Singular Perturbation Theory: A Viscous Flow out of Göttingen

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Abstract

This review describes how singular perturbation theory grew out of Prandtl's fluid dynamical boundary-layer theory of 1904. Developments were centered at Göttingen until 1933, when research spread worldwide. After that, singular perturbations developed more rapidly as the subject became centered within applied mathematics.

SINGULAR PERTURBATION THEORY: THE BEGINNING

Most experts would agree that the birth of singular perturbations occurred on August 12, 1904, at the Third International Congress of Mathematicians in Heidelberg when Ludwig Prandtl (1875–1953), an *ordentlicher* professor of mechanics from the Technical University of Hanover, spoke, in German, "On fluid motion with small friction." His seven-page paper in the 1905 proceedings states,

> The physical processes in the boundary layer (*Grenzschicht*) between fluid and solid body can be calculated in a sufficiently satisfactory way if it is assumed that the fluid adheres to the walls, so that the total velocity there is zero—or equal to the velocity of the body. If the viscosity is very small and the path of the fluid along the wall is not too long, the velocity will have again its usual value very near to the wall. In the thin transition layer (*Übergangsschicht*) the sharp changes of velocity, in spite of the small viscosity coefficient, produce noticeable effects. [English translation as NACA Memo. No. 452 (1928), reprinted in Ackroyd et al. 2001]

Evolving from the era of hydraulics (see Eckert 2006), "the paper marked an epoch in the history of fluid dynamics, opening the way for understanding the motion of real fluids" (Tani 1977). The essential understanding concerns a singular perturbation. The Navier-Stokes equations reduce to Euler's equations when the viscosity is zero, but their solutions do not reduce to those of Euler's equations.

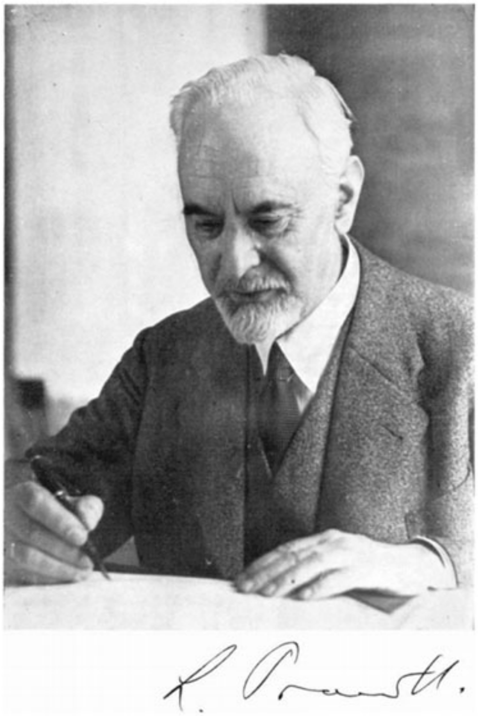
As Ting (2000) summarized, "Prandtl's boundary-layer theory initiated a systematic procedure for joining local and global expansions to form a uniformly valid approximation. The procedure, originally known as the boundary-layer technique, and later known as the method of matched asymptotic expansions, has been widely employed to resolve many singular perturbation problems in applied mechanics." (Those unfamiliar with the mathematics of singular perturbations might consult Cole 1968, O'Malley 1991, or Verhulst 2005.) Ting further describes

> the formulation of the boundary layer theory in four steps . . . in which the boundary layer solution and the inviscid solution are identified with the leading order inner and outer solutions respectively. The four steps are: I, the physical intuition or modeling of the flow field, II, the choice of scalings and the expansion scheme, III, derivation of the leading and higher order equations and the matching conditions, and IV, the construction of the inner and outer solutions and the study of their physical meaning.

The descriptions and diagrams invoked today are essentially the same as those that Prandtl used in 1904 (see Meier 2000, Meier et al. 2006, Ting 2000), although he considered only the leading-order terms in the asymptotic solutions. Prandtl's lecture occurred less than nine months after the Wright brothers' flight at Kitty Hawk, and its boundary-layer theory resolved d'Alembert's paradox that had suggested zero drag on a body in steady flow. Ting also observed, however, that results available from a more systematic matching needed to be hypothesized in Prandtl's related lifting-line theory of 1918 (see Friedrichs 1953), so the full story was unavailable at that time.

In summarizing fluid mechanics during the first half of the twentieth century, the fluid dynamicist Sydney Goldstein (1969) observed,

> In 1928 I asked Prandtl why he kept it so short, and he replied that he had been given ten minutes for his lecture at the Congress and that, being still quite young, he had thought he could publish only what he had time to say. The paper will certainly prove to be one of the most extraordinary papers of this century, and probably of many centuries.



L. Prandtl.

Figure 1

Ludwig Prandtl, 1875–1953.

