system. We first find:

$$C' = 0,\ C = \epsilon^2 B$$

In addition, we can observe that the actual field depends only on *x* = *εt*, *y*, and *x*, and write:

$$d\tau = d(x + \epsilon t) dy\, dz$$
$$d\tau' = dy' dy' dz' = kl^3 d\tau$$

hence

$$A' = kl^{-1}A, \quad B' = k^{-1}l^{-1}B, \quad A = \frac{lA'}{k}, \quad B = klB'.$$

In LORENTZ's hypothesis we have B' = 2A', and A ' (being inversely proportional to the radius of the electron) is a constant independent of the velocity of the real electron; we get for the total energy:

$$A + B + C = A'lk\left(3 + \epsilon^2\right)$$

and for the action (per unit time):

$$A + B - C = \frac{3A'l}{k}.$$

Now calculate the electromagnetic momentum; we find:

$$D = \int (g\gamma - h\beta)\, d\tau = -\epsilon \int \left(g^2 + h^2\right) d\tau = -2\epsilon B = -4\epsilon klA'.$$

But there must be some relation between the energy E = A + B + C, the action per unit time H = A + B - C, and the momentum D. The first of these relations is:

$$E = H - \epsilon \frac{dH}{d\epsilon},$$

the second is

$$\frac{dD}{d\epsilon} = -\frac{1}{\epsilon}\frac{dE}{d\epsilon};$$

hence

$$D = \frac{dH}{d\epsilon}, \quad E = H - \epsilon D. \tag{2}$$

The second of equations (2) is always satisfied; but the first is so only if

$$l = \left(1 - \epsilon^2\right)^{\frac{1}{6}} = k^{-\frac{1}{3}},$$

that is to say if the volume of the ideal electron is equal to that of the real electron; or if the volume of the electron is constant; that's the hypothesis of LANGEVIN.

This is in contradiction with the results of § 4 and with the result obtained by LORENTZ by another way. That is the contradiction which is to be explained.

Before addressing this explanation, I note that whatever is the hypotheses we have adopted

$$H = A + B - C = \frac{l}{k}(A' + B'),$$

or, because of C' = 0,

$$H = \frac{l}{k} H', \tag{3}$$

We can compare the result of the equation J = J' obtained in § 3.

We have indeed:

$$J = \int H\, dt, \quad J' = \int H'\, dt'.$$

We observe that the state of the system depends only on $x + \varepsilon t$, y and z, that is to say on x', y', z', and we have:

$$t' = \frac{l}{k} t + \epsilon x'$$

(4) $$dt' = \frac{l}{k} dt.$$

By comparing equations (3) and (4) we find J = J'.

Let us consider any hypothesis, which may be either that of LORENTZ, or that of ABRAHAM, or that of LANGEVIN, or an intermediate hypothesis.

Let

$$r, \; \theta r, \; \theta r$$

the three axes of the real electron; that of the ideal electron will be:

$$klr, \; \theta lr, \; \theta lr$$

Then A' + B' is the electrostatic energy of an ellipsoid with axes *klr, θlr, θlr*.

Let us suppose that the electricity is spread on the surface of the electron as it is known from an inductor, or uniformly distributed within the electron; than this energy will be of the form:

$$A' + B' = \frac{\varphi\left(\frac{\theta}{k}\right)}{klr},$$

where φ is a known function.

The hypothesis of ABRAHAM is to assume:

$$r = const.,\ \theta = 1.$$

That of LORENTZ:

$$l = 1,\ kr = const.,\ \theta = k.$$

That of LANGEVIN:

$$l = k^{-\frac{1}{3}}, \quad k = \theta, \quad klr = const..$$

We then find:

$$H = \frac{\varphi\left(\frac{\theta}{k}\right)}{k^2 r}.$$

ABRAHAM found, in different notation (*Göttinger Nachrichten*, 1902, p. 37)

$$H = \frac{a}{r}\frac{1-\epsilon^2}{\epsilon}\log\frac{1+\epsilon}{1-\epsilon},$$

a is a constant. However, in the hypothesis of ABRAHAM, we have θ = 1; then:

$$\varphi\left(\frac{1}{k}\right) = ak^2\frac{1-\epsilon^2}{\epsilon}\log\frac{1+\epsilon}{1-\epsilon} = \frac{a}{\epsilon}\log\frac{1+\epsilon}{1-\epsilon}, \tag{5}$$

which defines the function φ.

This granted, imagine that the electron is subject to a binding, so there is a relation between *r* and φ; in the hypothesis of LORENTZ this relation would be φr = const., in that of LANGEVIN $\varphi^2 r^2$ = const. We assume in a more general way

$$r = b\theta^m$$

b is a constant; hence:

$$H = \frac{1}{bk^2}\theta^{-m}\varphi\left(\frac{\theta}{k}\right).$$

What is the shape of the electron when the velocity become -*εt, if we do not suppose the involvement of forces other than the binding forces*? Its form will be defined by the equality:

$$\frac{\partial H}{\partial \theta} = 0, \tag{6}$$

or

$$-m\theta^{-m-1}\varphi + \theta^{-m}k^{-1}\varphi' = 0$$

or

$$\frac{\varphi'}{\varphi} = \frac{mk}{\theta}.$$

If we want equilibrium to occur so that θ = *k*, it is necessary that $\frac{\theta}{k} = 1$, the logarithmic derivative of φ is equal to *m*.

If we develop $\frac{1}{k}$ and the right-hand side of (5) in powers of ε, equation (5) becomes:

$$\varphi\left(1-\frac{\epsilon^2}{2}\right)=a\left(1-\frac{\epsilon^2}{3}\right)$$

neglecting higher powers of ε. By differentiating, we get:

$$-\epsilon\varphi'\left(1-\frac{\epsilon^2}{2}\right)=\frac{2}{3}\epsilon a.$$

For ε = 0, that is to say when the argument of φ is equal to 1, these equations become:

$$\varphi=a,\quad \varphi'=-\frac{2}{3}a,\quad \frac{\varphi'}{\varphi}=-\frac{2}{3}. \tag{7}$$

We must therefore have $m=-\frac{2}{3}$ in conformity with the hypothesis of LANGEVIN.

