COMMENT

Power laws in physics

James P. Sethna

Getting the most from power-law-type data can be challenging. James Sethna points out some of the pitfalls in studying power laws arising from emergent scale invariance, as well as important opportunities.

Power laws arise in many fields of knowledge — from word usage in linguistics, to income distributions in economics. There is an enormous literature observing and calculating power laws in nature. Publication of new, interesting results may involve data spanning one to two decades[1]: we need good tools to show that the power laws are real and accurate. In short, power laws are easy to fit, but challenging to measure and interpret well[2]. What are the particular challenges in studying power laws stemming from emergent scale invariance, a focus of much of statistical physics? And what opportunities exist to extract more science from the data?

Universal scaling functions

Many systems show fractal structure and scale-invariant fluctuations as they get large — the rules describing their behaviour look the same up to rescaling as one observes larger and larger systems. Continuous phase transitions (like the Curie point in ferromagnets), dynamical behaviour of disordered systems (depinning transitions, crackling noise and avalanches), the onset of chaos, earthquakes, fully developed turbulence, and the behaviour of the stock market all show clear symptoms of emergent scale invariance, and all exhibit power laws in various measures of their behaviour. In many of these systems, these power laws are convincingly explained using the renormalization group (RG)[3], which coarse-grains a system and then rescales (renorms) the parameters and observables to reach a fixed point. In some systems (turbulence, earthquakes) there is almost a consensus. In other systems (glasses[4], random matrix theory[5]) there are universal critical exponents and universal scaling functions, with no known RG explanation. The renormalization group predicts power laws relating various quantities, which are universal — shared between theory and experiment, and also shared between strikingly different experimental systems in the same 'universality class'. If Z depends on X, then $Z \sim X^{\beta}$ for some usually non-trivial, probably transcendental, universal critical exponent β.

The RG also predicts universal scaling functions for relations involving more than two parameters or observables. If Z depends on X and Y, then

$$Z \sim X^{\beta}\mathcal{Z}(X/Y^{\alpha}) \qquad (1)$$

where α is also a universal number and $\mathcal{Z}$ is a universal function. The challenges and most fruitful opportunities for experimentalists and simulators in measuring these power laws almost invariably involve corrections and modifications of the power laws due to these powerful universal scaling functions.

Finite-size scaling and scaling collapses

We start with finite-size scaling, describing the behaviour in a system confined to a cubic box of size L (or in a material with grains of size L). Suppose our system exhibits avalanches with sizes S spanning a large range. Then the fraction of the motion lying in avalanches with size between S and $S + \mathrm{d}S$ is

$$A(S,L) \sim S^{1-\tau}\mathcal{A}(S/L^{d_{\mathrm{f}}}) \qquad (2)$$

where d_{f} is the fractal dimension of the avalanche, so an avalanche spanning the system will have a typical size $S \sim L^{d_{\mathrm{f}}}$.

It is natural that avalanches larger than the box will be strongly suppressed! So $\mathcal{A}$ decreases quickly as its argument grows past one. Conversely, if $\mathcal{A}$ goes to a positive constant as its argument goes to zero, then small enough avalanches will have the predicted universal power law volume fraction $S^{1-\tau}$. But an experiment or simulation that measures avalanches in a size region where $\mathcal{A}$ is varying will often find a rather good — but incorrect — power-law fit (FIG. 1a).

A much better practice is to vary the system size (or the grain size) and do a scaling collapse to find $\mathcal{A}$: plot $S^{1-\overline{\tau}}A(S,L)$ against $S/L^{d_{\mathrm{f}}}$, and vary $\overline{\tau}$ and d_{f} until all the curves lie atop one another (FIG. 1b).