LUDWIG PRANDTL

Although his earlier education was in engineering, Prandtl (**Figure 1**) received his Ph.D. in mathematics from the University of Munich in 1900 on the torsional instability of beams with an extreme depth-width ratio. His thesis advisor, August Föppl (1854–1924), was a revolutionary spirit in engineering mechanics and a successful textbook writer. [His 1894 *Introduction to Maxwell's Theory of Electricity* influenced Einstein's 1905 concept of special relativity (see Holton 1988, which provides quite a bit of information on Föppl and his impact).] Prandtl worked as an engineer at MAN (Maschinenfabrik Augsburg-Nürnberg) for a year before becoming a professor at Hanover. He became interested in flow separation when designing diffusers for wood-cutting machines (so his fluid was wood shavings!) (Meier 2006). Anderson (2005) reports that Prandtl's design operated well using one-third the power of the design he improved. The Heidelberg International Congress of Mathematicians lecture (supplementing Föppl and others' earlier recommendations) reconfirmed the decision of the geometer Felix Klein (1849–1925) (acting as Geheimrat and ambitious university administrator) to offer Prandtl the (lower-level) position as an Extra Ordinarius (although with increased pay) at the Georg-August-Universität Göttingen (see Hanle 1982). Darrigold (2005) reports that Klein told Prandtl his was the "most beautiful" talk he heard at the Congress. Klein, indeed, also hired Carl Runge (1856–1927) from Hanover, and they formed the basis of an Institut für Angewandte Mathematik und Mechanik, consistent with Klein's wish to narrow the gulf between mathematics and technology. [Klein had essentially given up research after a breakdown at 33, and he came to Göttingen in 1886 and worked tirelessly until his retirement in 1913 to establish the university as the foremost mathematics research center in the world, centered around David Hilbert (1862–1943), among others.] Paraphrasing Hermann Weyl, Reid (1976) reports that "Klein ruled in Göttingen like a god, but his godlike power came from the force of his personality, his dedication and willingness to work, and his ability to get things done." In 1907, Prandtl formed the Aerodynamische Versuchsanstalt (proving ground) and, in 1925, the Kaiser Wilhelm-Institut für Strömungsforschung, which still live on under the Deutsches Gessellschaft für Luft- und Raumfahrt and Max Planck labels, down Bunsenstrasse from the Mathematics Institute, which Richard Courant (1888–1972) directed from 1922 to 1933. Prandtl's subsequent research involved many students, including Blasius (1908), and collaborators, lasting through two world wars (see Meier 2000).Three volumes of his collected papers, totaling over 1600 pages, have been published (see Prandtl 1961). One gets the impression that many routine administrative duties were handled by Albert Betz, a former student who succeeded Prandtl as director of the Max Planck Institute in 1947, but one has to wonder how corrupted Prandtl might have become during the Nazi years as his institutes expanded greatly, with Hermann Göring heading the Reich's Air Ministry (see Hirschel et al. 2004). Stanford engineering professors Flügge-Lotz & Flügge (1973), his former students, defended him. (This author, indeed, remembers a seminar "The Ludwig Prandtl I knew" by Flügge-Lotz at Stanford in the 1960s, conducted much like a memorial service.) Lienhard (1970) reports that the war's end in 1945 found "Prandtl complaining peevishly about bomb damage to his roof and asking how the Americans planned to support his ongoing work." Prandtl told von Kármán at that time that "he was not a Nazi, but had to defend his country" (see von Kármán & Edson 1967). After the war, Prandtl described himself as an "unpolitical German," according to Mehrtens & Kingsbury (1989).

The most famous student Prandtl had was, no doubt, the Hungarian (and, ultimately, American) Theodore von Kármán (1881–1963), who arrived in Göttingen in 1906 (**Figure 2**). The Kármán biographer Gorn (1992) reported, "Almost from the start, a thinly concealed rivalry developed between the thirty-one year-old mentor and the twenty-five year-old pupil. The Hungarian's *joie de vivre* contrasted sharply with the shy, formal, pedantic habits of Prandtl." Despite the German's brilliant achievement, von Kármán found it difficult to acknowledge Prandtl's superiority, and a clash of wills ensued. The competition intensified after von Kármán succeeded Hans Reissner as director of the Aeronautics Institute in Aachen in 1913. Indeed, in his autobiography, von Kármán noted,

> I came to realize that ever since I had come to Aachen my old professor and I were in a kind of world competition. The competition was gentlemanly, of course. But it was first-class rivalry nonetheless, a kind of Olympic Games, between Prandtl and me, and beyond that between Göttingen and Aachen. The "playing field" was the Congress of Applied Mechanics. Our "ball" was the search for a universal law of turbulence. (von Kármán & Edson 1967)

He further characterized Prandtl's life as "particularly full of overtones of naïveté." Indeed, in a story about Prandtl's marriage, he said,

> In 1909, for example, Prandtl decided that he really ought to marry; but he didn't know how to proceed. Finally, he wrote to Mrs. Föppl, asking for the hand of one of her daughters. But which one? Prandtl had not specified. At a family conference, the Föppls made the practical decision that he should marry the eldest daughter, Gertrude. He did and the marriage was apparently a happy one. (quoted in Lienhard 1970)