This result should come nearer to that which is connected to the first equation (a), and from which actually it does not differ. Indeed, suppose that every element *dτ* of the electron is subjected to a force X*d*τ parallel to the *x*-axis, X is the same for all elements; we will then have, in conformity with the definition of momentum:

$$\frac{dD}{dt}=\int Xd\tau.$$

In addition, the principle of least action gives us:

$$\delta J = \int X\delta U\, d\tau\, dt, \quad J = \int H\, dt, \quad \delta J = \int D\delta U\, dt,$$

δU is the displacement of the center of gravity of the electron; H depends on θ and on ε if we assume that *r* is related to θ by the equation of binding; we have thus:

$$\delta J = \int \left(\frac{\partial H}{\partial \epsilon} \delta\epsilon + \frac{\partial H}{\partial \theta} \right) dt.$$

In addition $\delta\epsilon = -\frac{d\delta U}{dt}$; where, by integrating by parts:

$$\int D\delta\epsilon\, dt = \int D\delta U\, dt$$

or

$$\int \left(\frac{\partial H}{\partial \epsilon} \delta\epsilon + \frac{\partial H}{\partial \theta} \delta\theta \right) dt = \int D\delta\epsilon\, dt,$$

hence

$$D = \frac{\partial H}{\partial \epsilon}, \quad \frac{\partial H}{\partial \theta} = 0.$$

But the derivative $\frac{dH}{d\epsilon}$, contained in the right-hand side of equation (2), is the derivative taken by supposing θ as a function of ε, so that

$$\frac{dH}{d\epsilon} = \frac{\partial H}{\partial \epsilon} + \frac{\partial H}{\partial \theta} \frac{d\theta}{d\epsilon}.$$

Equation (2) is therefore equivalent to equation (6).

The conclusion is that if the electron is subject to a binding between its three axes, *and if no other force intervenes except the binding forces*, the shape of that electron, when it is given a uniform velocity, may be such that the ideal electron corresponds to a sphere, except the case where the binding is such that the volume is constant, in conformity with the hypothesis of LANGEVIN.

We are led in this way to pose the following problem: what additional forces, other than the binding forces, are necessary to intervene to account for the law of LORENTZ or, more generally, any law other than that of LANGEVIN?

The simplest hypothesis, and the first that we should consider, is that these additional forces are derived from a special potential depending on the three axes of the ellipsoid, and therefore on θ and on *r*; let F(θ, *r*) be the potential; in which case the action will be expressed:

$$J = \int [H + F(\theta, r)]\, dt$$

and the equilibrium conditions are written:

$$\frac{dH}{d\theta} + \frac{dF}{d\theta} = 0, \quad \frac{dH}{dr} + \frac{dF}{dr} = 0. \tag{8}$$

If we assume *r* and θ are connected by $r = b\theta^m$, we can look at *r* as a function of θ, consider F as depending only on θ, and retain only the first equation (8) with:

$$H = \frac{\varphi}{bk^2\theta^m}, \quad \frac{dH}{d\theta} = \frac{-m\theta}{bk^2\theta^{m+1}} + \frac{\varphi'}{bk^3\theta^m}$$

For $k = \theta$ we need equation (8) to be satisfied; which gives, taking into account equations (7):

$$\frac{dF}{d\theta} = \frac{ma}{b\theta^{m+3}} + \frac{2}{3}\frac{a}{b\theta^{m+3}}$$

where:

$$F = \frac{-a}{b\theta^{m+2}}\frac{m+\frac{2}{3}}{m+2}$$

and in the hypothesis of LORENTZ, where $m = -1$:

$$F = \frac{a}{3b\theta}.$$

Now suppose that there is *no* connection and, considering r and θ as independent variables, retain the two equations (H); it follows:

$$H = \frac{\varphi}{k^2 r}, \quad \frac{dH}{d\theta} = \frac{\varphi'}{k^3 r}, \quad \frac{dH}{dr} = \frac{-\varphi}{k^2 r^2},$$

Equations (8) must be satisfied for $k = \theta$, $r = b\theta^m$; which gives:

(9) $$\frac{dF}{dr} = \frac{a}{b^2\theta^{2m+2}}, \quad \frac{dF}{d\theta} = \frac{2}{3}\frac{a}{b\theta^{m+3}}.$$

One way to satisfy these requirements is to pose:

(10) $$F = Ar^\alpha\theta^\beta$$

A, α, β are constants, the equations (9) must be satisfied for $k = \theta$, $r = b\theta^m$, which gives:

$$A\alpha b^{\alpha-1}\theta^{m\alpha-m+\beta} = \frac{a}{b^2\theta^{2m+2}}, \quad A\beta b^{\alpha}\theta^{m\alpha+\beta-1} = \frac{2}{3}\frac{a}{b\theta^{m+3}}.$$

By identifying we find

$$(11) \qquad \alpha = 3\gamma, \quad \beta = 2\gamma, \quad \gamma = -\frac{m+2}{3m+2}, \quad A = \frac{a}{\alpha b^{\alpha+1}}$$

But the volume of the ellipsoid is proportional to $r^3\theta^2$, so that the additional potential is proportional to the power γ of the volume of the electron.

In the hypothesis of LORENTZ, we have $m = 1$, $\gamma = 1$.

We thus come back to the hypothesis of LORENTZ, *under the condition of adding an additional potential proportional to the volume of the electron.*

The hypothesis of LANGEVIN corresponds to $\gamma = \infty$.

§ 7. — Quasi-stationary motion

It remains to see if this hypothesis of the contraction of electrons reflects the inability to demonstrate absolute motion, and I will begin by studying the quasi-stationary motion of an isolated electron, or which is subject only to the action of other distant electrons.

It is known that what is called quasi-stationary motion is the motion where the velocity changes are slow enough so that the electric and magnetic energy due to motion of the electron differ little from what they would be in uniform motion; we know also that ABRAHAM derived the transverse and longitudinal electromagnetic masses from the notion of quasi-stationary motion.

I think I should clarify. Let H be our action per unit time:

$$H = \frac{1}{2}\int \left(\sum f^2 - \sum \alpha^2\right) d\tau,$$

where we consider for the moment only the electric and magnetic fields due to the motion of an electron. In the preceding §, by considering the motion as uniform, we regarded H as dependent from the velocity ξ, η, ζ of the electrons' center of gravity (the three components in the preceding §, had as values -ε, 0, 0) and the parameters *r* and θ that define the shape of the electron.

But if the motion is more uniform, H depend not only on the values of ξ, η, ζ, *r*, θ at the instant in question, but on values of these quantities at other instants which may differ in quantities of the same order as the time by light to travel from one point to another of the electron; in other words, H depend not only on ξ, η, ζ, *r*, θ, but on their derivatives of all orders with respect to time.

Well, the motion is said to be quasi-stationary when the partial derivatives of H with respect to the successive derivatives of ξ, η, ζ, r, θ are negligible compared to the partial derivatives of H with respect to the quantities ξ, η, ζ, *r*, θ themselves.

The equations of such a motion can be written:

$$(1)\quad \left\{\begin{array}{l} \frac{dH}{d\theta} + \frac{dF}{d\theta} = \frac{dH}{dr} + \frac{dF}{dr} = 0, \\ \\ \frac{d}{dt}\frac{dH}{d\xi} = -\int X d\tau, \quad \frac{d}{dt}\frac{dH}{d\eta} = -\int Y d\tau, \quad \frac{d}{dt}\frac{dH}{d\zeta} = -\int Z d\tau. \end{array}\right.$$

In these equations, F has the same meaning as in the preceding §, X, Y, Z are the components of the force acting on the electron: this force is solely due to electric and magnetic fields produced by other electrons.

