IL NUOVO CIMENTO Vol. VI, N. 2 1° Agosto 1957

On the Generalized Ward Identity (*).

Y. Takahashi

Department of Physics, State University of Iowa - Iowa City, Iowa

(ricevuto il 30 Aprile 1957)

Ward's identity ([1]) which shows the relation between the vertex operator with equal electron momenta and the electron propagator has been generalized for the case where the electron momenta are not equal. The generalized identity has not been rigorously proved, in spite of the fact that it is extensively used by many authors. The proof is given in this paper *without recourse to perturbation expansion or Feynman's diagram.* It is shown to be a consequence of the conservation of the current.

One can express Ward's identity in the following form, namely

$$\frac{1}{i}\frac{\partial S_0(p)}{\partial p^\mu} = -S_0(p)\Gamma_\mu(p;p)S_0(p)\,, \tag{1}$$

where the function $S_0(p)$ is the renormalized electron propagator and the $\Gamma_\mu(p;p)$ the renormalized vertex operator with equal electron momenta.

It has been suggested that equation (1) be generalized in the following manner ([2]),

$$S_0(p) - S_0(q) = -i(p-q)^\mu S_0(p)\Gamma_\mu(p;q)S_0(q)\,. \tag{2}$$

The generalized relation (2) has not been proved in a rigorous manner. The aim of this note is to prove the relation (2) by the use of the equations of motion for the electron and the photon.

According to the gauge invariance of the theory, the renormalized photon propagator

$$D_{\mu\nu}(x-x') \equiv \langle T(\boldsymbol{A}_\mu(x), \boldsymbol{A}_\nu(x'))\rangle_0 \;(^{**})\,, \tag{3}$$

(*) This work is supported by the National Science Foundation.

([1]) J. C. Ward: *Phys. Rev.*, **77**, 293 (1950); **78**, 182 (1950); *Proc. Phys. Soc.*, **64**, 54 (1951).

([2]) T. D. Lee: *Phys. Rev.*, **95**, 1329 (1954); H. S. Green: *Proc. Phys. Soc.*, **66**, 873 (1953); L. D. Landau and I. M. Khalatnikov: *J.E.T.P.*, **29** 89 (1955); *English Translation*, **2**, 69 (1956).

(**) Bold-faced letters will be used for the renormalized Heisenberg field operators, throughout this note. For the notation in this note, see Jauch and Rohrlich: *Theory of Photons and Electrons* (Cambridge, Mass., 1955).

satisfies

$$\partial^{\mu} D_{\mu\nu}(x-x') = \partial_{\nu} D_c(x-x') , \tag{4}$$

where $D_c(x-x')$ is the Stückelberg-Feynman causal function defined as

$$D_c(x) = \frac{-i}{(2\pi)^4}\int d^4k \exp[ikx] \frac{1}{k^2 - i\varepsilon} . \tag{5}$$

The proof of the equation (4) is rather lengthy and will, therefore, be discussed later on.

The continuity equation of the current gives

$$\partial_y^{\mu}\langle T(\boldsymbol{J}_{\mu}(y), \psi(x), \bar{\psi}(x'))\rangle_0 = \\ = e_0\left\{\langle T(\psi(x), \bar{\psi}(y))\rangle_0\, \delta(y-x') - \delta(x-y)\langle T(\psi(y), \bar{\psi}(x'))\rangle_0\right\}, \tag{6}$$

where e_0 is the renormalized charge and use has been made of the relation

$$\delta(x_0 - x_0')[\psi(x), \boldsymbol{J}_0(x')] = e_0\, \delta(x-x')\psi(x') . \tag{7}$$

If one defines the vertex operator $\Gamma_{\mu}(x-y;\, y-x')$ by

$$\langle T(\psi(x), \bar{\psi}(x'), \boldsymbol{A}_{\mu}(y))\rangle_0 \equiv \\ \equiv -e_0\int d^4\xi\, d^4\eta\, d^4\zeta\, S_0(x-\xi)\Gamma^{\nu}(\xi-\eta;\, \eta-\zeta)S_0(\zeta-x')D_{\nu\mu}(\eta-y) , \tag{8}$$

then, the equation (6) is written, due to (4), in terms of Γ_{μ} and S_0 as follows