However, Klaus Gersten (personal communication), editor of the latest edition of Schlichting's *Boundary Layer Theory*, insists that Prandtl married the appropriate daughter since he was 34, and

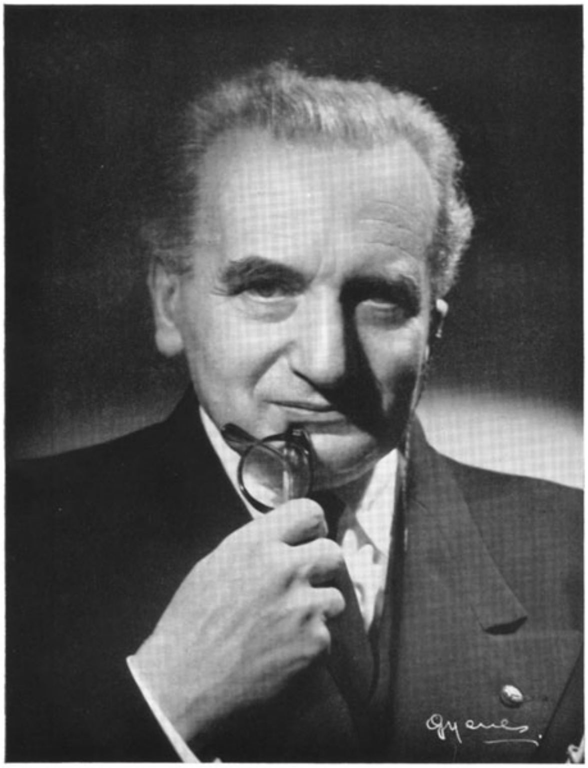


Figure 2

Theodore von Kármán, 1881–1963.

the daughters were then respectively in their late and early twenties. (The tradition of marrying the daughter of one's professor had been common in academic Germany, sometimes being suggested as the way mathematical talent was transferred between generations. Courant, for example, married Runge's daughter, and his daughters have also married prominent mathematicians, namely Jurgen Moser and Peter Lax, who certainly did not need family assistance to obtain success.)

In his history of aerodynamics, von Kármán (1954) summarized Prandtl's skills as follows: "His control of mathematical methods and tricks was limited.... But his ability to establish systems of simplified equations which expressed the essential physical relations and dropped the nonessentials was unique, I believe, even compared to his great predecessors in the field of mechanics—men like Leonhard Euler and d'Alembert." However, he also stated, "Prandtl had so precise a mind that he could not make a statement without qualifying it. This is a mistake. To be effective a teacher must see that a beginner in science grasps the basic principle before he can be expected to understand the exceptions." Furthermore, in his autobiography, von Kármán noted, "In my opinion Prandtl unraveled the puzzle of some natural phenomena of tremendous basic importance and was deserving of a Nobel prize" (von Kármán & Edson 1967). The leading British fluid dynamicist G.I. Taylor (1886–1975) apparently agreed. In a 1935 letter to Prandtl, Taylor wrote, "I feel very strongly that if the Nobel Prize is open to nonatomic physicists it is definitely insulting to our chief—and I think that in England and USA at any rate that means you—should never have been rewarded in this way" (quoted in Batchelor 1996). Reflecting this high regard, the University of Cambridge awarded Prandtl an honorary degree in June 1936. In his obituary of von Kármán, Taylor (1973), however, indicated some personal reservations:

> By the time the fourth Congress (of Applied Mechanics) was held in Cambridge, England in 1934, the German Jewish members were having a bad time. Theodore was well out of it but was doing a lot for his unfortunate fellow countrymen. Prandtl, who was not Jewish, appeared to be completely taken in by the Nazi propaganda and when the news of Roehm's murder came while he was in Cambridge he refused to believe it and said the papers had invented the story. In 1938 however when the fifth Congress was organized in Cambridge, Massachusetts, by Theodore and Jerome Hunsaker, conditions had changed. Prandtl and my wife and I were staying with Jerome at his home in Boston. The German delegation was strictly watched by political agents who had come as scientist members and Prandtl did not dare to be seen reading American papers. He used to ask my wife to tell him what was in them. After I had returned to Cambridge from the fifth Congress I had a letter from Prandtl telling me what a benevolent man Hitler was and including a newspaper cutting showing the Fuhrer patting children's heads. I imagine the poor man did this under pressure from the propaganda machine, for other people told me they had similar letters from him.

Prandtl had hoped the 1938 Congress would be held in Germany, assuming no distinction would be made between Jews and non-Jews. The Reich's Ministry of Education wrote Prandtl on August 28, 1938, that "Jews of foreign citizenship" who took part in the Congress would not be regarded "as Jews here," but that there would be no place for non-Aryans of German citizenship (see Mehrtens & Kingsbury1989).