Note that H is independent of ξ η ζ through the combination

$$V = \sqrt{\xi^2 + \eta^2 + \zeta^2},$$

that is to say, the magnitude of the velocity; therefore we still call D the momentum:

$$\frac{dH}{d\xi} = \frac{dH}{dV}\frac{\xi}{V} = -D\frac{\xi}{V},$$

where:

(2) $$-\frac{d}{dt}\frac{dH}{d\xi} = \frac{D}{V}\frac{d\xi}{dt} - D\frac{\xi}{V^2}\frac{dV}{dt} + \frac{dD}{dV}\frac{\xi}{V}\frac{dV}{dt},$$

(2^{bis}) $$-\frac{d}{dt}\frac{dH}{d\eta} = \frac{D}{V}\frac{d\eta}{dt} - D\frac{\eta}{V^2}\frac{dV}{dt} + \frac{dD}{dV}\frac{\eta}{V}\frac{dV}{dt},$$

with

(3) $$V\frac{DV}{dt} = \sum \xi\frac{d\xi}{dt}.$$

If we take the current direction of the velocity as the *x*-axis, we get:

$$\xi = V, \quad \eta = \zeta = 0, \quad \frac{d\xi}{dt} = \frac{dV}{dt};$$

equations (2) and (2_{bis}) become:

$$-\frac{d}{dt}\frac{dH}{d\xi} - \frac{dD}{dV}\frac{d\xi}{dt}, \quad -\frac{d}{dt}\frac{dH}{d\eta} = \frac{D}{V}\frac{d\eta}{dt}$$

and the last three equations (1):

$$\text{(4)} \quad \frac{dD}{dV}\frac{d\xi}{dt} = \int X d\tau, \quad \frac{D}{V}\frac{d\eta}{dt} = \int Y d\tau, \quad \frac{D}{V}\frac{d\zeta}{dt} = \int Z d\tau.$$

This is why ABRAHAM gave $\frac{dD}{dV}$ the name *longitudinal mass* and $\frac{D}{V}$ the name *transverse mass*; recall that $D = \frac{dH}{dV}$.

In the hypothesis of LORENTZ, we have:

$$D = -\frac{dH}{dV} = -\frac{\partial H}{\partial V},$$

$\frac{\partial H}{\partial V}$ represent the derivative with respect to V, after *r* and θ were replaced by their values as functions of V from the first two equations (1); we will also have, after the substitution,

$$H = +A\sqrt{1 - V^2}.$$

We choose units so that the constant factor A is equal to 1, and I pose $\sqrt{1-V^2}=h$, hence:

$$H=+h, \quad D=\frac{V}{h}, \quad \frac{dD}{dV}=h^{-3}, \quad \frac{dD}{dV}\frac{1}{V^2}-\frac{D}{V^3}=h^{-3}.$$

We will pose again:

$$M=V\frac{dV}{dt}=\sum\xi\frac{d\xi}{dt}, \quad X_1=\int X d\tau$$

and we find the equation for quasi-stationary motion:

$$h^{-1}\frac{d\xi}{dt}+h^{-3}\xi M=X_1. \tag{5}$$

Let's see what happens to these equations by the LORENTZ transformation. We will pose: $1+\xi\epsilon=\mu$, and we have first:

$$\mu\xi'=\xi+\epsilon, \quad \mu\eta'=\frac{\eta}{k}, \quad \mu\zeta'=\frac{\zeta}{k}$$

from which we derive easily

$$\mu h'=\frac{h}{k}.$$

We also have

$$dt' = k\,\mu\,dt$$

where:

$$\frac{d\xi'}{dt'} = \frac{d\xi}{dt}\frac{1}{k^3\mu^3}, \quad \frac{d\eta'}{dt'} = \frac{d\eta}{dt}\frac{1}{k^2\mu^2} - \frac{d\xi}{dt}\frac{\eta\epsilon}{k^2\mu^3}, \quad \frac{d\zeta'}{dt'} = \frac{d\zeta}{dt}\frac{1}{k^2\mu^2} - \frac{d\xi}{dt}\frac{\zeta\epsilon}{k^2\mu^3}$$

where again:

$$M' = \frac{d\xi}{dt}\frac{\epsilon h^2}{k^3\mu^4} + \frac{M}{k^3\mu^3}$$

and

$$(6) \qquad h'^{-1}\frac{d\xi'}{dt'} + h'^{-3}\xi' M' = \left[h^{-1}\frac{d\xi}{dt} + h^{-3}(\xi+\epsilon)M\right]\mu^{-1},$$

$$(7) \qquad h'^{-1}\frac{d\eta'}{dt'} + h'^{-3}\eta' M' = \left[h^{-1}\frac{d\eta}{dt} + h^{-3}\eta M\right]\mu^{-1}h^{-1}.$$

Let us return now to equations (11^{bis}) of § 1; we can regard X_1, Y_1, Z_1 as having the same meaning as in equations (5). On the other hand, we have $l = 1$ and $\dfrac{\rho'}{\rho} = k\mu$; these equations then become:

$$
(8) \qquad \begin{cases} X_1' = \mu^{-1}\left(X_1 + \epsilon \sum X_1 \xi\right), \\ Y_1' = k^{-1}\mu^{-1} Y_1. \end{cases}
$$

We calculate $\Sigma X_1 \xi$ using equation (5), we find:

$$
\Sigma X_1 \xi = h^{-3} M,
$$

where:

$$
(9) \qquad \begin{cases} X_1' = \mu^{-1}\left(X_1 + \epsilon h^{-3} M\right), \\ Y_1' = k^{-1}\mu^{-1} Y_1. \end{cases}
$$

Comparing equations (5) (6), (7) and (9), we finally find:

$$
(10) \qquad \begin{cases} h'^{-1} \frac{d\xi'}{dt'} + h'^{-3} \xi' M' = X_1', \\ h'^{-1} \frac{d\eta'}{dt'} + h'^{-3} \eta' M' = Y_1' \end{cases}
$$

This shows that the equations of quasi-stationary motion are not altered by the LORENTZ transformation, but it still does not prove that the hypothesis of LORENTZ is the only one that leads to this result.

To establish this point, we will restrict ourselves, as LORENTZ did, to certain particular cases; it will be obviously sufficient for us to show a negative proposal.

How do we first extend the hypotheses underlying the above calculation?

1° Instead of assuming l = 1 in the LORENTZ transformation, we assume any l.

2° Instead of assuming that F is proportional to the volume, and hence that H is proportional to h, we assume that F is any function of θ and r, so that [after replacing θ and r with their values as functions of V, from the first two equations (1)] H is any function of V.