Subdominant corrections and fitting functional forms

Finite-size scaling produces corrections important when the behaviour reaches the system size. But what about corrections important for small scales? Or when one is farther from the critical point? There are two types of 'subdominant' corrections, namely singular corrections to scaling and analytic corrections to scaling. For example, the liquid–gas critical point has a free energy of the form

$$F(T,P,u) \sim \tilde{t}^{\beta+\beta\delta}\mathcal{F}(\tilde{h}/\tilde{t}^{\beta\delta}, \tilde{u}/\tilde{t}^{-\Delta}) \qquad (3)$$

LASSP, Physics Department, Cornell University, Ithaca, NY, USA.

e-mail: sethna@lassp.cornell.edu

https://doi.org/10.1038/s42254-022-00491-x

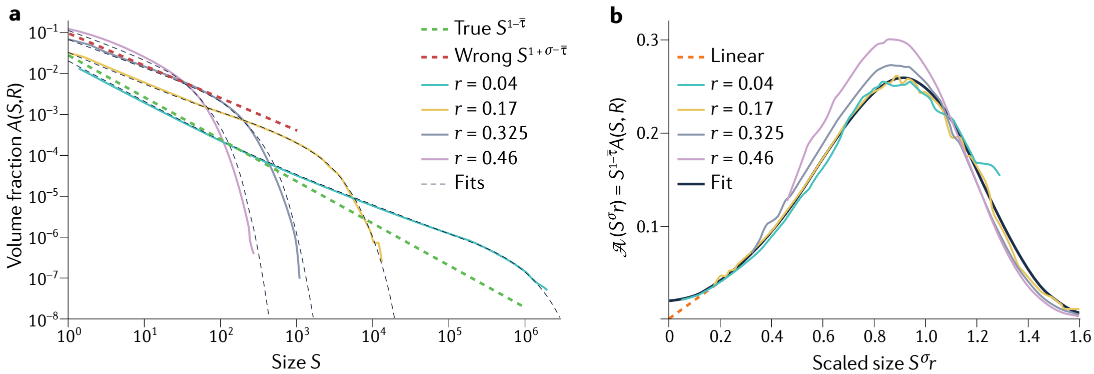

Fig. 1 | **Power laws and avalanche sizes in a random-field Ising model under an increasing field. a** | Avalanche probability distribution $A(S, R)$ that a site is in an avalanche of size S, for disorder R. Data are plotted at different values of $r = (R - R_c)/R$, where R_c is the critical disorder. The true power-law exponent is $1 - \bar{\tau}$; the apparent (wrong) power-law exponent is $1 + \sigma - \bar{\tau}$. Note that one needs over four decades of scaling to discover the correct power law. **b** | Scaling collapse of the same data, together with a fit to the scaling function $\mathcal{A}(S^{\sigma}r)$. Corrections to scaling are responsible for the deviations far from $r = 0$. Note that one needs simulations of a billion spins to discover that the asymptote of $\mathcal{A}$ is non-zero: smaller simulations give the wrong power law (dashed line). Data reproduced from REF.[9].

The variable u is an irrelevant control variable: it is multiplied by zero at $t = 0$, becoming less and less important as one approaches the critical temperature T_c and pressure P_c. The functions $\tilde{t}(T, P, u) = a(T - T_c) + b(P - P_c) + c(T - T_c)^2 + \cdots$, $\tilde{h}(P, T, u)$ and $\tilde{u}(u, T, P)$ are analytic power series that embody how temperature and pressure map onto the 'natural' RG parameters t, h and u in the Ising universality class.

By doing a Taylor expansion in u, $T - T_c$ and $P - P_c$, one gets corrections that go as higher powers of $T - T_c$. In particular, the irrelevant variable u causes a singular correction to scaling that is $\tilde{t}^{\Delta}$ times smaller than the dominant singularity.

When we have scaling functions with more than one variable as in Eq. 3, scaling collapses are no longer useful. A powerful, satisfying and numerically convenient approach is to do a multiparameter fit to the data[6–8], varying not only parameters like β, δ, u, T_c, P_c, a, b, c, and so on, but also a parameterized functional form for the scaling function $\mathcal{F}$.

Fitting functional forms have three additional benefits. First, they provide estimates not only of the universal critical exponents, but also of the equally universal scaling functions. Second, they allow for estimates of both statistical and systematic[6] errors in the exponents (which are often much larger than those of a straight power-law fit). Finally, these corrections, which are tiny near the critical point, become of increasing importance for describing precursor fluctuations in the surrounding phases. Indeed, here one imagines describing the (challenging) properties of liquids far into the phase diagram using analytic and singular corrections to the Ising critical point.