It is uncertain how much our hesitance regarding Prandtl's character results from von Kármán's large influence and personality and what Prandtl would say to defend himself. After Courant was placed on leave from Göttingen in the spring of 1933, Prandtl joined 27 others in a petition in support of Courant. As Friedrichs later reported, "Several of the names on the list are those of people who later were considered to be Nazis or near-Nazis, and even at the time some of them were known to be in sympathy with the regime." Prandtl accepted the Hermann Göring medal from the Academy of Aeronautical Research in 1939 and chaired an advisory board for the Air Ministry. His daughter (see Vogel-Prandtl 1993) portrayed Prandtl as hostile to the regime, however, noting his refusal to have a portrait of Hitler in his office. Von Kármán became very influential in the United States and was awarded the first National Medal of Science by President Kennedy in 1961. Caltech archives include his May 1945 report "German Scientists Recommended for Evacuation to U.S."

There were certainly roots of singular perturbations and the boundary-layer idea in the nineteenth century, for example, Laplace's and Rayleigh's work on the meniscus, Stokes' work on the drag of a sphere, Hertz's work on elastic bodies in contact, Maxwell's measurement of viscosity, Helmholtz and Kirchhoff's work on circular-disc capacitors, and Rayleigh and Love's work on thin shells (Van Dyke 1994; for other precedents, see Tani 1977). Most importantly, early work on the edge effect in shell theory (e.g., see Reissner 1912, 1949; Gol'denveizer 1960) paralleled some of Prandtl's boundary-layer analysis (although naturally interchanging the inner and outer terminology), according to Eric Reissner (personal communication), Hans Reissner's son. Hans Reissner told von Kármán that "[p]eople attribute to Prandtl and to you discoveries which neither of you ever made" (von Kármán & Edson 1967). The French fluid dynamicist Germain (2000) observes that "Prandtl appears to be the first visionary discoverer of what we may, now, call fluid dynamics inspired by asymptotics." He goes on to write,

> One must stress that forty lines only are sufficient to Prandtl for delivering the essential of a number of great discoveries: the boundary layer concept itself, the equations which rule it and how they may

> be used, their self-similar solutions, the basic law that the boundary layer thickness goes like the square root of the viscosity. It is impossible to announce such major achievements in a shorter way.

Prandtl (1948) also emphasized the centrality of asymptotics. In explaining his approach, he stated,

> When the complete mathematical problem looks hopeless, it is recommended to enquire what happens when one essential parameter of the problem reaches the limit zero. It is assumed that the problem is strictly solvable when the parameter is set to zero from the start and that for very small values of the parameter a simplified approximate solution is possible. Then it must still be checked whether the limit process and the direct way lead to the same solution. Let the boundary condition be chosen so that the answer is positive. The old saying "Natura non facit soltus" decides the physical soundness of the solution: in nature the parameter is arbitrarily small, but it never vanishes. Consequently, the first way (the limiting process) is the physically correct one!

Tani (1977) observed that "the genesis of the boundary layer theory stood in sublime isolation: nothing similar had ever been suggested before, and no publications on the subject followed except for a small number of papers due to Prandtl's students for almost two decades." Only a brief reference to boundary-layer theory appeared in the fifth edition of Lamb's *Hydrodynamics* (1924). (The third and fourth editions, published in 1906 and 1916, respectively, did not mention the theory.) Prandtl's student Hermann Schlichting's lectures at Braunschweig from 1941–1942 lived on in mimeographed versions until they were revised and published by G. Braun of Karlsruhe as *Grenzschicht-Theorie* in 1951. The current ninth and eighth editions in German and English, respectively, are still published by Springer-Verlag. The slow acceptance and development of boundary-layer theory are considered by Dryden (1955) and Darrigold (2005).

In one fascinating section, Schlichting & Gersten (2000) quote extensively from Prandtl's lectures from the winter semester in 1931–1932 on intuitive and useful (*anschauliche und nützliche*) mathematics, describing the oscillations of a very small mass with damping. The outlines of matched asymptotic expansions are crystal clear, 10 years before Friedrichs' mimeographed 1941 Brown University lecture notes that motivated much later fundamental asymptotics at Caltech, New York University, and elsewhere (e.g., Lagerstrom 1988, von Mises & Friedrichs 1971), using a similar example. It describes the asymptotic solution of the two-point boundary value problem

$$\varepsilon y'' + y' + y = 0, \quad 0 \le x \le 1, \quad \text{with } y(0) \text{ and } y(1) \text{ prescribed},$$

where ε is a small positive parameter, reminiscent of the square root of the reciprocal of the physically relevant Reynolds number.