I note first that, assuming H = h, we must have l = 1; and in fact the equations (6) and (7) remain, except that the right-hand sides will be multiplied by $\frac{1}{l}$; so do equations (9), except that the right-hand sides will be multiplied by $\frac{1}{l^2}$; and finally the equations (10), except that the right-hand sides will be multiplied by $\frac{1}{l}$. If we want that the equations of motion are not altered by the LORENTZ transformation that is to say that the equations (10) only differ from equations (5) by the accentuation of the letters, it must be assumed:

$$l = 1.$$

Suppose now that we have η = ζ = 0, where ξ = V, $\frac{d\xi}{dt} = \frac{dV}{dt}$; the equations (5) take the form:

$$(5^{\text{bis}}) \quad -\frac{d}{dt}\frac{dH}{d\xi} = \frac{dD}{dV}\frac{d\xi}{dt} = X_1, \quad -\frac{d}{dt}\frac{dH}{d\eta} = \frac{D}{V}\frac{d\eta}{dt} = Y_1.$$

We can also pose:

$$\frac{dD}{dV} = f(V) = f(\xi), \quad \frac{D}{V} = \varphi(V) = \varphi(\xi).$$

If the equations of motion are not altered by the LORENTZ transformation, we must have:

$$f(\xi)\frac{d\xi}{dt} = X_1,$$

$$\varphi(\xi)\frac{d\eta}{dt} = Y_1,$$

$$f(\xi')\frac{d\xi'}{dt'} = X_1' = l^{-2}\mu^{-1}(X_1 + \epsilon \sum X_1 \xi) = l^{-2}\mu^{-1}X_1(1+\epsilon\xi) = l^{-2}X_1,$$

$$\varphi(\xi')\frac{d\eta'}{dt'} = Y_1' = l^{-2}k^{-1}\mu^{-1}Y_1,$$

and therefore:

$$(11) \qquad \begin{cases} f(\xi)\frac{d\xi}{dt} = l^2 f(\xi')\frac{d\xi'}{dt'}, \\ \\ \varphi(\xi)\frac{d\eta}{dt} = l^2 k\mu\varphi(\xi')\frac{d\eta'}{dt'}. \end{cases}$$

But we have:

$$\frac{d\xi'}{dt'} = \frac{d\xi}{dt}\frac{1}{k^3\mu^3}, \quad \frac{d\eta'}{dt'} = \frac{d\eta}{dt}\frac{1}{k^2\mu^2},$$

where:

$$f(\xi') = f\left(\frac{\xi+\epsilon}{1+\xi\epsilon}\right) = f(\xi)\frac{k^3\mu^3}{l^2},$$

$$\varphi(\xi') = \varphi\left(\frac{\xi+\epsilon}{1+\xi\epsilon}\right) = \varphi(\xi)\frac{k\mu}{l^2};$$

whence, by eliminating l^2, we find the functional equation:

$$k^2\mu^2\frac{\varphi\left(\frac{\xi+\epsilon}{1+\xi\epsilon}\right)}{\varphi(\xi)} = \frac{f\left(\frac{\xi+\epsilon}{1+\xi\epsilon}\right)}{f(\xi)},$$

or by posing

$$\frac{\varphi(\xi)}{f(\xi)} = \Omega(\xi) = \frac{D}{V\frac{dD}{dV}},$$

that is:

$$\Omega\left(\frac{\xi+\epsilon}{1+\xi\epsilon}\right) = \Omega(\xi)\frac{1+\epsilon^2}{(1+\xi\epsilon)^2},$$

an equation that must be satisfied for all values of ξ and ε. For $\zeta = 0$ we find:

$$\Omega(\epsilon) = \Omega(0)\left(1-\epsilon^2\right),$$

where:

$$D = A\left(\frac{V}{\sqrt{1-V^2}}\right)^m,$$

A is a constant, and I set $\Omega(0) = \dfrac{1}{m}$.

We then find:

$$\varphi(\xi) = \frac{A}{\xi}\left(\frac{\xi}{\sqrt{1-\xi^2}}\right)^m, \quad \varphi(\xi') = \frac{A\mu}{\xi+\epsilon}\left(\frac{\xi+\epsilon}{\sqrt{1-\xi^2}\sqrt{1-\epsilon^2}}\right)^m.$$

Now $\varphi(\xi') = \varphi(\xi)\dfrac{k\mu}{l^2}$; so we have:

$$(\xi+\epsilon)^{m-1}\left(1-\epsilon^2\right)^{-\frac{m}{2}} = -\xi^{m-1}\left(1-\epsilon^2\right)^{-\frac{1}{2}} l^{-2}.$$

As *l* should depend only on ε (since, if there are more electrons, *l* must be the same for all electrons whose velocities ξ may be different), this identity can take place only if we have:

$$m = 1,\ l = 1.$$

Thus LORENTZ's hypothesis is the only one consistent with the inability to demonstrate absolute motion; if we accept this impossibility, we must admit that the moving electrons contract and become ellipsoids of revolution where two axes remain constant; it must be admitted, as we have shown in the previous §, the existence of an additional potential which is proportional to the volume of the electron.

The analysis of LORENTZ is therefore fully confirmed, but we can better give us an account of the true reason of the fact which occupies us; and this reason must be sought in the considerations of § 4. *The transformations that do not alter the equations of motion must form a group, and this can take place only if l = 1.* As we do not recognize if an electron is at rest or in absolute motion, it is necessary that, when in motion, it undergoes a distortion that must be precisely that which imposes the corresponding transformation of the group.

§ 8. — Arbitrary motion

The above results apply only to quasi-stationary motion, but it is easy to extend them to the general case; it suffices to apply the principles of § 3, that is to say, the principle of least action.

For the expression of the action

$$J = \int dt\, d\tau \left(\frac{\sum f^2}{2} - \frac{\sum \alpha^2}{2} \right),$$

it is convenient to add a term representing the additional potential F of § 6; this term will obviously have the form:

$$J_1 = \int \sum (F) dt$$

where $\Sigma(F)$ represents the sum of the additional potential due to the different electrons, each of which is proportional to the volume of the corresponding electron.

I write (F) in brackets to avoid confusion with the vector F, G, H.

The total action is then $J + J_1$. We saw in § 3 that J is not altered by the LORENTZ transformation, we must show now that it is the same for J_1.

We have for one electron,

$$(F) = \omega_0 \tau$$

ω_0 being a special coefficient of the electron and τ its volume; so I can write:

$$\sum (F) = \int \omega_0 d\tau,$$

the integral has to be extended to the entire space, but so that the coefficient ω_0 is zero outside the electrons, and that within each electron it is equal to the special coefficient of that electron. Then we have:

$$J_1 = \int \omega_0 d\tau \, dt$$

and after the LORENTZ transformation:

$$J_1' = \int \omega_0' d\tau' \, dt' .$$

Now we have $\omega_0 = \omega'_0$; for if a point belong to an electron, the corresponding point after the LORENTZ transformation still belongs to the same electron. On the other hand, we found in § 3;

$$d\tau' dt' = l^4 d\tau \, dt$$

and since we now assume $l = 1$

$$d\tau' dt' = d\tau \, dt$$

We have therefore

$$J_1 = J_1' . \qquad \text{C.Q.F.D.}$$

The theorem is thus general, it gives us at the same time a solution of the question we posed at the end of § 1: finding the complementary forces which are unaltered by the LORENTZ transformation. The additional potential (F) satisfies this condition.