$$e_0\Box_y\, \partial_y^{\mu}\int d^4\xi\, d^4\eta\, d^4\zeta\, S_0(x-\xi)\Gamma^{\nu}(\xi-\eta;\, \eta-\zeta)S_0(\zeta-x')D_{\nu\mu}(\eta-y) = \\ = i\, e_0\int d^4\xi\, d^4\eta\, d^4\zeta\, S_0(x-\xi)\Gamma_{\nu}(\xi-\eta;\, \eta-\zeta)S_0(\zeta-x')\, \partial_y^{\nu}\, \delta(\eta-y) = \\ = i\, e_0\int d^4\xi\, d^4\eta\, d^4\zeta\, S_0(x-\xi)\, \partial_y^{\nu}\Gamma_{\nu}(\xi-y;\, y-\zeta)S_0(\zeta-x') = \\ = e_0\{S_0(x-y)\, \delta(y-x') - \delta(x-y)S_0(y-x')\} . \tag{9}$$

Upon introducing the Fourier transform of the equation (9), one gets

$$-i(p-q)^{\nu}S_0(p)\Gamma_{\nu}(p;\, q)S_0(q) = S_0(p) - S_0(q) . \tag{10}$$

This proves the generalized Ward identity (*).

Let us return to the equation (4).

(*) The conjecture that Ward's identity would be a consequence of the gauge invariance of the theory was stated by ROHRLICH. F. ROHRLICH: *Phys. Rev.*, **80**, 666 (1950).

The total electromagnetic potential $\boldsymbol{A}_\mu(x)$ is split into two parts

$$\boldsymbol{A}_\mu(x) = \boldsymbol{a}_\mu(x) + \partial_\mu \Lambda(x) , \tag{11}$$

where

$$\left\{ \begin{aligned} &\partial^\mu \boldsymbol{a}_\mu(x) &= 0 , \\ &(\Box \Lambda(x))^{(+)} \Phi &= 0 . \end{aligned} \right. \tag{12}$$

From (12), the $\boldsymbol{a}_\mu(x)$ must satisfy the commutation relation

$$\langle [\boldsymbol{a}_\mu(x), \boldsymbol{a}_\nu(x')] \rangle_0 = -i \int_0^\infty \mathrm{d}a \varrho(a) \left(g_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{a} \right) \Delta(x-x', a) ,$$

where $\varrho(a)$ is a spectral function introduced by KÄLLÉN, LEHMANN, and GELL-MANN and LOW ([3]).

In a similar fashion, the $\Lambda(x)$ satisfies

$$\langle [\Lambda(x), \Lambda(x')] \rangle_0 = -i \int \mathrm{d}a \varrho_1(a)\, \Delta(x-x', a) . \tag{13}$$

The commutation relation of the total potential $\boldsymbol{A}_\mu(x)$ is

$$\begin{aligned} \langle [\boldsymbol{A}_\mu(x), \boldsymbol{A}_\nu(x')] \rangle_0 &= \langle [\boldsymbol{a}_\mu(x), \boldsymbol{a}_\nu(x')] \rangle_0 + \partial_\mu \partial'_\nu \langle [\Lambda(x), \Lambda(x')] \rangle_0 = \\ &= -i \int_0^\infty \mathrm{d}a \varrho(a) \left(g_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{a} \right) \Delta(x-x', a) + i \int_0^\infty \mathrm{d}a \varrho_1(a)\, \partial_\mu \partial_\nu \Delta(x-x', a) . \end{aligned} \tag{14}$$

If one compares (14) with the canonical commutation relation at $t = t'$ (*), the following relation will be obtained:

$$\left| \begin{aligned} & M \equiv \int_0^\infty \mathrm{d}a \varrho(a)/a = -\int_0^\infty \mathrm{d}a \varrho_1(a) , \\ & Z_3^{-1} = \int_0^\infty \mathrm{d}a \varrho(a) , \\ & -1 = \int_0^\infty \mathrm{d}a \cdot a \cdot \varrho_1(a) . \end{aligned} \right. \tag{15}$$

([3]) G. KÄLLÉN: *Helv. Phys. Acta*, **25**, 417 (1952); H. LEHMANN: *Nuovo Cimento*, **11**, 342 (1954); M. GELL-MANN and F. E. LOW: *Phys. Rev.*, **95**, 1300 (1954).

(*) For instance, see KÄLLÉN's article ([3]).