GÖTTINGEN

Before 1933, Göttingen was a Mecca for mathematicians worldwide, and the German influence was pervasive and dominant internationally, especially in pure mathematics (e.g., see Mac Lane 2005). Thus, it was natural that the early work of the prominent Japanese mathematician Mitio Nagumo (1905–1993) be published in German and that he would visit Göttingen from February 1932 to March 1934 (**Figure 3**). His lifelong research is said to be much influenced by that stay. A 1939 paper on initial value problems for singularly perturbed second-order ordinary differential equations was motivated by a chemist's question. As Sibuya observed in Nagumo's selected works, "when this paper was published, singular perturbations didn't exist" (Yamaguti et al. 1993). With

(Photographed by K.S., 1992)

南雲道夫

Figure 3

Mitio Nagumo, 1905–1993.

the exception of a 1959 paper on partial differential equations, however, none of Nagumo's other work seems to concern singular perturbations. The later critical use of differential inequalities to analyze singularly perturbed two-point boundary value problems (see Chang & Howes 1984, DeCoster & Habets 2006) is directly based on Nagumo's early use of upper and lower solutions (or "bounding functions") beginning about 1937.

Somewhat similarly, the Chinese mathematician Yu-Why Chen (1910–1995) came to Göttingen in 1928 at the suggestion of S.L. Wei, an earlier student of Courant (**Figure 4**). He attended lectures by Courant, Weyl, and Herglotz, with Courant being his Doktorvater, and Franz Rellich and Gustav Herglotz his referees. Rellich was the examiner on Mathematical Analysis in November 1933, and the doctorate was granted in 1935 for a thesis coinciding with the *Compositio Mathematica* paper (Tschen 1935). The work was generally forgotten, although it overlaps somewhat with Wasow's (1942) influential NYU thesis. Although Chen spent most of his career in China and Taiwan, where he was a member of Academia Sinica (see Math. Res. Center 1970), he ended his academic career at the University of Massachusetts in Amherst. When this author phoned him about 20 years ago, Chen said no one had asked him about his thesis for 50 years and confirmed that Courant had suggested a thesis topic to him on an ordinary differential equation model featuring boundary-layer behavior because he wished to encourage a mathematical analysis of the boundary-layer phenomenon.

Quite naturally, the ambitious new Cambridge Ph.D. Sydney Goldstein (1903–1989) visited Göttingen for the 1928–1929 year as a Rockefeller Research Fellow (**Figure 5**), and he studied problems involved in numerical boundary-layer calculations, a topic later pursued by his Manchester and Cambridge colleague D.R. Hartree (Goldstein 1930). Goldstein's readable two-volume *Modern Developments in Fluid Dynamics* (1938) was influential in promoting the use and study of boundary-layer theory. Quoting Sir Francis Bacon, its motto was "[f]or when propositions are denied, there is an end of them, but if they bee allowed, it requireth a new worke." Goldstein had, indeed, taken on this substantial editorial task upon the death of Sir Horace Lamb (1849–1934),

程毓淮

Yu Why Chen

Figure 4

Yu-Why Chen, 1903–1989.

S. Goldstein.

Figure 5

Sydney Goldstein, 1903–1989.

Figure 6

Erich Rothe, 1895–1998.

who had finally treated boundary layers extensively in the sixth edition of *Hydrodynamics* of 1932. An updated volume on laminar boundary layers appeared later (see Rosenhead 1963).

Applied mathematics was also actively developed at other German universities, especially at Berlin under the influence of Richard von Mises (1883–1953). One person so influenced was Erich Rothe (1895–1988), who wrote several papers on the asymptotic solution of partial differential equations and on the skin effect in electrical conductors (**Figure 6**) (see Rothe 1933a,b; 1936). After moving to America from Breslau and Berlin, Rothe (1939) published a very good paper on the solution of a singularly perturbed two-point boundary value problem in the obscure *Iowa State College Journal of Science*. He was later prominent at the University of Michigan (see Cesari et al. 1978), known largely for his work on nonlinear functional analysis. His transition between continents was not smooth, however. Rothe lived in Iowa, together with a young son and wife, Hildegard, also a refugee mathematician, who died of cancer in 1942. He lived temporarily on a $100 per month stipend and was also temporarily employed as a school teacher with only room and board and an allowance of $500 per year (Siegmund-Schultze 2009).

Göttingen and its prominence in the mathematical world changed abruptly with National Socialism. Adolph Hitler took dictatorial powers on March 23, 1933, and on April 7th, the "Law for the Reorganization of the Civil Service" was promulgated, dismissing Jews (except those appointed prior to 1914 or who had served in the front lines during World War I), political unreliables, and those whose positions were judged supernumerary (see Segal 2003). This explains why Chen's thesis direction switched that year from Courant to Rellich. As of 1937, anyone married to a Jew also lost his or her university position. Hitler's victims consequently spread worldwide. For example, among some who were later involved in asymptotics, Wiktor Eckhaus survived the war in Poland (see Eckhaus 1997), Arthur Erdélyi emigrated to Edinburgh (see Colton 1979), von Mises went to Istanbul and then Boston (after the death of Atatürk) (see Goldstein 1963), and Abraham Robinson went to France and then England (see Daubin 1995). Von Mises and his future wife, Hilda Geiringer, actually came to Harvard in 1939 without pay (although he became the Gordon-McKay Professor of Aerodynamics and Applied Mathematics there in 1944). The Dean of Harvard's Graduate School of Engineering wrote to him: "I was pleased to receive your cablegram according to which you will accept an appointment as Lecturer of Applied Mathematics for the year 1939–40, without salary but with an obligation to present some Lectures bearing on mechanics. I regret that our funds do not permit us to offer a salary, but we shall welcome you here" (quoted in Siegmund-Schultze 2009). Beyerchen (1977) reports that Hilbert was asked at a banquet by the Nazi minister of science, "And how is mathematics in Göttingen now that it has been freed of the Jewish influence." Hilbert replied, "Mathematics in Göttingen? There is really none anymore."