So we can generalize the result announced at the end of § 1 and write:

If the inertia of electrons is exclusively of electromagnetic origin, if they are only subject to forces of electromagnetic origin, or to forces generated by the additional potential (F), no experiment can demonstrate absolute motion.

So what are these forces that create the potential (F)? They can obviously be compared to a pressure which would reign inside the electron; all occurs as if each electron were a hollow capacity subjected to a constant internal pressure (volume independent); the work of this pressure would be obviously proportional to the volume changes.

In any case, I must observe that this pressure is negative. Remember the equation (10) of § 6, according to LORENTZ's hypothesis we write:

$$F = Ar^3\theta^2;$$

equations (11) of § 6 give us:

$$A = \frac{a}{3b^4}.$$

Our pressure is equal to A, with a constant coefficient, which is indeed negative.

Now assessing the mass of the electron – I mean the "experimental mass", that is to say the mass for low velocities – we have (cf. § 6):

$$H = \frac{\varphi\left(\frac{\theta}{k}\right)}{k^2 r}, \quad \theta = k, \quad \varphi = a, \quad \theta r = b;$$

hence

$$H = \frac{a}{bk} = \frac{a}{b}\sqrt{1 - V^2},$$

I can write for very small V

$$H = \frac{a}{b}\left(1 - \frac{V^2}{2}\right),$$

so that the mass, both longitudinal and transverse, will be $\frac{a}{b}$.

Now a is a numerical constant which shows that: *the pressure that creates our additional potential is proportional to the* 4^{th} *power of the experimental mass of the electron.*

As NEWTON's law is proportional to the experimental mass, we are tempted to conclude that there is some relation between the cause that generates gravitation and the one that generates the additional potential.

§ 9. — Hypotheses on gravitation

Thus LORENTZ's theory would completely explain the impossibility to demonstrate absolute motion, if all forces are of electromagnetic origin.

But there are forces which we can not assign an electromagnetic origin, as for example gravitation. It could happen, indeed, that two systems of bodies produce equivalent electromagnetic fields, that is to say, exerting the same action on the electrified bodies and on the currents, and yet these two systems do not exercise the same gravitational action on the NEWTONian mass. The gravitational field is thus distinct from the electromagnetic field. LORENTZ was thus forced to complete his hypothesis by assuming that *forces of any origin, and in particular gravitation, are affected by translation* (or, if preferred, by the LORENTZ transformation) *the same way as electromagnetic forces.*

It is now convenient to enter into details and look more closely at this hypothesis. If we want that the NEWTONian force is affected in this way by the LORENTZ transformation, we can not accept that the force depends only on the relative position of the attracting body and of the body attracted at the instant considered. It will also depend on the velocities of the two bodies. And that's not all: it is natural to assume that the force acting at time *t* on the attracted body, depends on the position and velocity of this body at the same time *t*; but it will depend, in addition, on the position and velocity of the *attracting* body, not at time *t*, but *a moment earlier*, as if gravitation needs a certain time to propagate.

Consider therefore the position of the attracted body at the instant t_0 and, at this point, x_0, y_0, z_0 are the coordinates, ξ, η, ζ the components of its velocity; consider the other attracting body at the corresponding time $t_0 + t$ and, at this point, $x_0 + x$, $y_0 + y$, $z_0 + z$ are the coordinates, ξ_1, η_1, ζ_1 the components of its velocity.

We must first have a relationship

$$(1) \qquad \phi(t,\ x,\ y,\ z,\ \xi,\ \eta,\ \zeta,\ \xi_1,\ \eta_1,\ \zeta_1) = 0$$

to define the time *t*. This relation will define the law of propagation of the gravitational action (I do not impose on me the condition that the propagation takes place with the same speed in all directions).

Now let X_1, Y_1, Z_1 the 3 components of the action exerted at time t_0 on the body; we have to express X_1, Y_1, Z_1 as functions of

$$(2) \qquad t,\ x,\ y,\ z,\ \xi,\ \eta,\ \zeta,\ \xi_1,\ \eta_1,\ \zeta_1$$

What are the conditions to fulfill?

1° The condition (1) shall not be altered by transformations of the LORENTZ group.

2° The components X_1, Y_1, Z_1 will be affected by the LORENTZ transformations the same way as electromagnetic forces designated by the same letters, that is to say, according to equations (11^{bis}) of § 1.

3° When two bodies are at rest, we will fall back to the ordinary law of attraction.

It is important to note that in the latter case, the relation (1) disappears, because time does not play any role if the two bodies are at rest.

The problem thus posed is obviously undetermined. We will thus seek to satisfy as many as possible other additional conditions:

4° Astronomical observations do not appear to show significant derogation to NEWTON's law, we will choose the solution that deviates the least of this law, for low velocities of two bodies.

5° We will endeavor to arrange that T is always negative; if indeed it is conceived that the effect of gravitation takes a certain time to be propagated, it would be more difficult to understand how this effect could depend on the position *not yet attained* by the attracting body.

There is one case where the indeterminacy of the problem disappears; it is where the two bodies are at rest *relative* to each other, that is to say that:

$$\xi = \xi_1, \ \eta = \eta_1, \ \zeta = \zeta_1 \, ;$$

this is the case we will consider first, assuming that these velocities are constant, so that the two bodies are drawn into a common translational motion, rectilinear and uniform.

We can assume that the axis of *x* has been taken parallel to the translation, so that $\eta = \zeta = 0$, and we take $\varepsilon = -\xi$.

If in these circumstances we apply the LORENTZ transformation, after the transformation the two bodies are at rest and we have:

$$\xi' = \eta' = \zeta' = 0$$

Then the components x'_0, Y'_0, Z'_0 must conform to NEWTON's law and we will have a constant factor:

$$(3) \qquad \begin{cases} X_1' = -\frac{x'}{r'^3}, \quad Y_1' = -\frac{y'}{r'^3}, \quad Z_1' = -\frac{z'}{r'^3}, \\ r'^2 = x'^2 = y'^2 + z'^2. \end{cases}$$

But we have, according to § 1:

$$\begin{aligned} & x' = k(x + \epsilon t), \quad y' = y, \quad z' = z, \quad t' = k(t + \epsilon x), \\ & \frac{\rho'}{\rho} = k(1 + \xi\epsilon) = k\left(1 - \epsilon^2\right) = \frac{1}{k}, \quad \sum X_1 \xi = -X_1 \epsilon, \\ & X_1' = k\frac{\rho}{\rho'}\left(X_1 + \epsilon \sum X_1 \xi\right) = k^2 X_1 \left(1 - \epsilon^2\right) = X_1, \\ & Y_1' = \frac{\rho}{\rho'} Y_1 = k Y_1, \\ & Z_1' = k Z_1. \end{aligned}$$