Singular scaling functions and dangerous irrelevant variables

Being careful to measure properties at intermediate sizes large compared to microscopic and small compared to the system, will one find the correct power laws? Not if our scaling function is itself singular — going to zero or infinity as its argument goes to zero. In a study[9] by our group of the random-field Ising model in 3D, this almost happened (FIG. 1). We were measuring the fractional coverage of avalanches $A(S, R) \sim S^{1-\bar{\tau}}\mathcal{A}(S^{\sigma}r)$, where $r = (R - R_c)$ is the distance to a critical disorder. We found excellent scaling collapses, but $\mathcal{A}$ seemed to go linearly to zero as $S^{\sigma}r$ went to zero (FIG. 1b) — leaving us with an effective power law $A(S, R) \sim S^{1-\bar{\tau}+\sigma}$ (FIG. 1a) that disagreed with the 'RG' exponent $1 - \bar{\tau}$ extracted from the scaling collapse. In the end, we used (at the time) heroic billion-site simulations to discover that $\mathcal{A}$ only nearly vanishes — it rises by a factor of ten from its small initial value.

Singular scaling functions also arise in the important case of dangerous irrelevant variables — quantities like u in Eq. 3 that vanish under rescaling (are irrelevant), but for which the scaling function for a physical property diverges as it vanishes. This happens in some glassy systems, in which the freezing on long length scales is not the usual competition between temperature and coupling between particles, but instead a competition between random disorder and coupling. Temperature allows hopping over barriers, allowing the system to relax. Because temperature is an irrelevant variable at the glass transition, the relaxation time (and its scaling function) diverges as the system is cooled through the transition.

Crossover scaling, nonlinear RG flows, and all that

There are many more fascinating implications and uses for universal scaling functions, and associated warnings that fitting power laws can lead you astray. Many systems exhibit crossovers, going smoothly from one power law to another as the scales become large — commonly arising for quantum critical points observed at finite temperatures, but also observed, for example, in magnetic avalanches[10] and fracture and depinning transitions[7]. Other systems exhibit more complex scaling behaviour, because their RG flows are intrinsically nonlinear[8]. This is remarkably common, for example, at critical points in phase transitions, where all systems in 2D and 4D have logarithms, exponentials, or essential singularities.

Thus the pitfalls of trusting a power-law fit should be viewed not as an obstacle, but an opportunity. It is challenging, but intellectually and scientifically fruitful, to use universal scaling functions to extract the most from your data.

1. Avnir, D., Biham, O., Lidar, D. & Malcai, O. Is the geometry of nature fractal? *Science* **279**, 39–40 (1998).
2. Newman, M. Power laws, pareto distributions and Zipf's law. *Contemp. Phys.* **46**, 323–351 (2005).
3. Sethna, J. P., Dahmen, K. A. & Myers, C. R. Crackling noise. *Nature* **410**, 242–250 (2001).
4. Liarte, D. B. et al. Universal scaling for disordered viscoelastic matter I: Dynamic susceptibility at the onset of rigidity. Preprint at https://arxiv.org/abs/2103.07474 (2021).
5. Adam, S., Brouwer, P. W., Sethna, J. P. & Waintal, X. Enhanced mesoscopic fluctuations in the crossover between random-matrix ensembles. *Phys. Rev. B* **66**, 165310 (2002).
6. Chen, Y., Papanikolaou, S., Sethna, J. P., Zapperi, S. & Durin, G. Avalanche spatial structure and multivariable scaling functions; sizes, heights, widths, and views through windows. *Phys. Rev. E* **84**, 061103 (2011).
7. Chen, Y.-J., Zapperi, S. & Sethna, J. P. Crossover behavior in interface depinning. *Phys. Rev. E* **92**, 022146 (2015).
8. Hayden, L. X., Raju, A. & Sethna, J. P. Unusual scaling for two-dimensional avalanches: Curing the faceting and scaling in the lower critical dimension. *Phys. Rev. Res.* **1**, 033060 (2019).
9. Perkovic, O., Dahmen, K. & Sethna, J. P. Avalanches, Barkhausen noise, and plain old criticality. *Phys. Rev. Lett.* **75**, 4528–4531 (1995).
10. Sethna, J. P. Crackling crossover. *Nat. Phys.* **3**, 518–519 (2007).

Competing interests

The author declares no competing interests.