Eighty-seven of the 143 German émigré mathematicians came to the United States, and before 1945, all but seven had reached positions in the university system (Pinl & Furtmüller 1973, Siegmund-Schultze 2009). The United States benefited tremendously from the resulting influx of these émigrés, especially regarding the development of applied mathematics. Indeed, their immediate contribution to the war against Germany was substantial (see Greenberg & Goldstein 1983). Already in 1934, Courant wrote from his intermediate stop at Cambridge University,

> Germany's best friends such as Hardy, Flexner, Lord Rutherford, the Rockefeller Foundation get alienated while our institutions, which were unequalled in the world, are destroyed—even Cambridge cannot compare to the old Göttingen. The foreign countries take advantage of the situation and employ people, particularly physicists and chemists, who will in the long run give science and its applications there a huge boost. (quoted in Siegmund-Schultze 2009)

(Courant's transition from director at Göttingen to founder of the graduate mathematics program at New York University, now known as the Courant Institute, is described in Reid 1976.) When Max Planck, as president of the Kaiser Wilhelm Society, expressed his concerns, Hitler replied, "If the dismissal of Jewish scientists means the annihilation of contemporary German science, then we will do without science for a few years" (quoted in Cornwell 2003).

The abrupt decline in German mathematical dominance was quantified by Wilhelm Süss, Führer of the Deutsche Mathematiker-Vereinigung from 1938 to 1945, who reported to a conference of German university presidents in 1942,

> From all mathematical journals of the world of 1937 . . . all citations have been counted and ordered according to countries and years. Of all papers cited in 1937 which were published until 1870 about 46% were written by German authors, 20% by English and only 1% by American mathematicians. For the period 1931–1935 the numbers are 28, 13, and 25, and meanwhile the development may well have become even more unfortunate for us. (quoted in Siegmund-Schultze 1997)

Mehrtens & Kingsbury (1989) report that the number of math students at Göttingen decreased from 432 in 1932 to 37 in 1939.

A summer school for applied mechanics began at Brown University in 1941, before the United States entered the war, with a faculty consisting largely of new immigrants. Brown University's Division of Applied Mathematics, indeed, had its origin in the corresponding National Research Council's Program of Advanced Instruction and Research in Mechanics, as did its journal

Quarterly of Applied Mathematics (1943 to present). Roland G.D. Richardson (1878–1949), Dean of Brown's Graduate School and long-time Secretary of the American Mathematical Society, drove much of this development, which was initially centered on continuum mechanics. Reingold (1981) reports on the natural clash between Richardson and Courant, who were both ambitious to launch applied mathematics programs in America, whereas Reid (1976) reports on relations between Richardson and Courant back to 1908, including the story of a letter of Richardson's stolen by Courant.

The most prominent American differential equations expert (and Dean of Harvard's Faculty of Arts and Sciences), George David Birkhoff, warned, as the leading American mathematician in the late depression, that this big migration from Europe would force some young American mathematicians to become "hewers of wood and drawers of water" (see Birkhoff 1938). Curiously, the Deutsche Mathematiker-Vereinigung had asked Birkhoff, who they considered "deutschfreundlich," to represent them at the 50th anniversary of the American Mathematical Society in 1938 (see Siegmund-Schultze 2009). Einstein is said to have labeled Birkhoff "one of the world's great anti-Semites," whereas Mac Lane (1994) defended him, saying that whatever "diffuse and varied versions of anti-Semitism" Birkhoff may have had "were undoubtedly shared by many of his contemporaries." Courant told Reid (1976), "I don't think he was any more antisemitic than good society in Cambridge, Massachusetts." Birkhoff's Chicago thesis was indeed about the asymptotic integration of linear ordinary differential equations, relevant to the later development of singular perturbation theory (similar to much related work of his prominent student Rudolph Langer) (see Birkhoff 1908).