We have also:

$$x + \epsilon t = x - \xi t, \quad r'^2 = k^2 (x - \xi t)^2 + y^2 + z^2$$

and

$$(4) \qquad X_1 = \frac{-k(x-\xi t)}{r'^3}, \quad Y_1 = \frac{-y}{kr'^3}, \quad Z_1 = \frac{-z}{kr'^3};$$

which can be written:

$$(4^{\text{bis}}) \qquad X_1 = \frac{dV}{dx}, \quad Y_1 = \frac{dV}{dy}, \quad Z_1 = \frac{dV}{dz}; \quad V = \frac{1}{kr'}.$$

It seems at first sight that the indetermination remains, since we have made no hypothesis about the value of *t*, that is to say about the speed of transmission; and that also *x* is a function of *t*, but it is easy to see that $x - \xi t$, y, z (which appear in our formulas) do not depend on *t*.

We see that if two bodies are simply in motion by a common translation, the force acting on the body is drawn normal to an ellipsoid with its center at the attracting body.

To go further we must look for the *invariants of the Lorentz group*.

We know that the substitutions of this group (assuming $l = 1$) are linear substitutions which do not affect the quadratic form

$$x^2 + y^2 + z^2 - t^2.$$

Let on the other hand:

$$\xi = \frac{\delta x}{\delta t}, \quad \eta = \frac{\delta y}{\delta t}, \quad \zeta = \frac{\delta z}{\delta t};$$
$$\xi_1 = \frac{\delta_1 x}{\delta_1 t}, \quad \eta_1 = \frac{\delta_1 y}{\delta_1 t}, \quad \zeta_1 = \frac{\delta_1 z}{\delta_1 t};$$

we see that the LORENTZ transformation will cause to make δx, δy, δz and $\delta_1 x$, $\delta_1 y$, $\delta_1 z$, $\delta_1 t$ undergo the same linear substitutions as with x, y, z, t.

We regard

$$\begin{array}{cccc} x, & y, & z, & t\sqrt{-1}, \\ \delta x, & \delta y, & \delta z, & \delta t\sqrt{-1}, \\ \delta_1 x, & \delta_1 y, & \delta_1 z, & \delta_1 t\sqrt{-1}, \end{array}$$

as the coordinates of three points P, P', P'' in a 4-dimensional space. We see that the LORENTZ transformation is a rotation of that space around the origin, regarded as fixed. We shall therefore have no other distinct invariants than 6 distances of the 3 points P, P', P'' between them and the origin, or, if you like it better, than the 2 expressions:

$$x^2 + y^2 + z^2 - t^2, \; x\delta x + y\delta y + z\delta z - t\delta t$$

or the 4 expressions of the same form, deduced from permuting (in an arbitrary way) the three points P, P', P''.

But what we look for are the functions of 10 variables (2) that are invariants; so we must, among the combinations of our 6 invariants, seek those which depend only on these 10 variables, that is to say those that are homogeneous of degree 0 as compared to δx, δy, δz, δt, as compared to $\delta_1 x$, $\delta_1 y$, $\delta_1 z$, $\delta_1 t$. We will thus have 4 distinct invariants, which are:

$$(5)\quad \sum x^2 - t^2, \quad \frac{t-\sum x\xi}{\sqrt{1-\sum \xi^2}}, \quad \frac{t-\sum x\xi_1}{\sqrt{1-\sum \xi_1^2}}, \quad \frac{t-\sum \xi\xi_1}{\sqrt{\left(1-\sum \xi^2\right)\left(1-\sum \xi_1^2\right)}}$$

Let us now consider the transformations undergone by the components of the force; resume the equations (11) of § 1, which relate not to the force X_1, Y_1, Z_1, which we consider here, but to the force X, Y, Z referred to unit volume. We pose also:

$$T = \sum X\xi;$$

we see that these equations (11) can be written as (l = 1):

$$(6)\quad \begin{cases} X' = k(X+\epsilon T), & T' = k(T+\epsilon X), \\ Y' = Y, & Z' = Z; \end{cases}$$

so that X, Y, Z, T undergo the same transformation as x, y, z, t. The invariants of the group are therefore

$$\sum X^2 - T^2, \quad \sum Xx - Tt, \quad \sum X\delta x - T\delta t, \quad \sum X\delta_1 x - T\delta_1 t.$$

But this is not X, Y, Z which we need, it is X_1, Y_1, Z_1 with

$$T_1 = \sum X_1 \xi.$$

We see that

$$\frac{X_1}{X} = \frac{Y_1}{Y} = \frac{Z_1}{Z} = \frac{T_1}{T} = \frac{1}{\rho}.$$

So the LORENTZ transformations act on X_1, Y_1, Z_1, T_1 in the same manner as X, Y, Z, T, with the difference that these expressions are also multiplied by

$$\frac{\rho}{\rho'} = \frac{1}{k(1+\xi\epsilon)} = \frac{\delta t}{\delta t'}.$$

Similarly it would act on ξ, η, ζ, 1, in the same manner as δx, δy, δz, δt, with the difference that these expressions are also multiplied by the same factor:

$$\frac{\delta t}{\delta t'} = \frac{1}{k(1+\xi\epsilon)}.$$

Consider then $X, Y, Z, T\sqrt{-1}$ as the coordinates of a fourth point Q, then the invariants are functions of mutual distances of five points

$$0,\ P,\ P',\ P'',\ Q$$

and among these functions we must retain only those that are homogeneous of degree 0, on the one hand in relation to

$$X,\ Y,\ Z,\ T,\ \delta x,\ \delta y,\ \delta z,\ \delta t$$

(variables that can then be replaced by X_1, Y_1, Z_1, T_1, ξ, η, ζ, 1), on the other hand in relation to

$$\delta_1 x,\ \delta_1 y,\ \delta_1 z,\ 1$$

(variables that can be replaced later by ξ_1, η_1, ζ_1, 1).

Thus we find in addition to the four invariants (5), four new distinct invariants, which are:

$$(7) \quad \frac{\sum X_1^2 - T_1^2}{1 - \sum \xi^2}, \quad \frac{\sum X_1 x - T_1 t}{\sqrt{1 - \sum \xi^2}}, \quad \frac{\sum X_1 \xi_1 - T}{\sqrt{1 - \sum \xi^2}\sqrt{1 - \sum \xi_1^2}}, \quad \frac{\sum X_1 \xi - T_1}{1 - \sum \xi^2}.$$

The last invariant is always zero, according to the definition of T_1.

This granted, what are the requirements?