Consequently,

$$(16)\qquad \begin{cases} \varrho_1(a) = -\left(\dfrac{1}{a} + N\right)\delta(a)\,, \\ N = M - \displaystyle\int_0^\infty \mathrm{d}a\,\delta(a)/a\,. \end{cases}$$

The total photon propagator is now

$$(17)\qquad D_{\mu\nu}(x-x') = \langle T(\boldsymbol{a}_\mu(x), \boldsymbol{a}_\nu(x'))\rangle_0 + \langle T(\partial_\mu \Lambda(x), \partial'_\nu \Lambda(x'))\rangle_0 =$$

$$= \int_0^\infty \mathrm{d}a\varrho(a)\left(g_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{a}\right)\Delta_c(x-x', a) - iMn_\mu n_\nu\,\delta(x-x') -$$

$$- \int_0^\infty \mathrm{d}a\varrho_1(a)\,\partial_\mu\partial_\nu\,\Delta_c(x-x', a) + iMn_\mu n_\nu\,\delta(x-x') =$$

$$= \int_0^\infty \mathrm{d}a\left\{\varrho(a)\left(g_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{a}\right) - \varrho_1(a)\,\partial_\mu\partial_\nu\right\}\Delta_c(x-x', a)\,.$$

Therefore,

$$(18)\qquad \partial^\mu D_{\mu\nu}(x-x') = \int_0^\infty \mathrm{d}a\left\{\varrho(a)\,\partial_\nu\left(1 - \frac{\Box}{a}\right) - \varrho_1(a)\,\partial_\nu\Box\right\}\Delta_c(x-x', a) =$$

$$= \partial_\nu\left\{-i\int_0^\infty \mathrm{d}a\,\frac{\varrho(a)}{a}\,\delta(x-x') - i\int_0^\infty \mathrm{d}a\varrho_1(a)\,\delta(x-x')\right\} -$$

$$- \partial_\nu\int_0^\infty \mathrm{d}a\varrho_1(a)\cdot a\cdot\Delta_c(x-x', a) = \partial_\nu\,D_c(x-x'),$$

where the relations (15) and (16) have been used and the n_μ is the time-like unit vector.

We can further derive a relation between the radiative correction of the vertex part and the *improper* self-energy part as follows:

$$(19)\qquad \partial^\mu_y\langle T(\boldsymbol{J}_\mu(y), \boldsymbol{I}(x), \bar{\boldsymbol{I}}(x'))\rangle_0 =$$

$$= e_0\left\{\langle T(\boldsymbol{I}(x), \bar{\boldsymbol{I}}(y))\rangle_0\,\delta(y-x') - \delta(x-y)\langle T(\boldsymbol{I}(y), \bar{\boldsymbol{I}}(x'))\rangle_0\right\},$$

where

$$(20)\qquad \boldsymbol{I}(x) \equiv ie_0\boldsymbol{A}_\mu(x)\gamma^\mu\boldsymbol{\psi}(x) + \delta m\boldsymbol{\psi}(x)\,,$$

2234

and

$$\delta(x_0 - x_0')[\boldsymbol{I}(x), \boldsymbol{J}_0(x')] = e_0\,\delta(\boldsymbol{x} - \boldsymbol{x}')\,\boldsymbol{I}(x')\,. \tag{21}$$

The insertion of the expression

$$\begin{cases} \langle T(\boldsymbol{I}(x), \bar{\boldsymbol{I}}(x'), \boldsymbol{J}_\mu(y))\rangle_0 \equiv \dfrac{ie_0}{(2\pi)^8}\displaystyle\int \mathrm{d}^4p\,\mathrm{d}^4q \exp[ip(x-y)] \exp[iq(y-x')]\Lambda_\mu(p;q)\,, \\ \langle T(\boldsymbol{I}(x), \boldsymbol{I}(x'))\rangle_0 = \dfrac{i}{(2\pi)^4}\displaystyle\int \mathrm{d}^4p \exp[ip(x-x')] \sum_0 (i\gamma p)\,, \end{cases} \tag{22}$$

gives

$$-i(p-q)^\mu \Lambda_\mu(p;q) = \sum\nolimits_0 (i\gamma p) - \sum\nolimits_0 (i\gamma q)\,. \tag{23}$$

An application of the equations (10) and (23) will be presented in a forthcoming paper.

* * *

The author thanks Professor J. M. JAUCH for his valuable discussion.

2235