Wolfgang Wasow (1909–1993) studied mathematics in Göttingen, seeking to pass his Staatsexamen to become a teacher (**Figure 7**) (see O'Malley 1993, Wasow 1986). He passed his orals in January 1933 (two days before Hitler became chancellor) and applied for practice teaching, but was not employed. After some wandering and teaching in a German boarding school in Florence and at Goddard College in Vermont, he took a fellowship at New York University in the fall of 1940, arranged by Courant. (The fellowship was for $600, though almost half was required for tuition. Wasow's wife continued to teach art at Goddard. During the following year, he commuted from New York to New London to teach at the Connecticut College for Women.) His 133-page thesis was nearly complete the following summer under the direction of Kurt O. Friedrichs (1901–1982). It described many singular perturbation examples and made Prandtl's boundary-layer theory into a mathematical subject. The topic was not immediately well received, however. Papers based on it were rejected by the *Transactions* and the *Annals*, but a 10-page paper ultimately appeared in 1944 in MIT's *Journal of Mathematics and Physics* ["with some behind the scenes support from Courant," according to Wasow (1986)]. Wasow continued important work on asymptotics throughout his life, summarized in *Asymptotic Expansions for Ordinary Differential Equations* (1965) and *Linear Turning Point Theory* (1985). After the 1960s, Wasow received compensation from the German government because he had "fulfilled all professional conditions for a government position and had applied but was rejected for such [racial and political] reasons." He wrote, "My grades and qualifications were good, and I was excluded because of my Jewish ancestors. The point that in 1933 the teaching profession was so overcrowded in Germany that I might not have been hired even under normal conditions, fortunately was not raised" (Wasow 1986).

Courant's student Friedrichs (**Figure 8**) left a professorship at Braunschweig in 1937 to come to New York University to marry a non-Aryan. He encountered boundary layers in plate theory, in joint work with J.J. Stoker (see Friedrichs & Stoker 1941). Moreover, in studying nonlinear oscillators, such as those described by the van der Pol equation, Friedrichs & Wasow (1946) introduced the term singular perturbation, in contrast to the more common situation of a regular perturbation in which a simple asymptotic power series suffices. Wasow (1986) observed, "In later

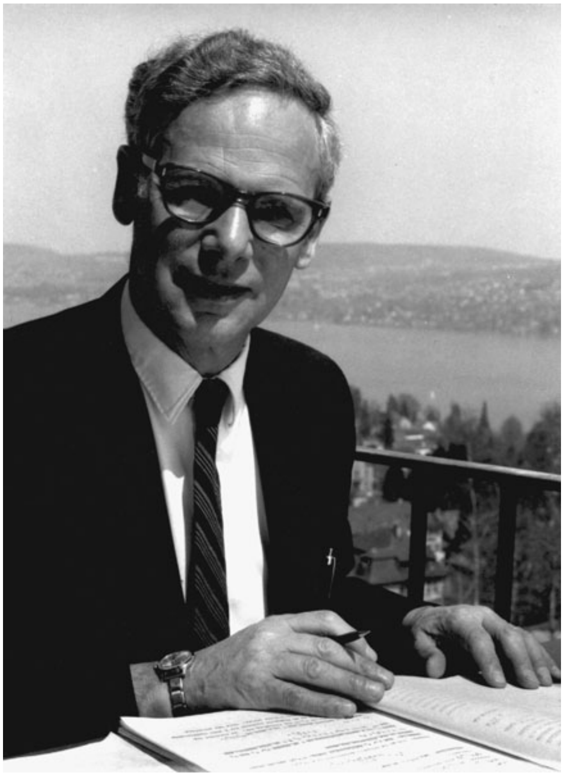


Figure 7

Wolfgang Wasow, 1909–1993.

years, neither of the two authors could remember to which of them belongs the credit for coining this terminology, but I believe it was Friedrichs." That memoir modestly states that this post-thesis research "was soon overshadowed by two articles by MIT's Norman Levinson (1912–1975) who obtained more general results by different methods." (Levinson's important work on singular perturbations is summarized in Nohel 1998.) Friedrichs' enthusiasm and a magnificent overview of asymptotic analysis are clearly displayed in his 1954 Gibbs' Lecture (see Friedrichs 1955). His broad contributions to analysis and differential equations more generally overwhelm those in asymptotics alone (see Morawetz 1968), but many of his students and colleagues, especially Joseph B. Keller, have contributed enormously to the field and its many important applications.

Courant had convinced the Rockefeller Foundation's International Education Board (IEB) to spend $275,000 to build a new Mathematical Institute in Göttingen (balancing the Institut Henri Poincaré it funded in Paris). It was completed in 1929, making its anticipated impact short-lived. The IEB's rationale (see Siegmund-Schultze 2001) emphasized,

> The Board would be *not* interested in . . . housing or even helping to house the mathematical department in more agreeable quarters, unless thereby there was a practical certainty that greater and much greater usefulness to a group of sciences would result.
>
> In fact, the new mathematical institute was finally erected on Bunsenstraβe in Göttingen close to the physical and chemical institutes and the aerodynamical institutes of Ludwig Prandtl (1875–1953). These scientists in turn now had better access to the mathematicians and to the mathematical library.