1° The left-hand side of relation (1), which defines the velocity of propagation must be a function of the four invariants (5)

One can obviously make a lot of hypotheses, we only look at two:

A) It may be

$$\sum x^2 - t^2 = r^2 - t^2 = 0,$$

where $t = \pm r$, and since t must be negative, $t = -r$. This means that the propagation velocity is equal to that of light. At first it seems that this hypothesis should be rejected without consideration. LAPLACE has indeed shown that this propagation is either instantaneous, or much faster than light. But LAPLACE had considered the hypothesis of finite speed of propagation, *ceteris non mutatis*; here, however, this hypothesis is complicated by many others, and it may happen that there is a more or less perfect compensation, as the applications of the LORENTZ transformation gave us already so many examples.

B) It may be

$$\frac{t - \sum x\xi_1}{\sqrt{1 - \sum \xi_1^2}} = 0, \quad t = \sum x\xi.$$

The propagation velocity is much faster than that of light, but in some cases *t* may be negative, which, as we have said, seems hardly acceptable. *We will add this to hypothesis (A)*.

2° The four invariants (7) must be functions of the invariants (5).

3° When the two bodies are in absolute rest, X_1, Y_1, Z_1 must have the value deduced from the law of NEWTON, and when they are in relative rest, the value deduced from the equations (4).

Under the hypothesis of absolute rest, the first two invariants (7) must be reduced to

$$\sum X_1^2, \quad \sum X_1 x,$$

or by NEWTON's law at

$$\frac{1}{r^4}, \quad -\frac{1}{r}$$

secondly, in hypothesis (A), the 2nd and 3rd of the invariants (5) become:

$$\frac{-r-\sum x\xi}{\sqrt{1-\sum \xi^2}}, \quad \frac{-r-\sum x\xi_1}{\sqrt{1-\sum \xi_1^2}}$$

that is to say, for absolute rest, to

$$-r, \; -r.$$

We may therefore assume for example that the first two invariants (4) are reduced to

$$\frac{\left(1-\sum x\xi_1^2\right)^2}{\left(r+\sum x\xi_1\right)^4}, \quad -\frac{\sqrt{1-\sum \xi_1^2}}{r+\sum x\xi_1},$$

but other combinations are possible.

We must choose between these combinations, and secondly, in order to define X_1, Y_1, Z_1 we need a third equation. For such a choice, we must endeavor to bring us closer as much as possible to the law to NEWTON. Let's see what happens when (always making $t = -r$) we neglect the squares of the velocities ξ η etc.. The 4 invariants (5) then become:

$$0, \quad -r-\sum x\xi, \quad -r-\sum x\xi_1, \quad 1$$

and the 4 invariant (7):

$$\sum X_1^2, \quad \sum X_1(x+\xi r), \quad \sum X_1\left(\xi_1-\xi\right), \quad 0.$$

But to be able to compare it with the law of NEWTON, another transformation is needed; here $x_0 + x$, $y_0 + y$, $z_0 + z$ are the coordinates of the attracting body at the instant $t_0 + x$, and $r = \sqrt{\sum x^2}$; in the law of Newton it is necessary to consider the coordinates $x_0 + x1$, $y_0 + y_1$, $z_0 + z_1$ of the attracting body at the instant t_0, and the distance $r_1 = \sqrt{\sum x_1^2}$.

We can neglect the square of time t required for the propagation and therefore proceed as if the movement was uniform, then we have:

$$x = x_1 + \xi_1 t, \quad y = y_1 + \eta_1 t, \quad z = z_1 + \zeta_1 t, \quad r\left(r - r_1\right) = \sum x \xi_1 t$$

or, since $t = -r$,

$$x = x_1 - \xi_1 r, \quad y = y_1 - \eta_1 r, \quad z = z_1 - \zeta_1 r, \quad r = r_1 - \sum x \xi_1;$$

so that our 4 invariants (5) become:

$$0, \quad -r_1 + \sum x \left(\xi_1 - \xi\right), \quad -r_1, \quad 1$$

and our 4 invariants (7):

$$\sum X_1^2, \quad \sum X_1 \left[x_1 + \left(\xi - \xi_1\right) r_1\right], \quad \sum X_1 \left(\xi_1 - \xi\right), \quad 0.$$

In the second of these expressions I wrote r_1 instead of r, because r is multiplied by $\xi - \xi_1$ and I neglect the square of ξ.

On the other hand, Newton's law would us give for these 4 invariants (7)

$$\frac{1}{r_1^4}, \quad -\frac{1}{r_1} - \frac{\sum x_1 (\xi - \xi_1)}{r_1^2}, \quad \frac{\sum x_1 (\xi - \xi_1)}{r_1^3}, \quad 0.$$

So if we denote the 2nd and 3rd invariants (7) by A and B, and the 3 first invariants (7) by M, N, P, we will satisfy NEWTON's law up to terms of order of the square velocities, by:

(8) $$M = \frac{1}{B^4}, \quad N = \frac{+A}{B^2}, \quad P = \frac{A - B}{B^3}.$$

This solution is not unique. Indeed, let C be the fourth invariant (5), C - 1 is of the order of the square of ξ, and it is equal to $(A - B)^2$.

So we could add to the 2ds members of each of equations (8) a term consisting of C - 1 multiplied by an arbitrary function of A, B, C, and a term of the form of $(A - B)^2$ also multiplied by a function of A, B, C.

At first sight, the solution (8) seems the most straightforward, it may nevertheless be adopted and in effect – since M, N, P are functions of X_1, Y_1, Z_1 and $T_1 = \sum X_1 \xi$ – we can draw from these three equations (8) the values of X_1, Y_1, Z_1, but in some cases these values become imaginary.

To avoid this, we will operate in another way. Let:

$$k_0 = \frac{1}{\sqrt{1 - \sum \xi^2}}, \quad k_1 = \frac{1}{\sqrt{1 - \sum \xi_1^2}}$$

This is justified by the analogy with the notation

$$k = \frac{1}{\sqrt{1-\epsilon^2}}.$$

which appears in the substitution of LORENTZ.

In this case, and because of the condition, -r = t, the invariants (5) become:

$$0, \quad A = -k_0(r + \sum x\xi), \quad B = -k_1\left(r + \sum x\xi_1\right), \quad C = k_0 k_1\left(1 - \sum \xi\xi_1\right).$$

On the other hand, we see that the following systems of quantities:

$$\begin{array}{cccc} x, & y, & z, & -r = t \\ \\ k_0 X_1, & k_0 Y_1, & k_0 Z_1, & k_0 T_1 \\ \\ k_0\xi, & k_0\eta, & k_0\zeta, & k_0 \\ \\ k_1\xi_1, & k_1\eta_1, & k_1\zeta_1, & k_1 \end{array}$$

undergo the *same* linear substitutions when we apply the transformations of the LORENTZ group. We are thus led to pose:

$$
(9) \qquad \begin{cases} X_1 = x\frac{\alpha}{k_0} + \xi\beta + \xi_1\frac{k_1}{k_0}\gamma, \\ \\ Y_1 = y\frac{\alpha}{k_0} + \eta\beta + \eta_1\frac{k_1}{k_0}\gamma, \\ \\ Z_1 = z\frac{\alpha}{k_0} + \zeta\beta + \zeta_1\frac{k_1}{k_0}\gamma, \\ \\ T_1 = -r\frac{\alpha}{k_0} + \beta + \frac{k_1}{k_0}\gamma, \end{cases}
$$

It is clear that if α, β, γ are invariants, X_1, Y_1, Z_1, T_1 satisfy the basic condition, that is to say, it will undergo, by the effect of the LORENTZ transformations, a suitable linear substitution.

But for the equations (9) to be consistent, we must have:

$$\sum X_1\xi - T_1 = 0,$$

which, by replacing X_1, Y_1, Z_1, T_1 by their values (9) and multiplying by k_0^2, becomes:

$$(10) \qquad -A\alpha - \beta - C\gamma = 0.$$

What we want is, if we neglect the square of speed of light, the squares of the velocities ξ, etc., as well as the product of accelerations by the distances as we did above, so that the values of X_1, Y_1, Z_1 remain in conformity with the law of NEWTON.

We can take:

$$\beta = 0, \quad \gamma = -\frac{A\alpha}{C}.$$

With the order of approximation adopted, we have:

$$k_0 = k_1 = 1, \quad C = 1, \quad A = -r_1 + \sum x\left(\xi_1 - \xi\right), \quad B = -r_1,$$
$$x = x_1 + \xi_1 t = x_1 - \xi_1 r$$

The first equation (9) becomes:

$$X_1 = \alpha\left(x - A\xi_1\right)$$

But if we neglect the square of ξ, we can replace $A\xi_1$ by $-r_1\xi_1$, or by $-r\xi_1$, which gives:

$$X_1 = \alpha\left(x + \xi_1 r\right) = \alpha x_1$$

NEWTON's law would give:

$$X_1 = -\frac{x_1}{r_1^3}.$$

We must therefore choose, for the invariant α, one that reduces to $-\frac{1}{r_1^3}$ to the order of approximation adopted, that is to say $\frac{1}{B^3}$. The equations (9) become:

$$
(11) \qquad \begin{cases}
X_1 = \frac{x}{k_0 B^3} - \xi_1 \frac{k_1}{k_0} \frac{A}{B^3 C}, \\
Y_1 = \frac{y}{k_0 B^3} - \eta_1 \frac{k_1}{k_0} \frac{A}{B^3 C}, \\
Z_1 = \frac{z}{k_0 B^3} - \zeta_1 \frac{k_1}{k_0} \frac{A}{B^3 C}, \\
T_1 = -\frac{r}{k_0 B^3} - \frac{k_1}{k_0} \frac{A}{B^3 C}.
\end{cases}
$$

We first see that the corrected attraction is composed of two components, one parallel to the vector joining the positions of the two bodies, the other parallel to the velocity of the attracting body.

Recall that when we talk about the position or velocity of the attracting body, it is its position or its velocity when the gravitational wave leaves; for the body attracted, on the contrary, it is its position or its velocity when the gravitational wave reaches it, the wave is assumed to propagate with the speed of light.

I think it would be premature to push further discussion of these formulas, I will confine myself to a few remarks.

1° The solutions (11) are not unique; we can indeed replace $\frac{1}{B^3}$ which enters in the factor everywhere, by

$$
\frac{1}{B^3} + (C-1) f_1(A, B, C) + (A-B)^2 f_2(A, B, C),
$$

f_1 and f_2 are arbitrary functions of A, B, C; or we are taking β no longer as zero but adding arbitrary complementary terms to α β γ, provided they satisfy the condition (10) and are of the 2nd order with regard to ξ as far as α is concerned, and of the 1st order as far as β and γ are concerned.

2° The first equation (11) can be written:

$$(11^{\text{bis}}) \qquad X_1 \frac{k_1}{B^3 C} \left[x \left(1 - \sum \xi\xi_1\right) + \xi_1 \left(r + \sum x\xi\right)\right]$$

and the quantity in brackets can, itself, written as:

$$(12) \qquad (x + r\xi_1) + \eta\,(\xi_1 y - x\eta_1) + \zeta\,(\xi_1 z - x\zeta_1)\,,$$

so that the total force can be divided into three components corresponding to the three brackets of expression (12); the first component has a vague analogy with the mechanical force due to the electric field, the other two with mechanical forces due to a magnetic field; to complete the analogy I can, under the first point, replace $\frac{1}{B^3}$ by $\frac{C}{B^3}$ in equations (11), so that X_1, Y_1, Z_1 only depend linearly on the velocity ξ, η, ζ of the attracted body, since C has disappeared from the denominator of (11^{bis}).

We pose then:

$$(13) \quad \begin{cases} k_1(x + r\xi_1) = \lambda, \quad k_1(y + r\eta_1) = \mu, \quad k_1(z + r\zeta_1) = \nu, \\ k_1\,(\eta_1 z - \zeta_1 y) = \lambda', \quad k_1\,(\zeta_1 x - \xi_1 z) = \mu', \quad k_1\,(\xi_1 y - x\eta_1) = \nu'; \end{cases}$$

it follows that C had disappeared from the denominator of (11a):

(14) $$\begin{cases} X_1 = \frac{\lambda}{B^3} + \frac{\eta\nu' - \zeta\mu'}{B^3}, \\ Y_1 = \frac{\mu}{B^3} + \frac{\zeta\lambda' - \xi\nu'}{B^3}, \\ Z_1 = \frac{\nu}{B^3} + \frac{\xi\mu' - \eta\lambda'}{B^3}, \end{cases}$$

and there will also:

(15) $$B^2 = \sum \lambda^2 - \sum \lambda'^2.$$

Then λ, μ, ν or $\frac{\lambda}{B^3}$, $\frac{\mu}{B^3}$, $\frac{\nu}{B^3}$ is a kind of electric field, while λ', μ', ν' or rather $\frac{\lambda'}{B^3}$, $\frac{\mu'}{B^3}$, $\frac{\nu'}{B^3}$ is a kind of magnetic field.

3° The postulate of relativity would require us to adopt solution (11) or solution (14) or any solution that would inferred by using the first remark; but the first question that arises is whether they are compatible with astronomical observations; the discrepancy with NEWTON's law is of the order ξ^2, that is to say, 10000 times smaller when it were of order ξ, that is to say, if the propagation happens with the speed of light, *ceteris non mutatis*; it is permissible to hope that it will not be too great. But only a thorough discussion will be able to teach it to us.

Paris, July 1905.

H. POINCARÉ

1. ↑ LANGEVIN was preceded by M. BUCHERER from Bonn, who had put forward the same theory before. (*See*: BUCHERER,

Mathematische Einführung in die Elektronentheorie; August 1904. Teubner, Leipzig).

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