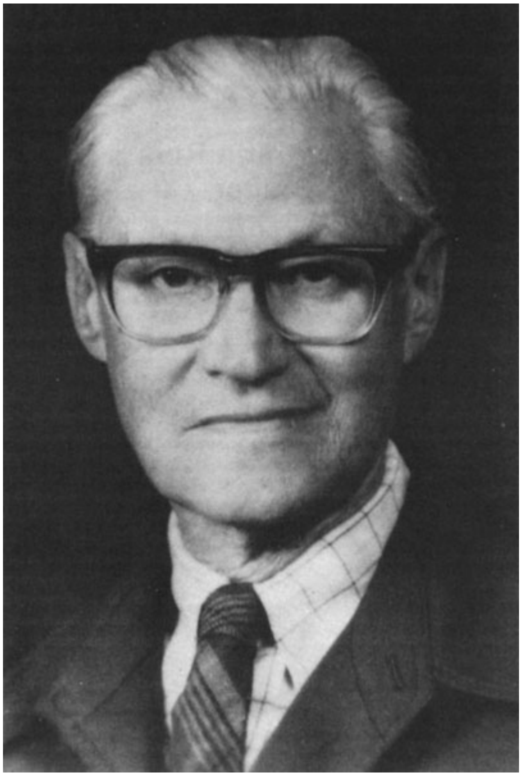


Figure 8

Kurt O. Friedrichs, 1901–1982.

> Although Prandtl was not involved in the negotiations proper, and at one point was mentioned in a slightly disparagingly way as "more of an engineer than a physicist" (but also a professor in the University mathematics department), his institutes and the technical sciences are expressly included in the "group of sciences" which were of interest to (the IEB's) Trowbridge. In fact this was completely in the tradition of Felix Klein who had called Prandtl to Göttingen in 1904 in order to enrich its scientific environment. Trowbridge's report includes a longer passage on his visit to Prandtl's AVA (aerodynamical proving ground) with its wind tunnel, and mentions the new and more theoretical Kaiser-Wilhelm-Institut.

Curiously, Birkhoff participated in the IEB site visit to Göttingen. When Courant visited Göttingen in 1947, he reported that the aerodynamics institute had become "a veritable fortress." Although ill and depressed, Prandtl was mentally alive. He had given much thought to analog computing machines with a view toward meteorological computations.

CONCLUSION

By 1950, singular perturbations was being studied by a wide variety of mathematicians and engineers worldwide (most prominently in Cambridge, Massachusetts; Cambridge, England; Pasadena; Moscow; and New York; for example). Comparatively little, however, was happening in Göttingen. Notable asymptotics concerning nonlinear oscillations, but independent of boundary-layer theory, had been done by the Kiev School (see Krylov & Bogoliubov 1943). This study of early boundary-layer theory should, indeed, be followed by another about the work on singular perturbations at Caltech in the 1950s by Kaplun, Lagerstrom, Cole, and others (see Cole 1994, Kaplun 1967), mostly in the Guggenheim Aeronautical Laboratory, which von Kármán headed since 1930. Fluid dynamics continued to motivate much of the ongoing work.

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LITERATURE CITED

Ackroyd JAD, Axcell BP, Ruban AJ. 2001. *Early Developments of Modern Aerodynamics*. Oxford: Butterworth-Heinemann

Anderson JD Jr. 2005. Ludwig Prandtl's boundary layer. *Phys. Today* 58:42–48

Batchelor G. 1996. *The Life and Legacy of G. I. Taylor*. Cambridge, UK: Cambridge Univ. Press

Beyerchen A. 1977. *Scientists Under Hitler*; *Politics and the Physics Community in the Third Reich*. New Haven, CT: Yale Univ. Press

Birkhoff GD. 1908. On the asymptotic character of the solutions of certain linear differential equations containing a parameter. *Trans. Am. Math. Soc.* 9:219–31

Birkhoff GD. 1938. Fifty years of American mathematics. In *Semicentennial Addresses of the American Mathematical Society*, 2:270–315. New York: Am. Math. Soc.

Blasius H. 1908. Grenzschlichten in Flüssigkeiten mit Kleiner Reibung. *Z. Math. Phys.* 56:1–37

Cesari L, Kannan R, Weinberger H, eds. 1978. *Nonlinear Analysis: A Collection of Papers in Honor of Erich H. Rothe*. New York: Academic

Chang KW, Howes FA. 1984. *Nonlinear Singular Perturbation Phenomena: Theory and Application*. New York: Springer-Verlag

Cole JD. 1968. *Perturbation Methods in Applied Mathematics*. Waltham: Blaisdell

Cole JD. 1994. The development of perturbation theory at GALCIT. *SIAM Rev.* 36:425–30

Colton D. 1979. Arthur Erdélyi. *Bull. Lond. Math. Soc.* 11:191–207

Cornwell J. 2003. *Hitler's Scientists*. New York: Viking

Darrigold O. 2005. *Worlds of Flow: A History of Hydrodynamics from the Bernoullis to Prandtl*. Oxford, UK: Oxford Univ. Press

Dauben JW. 1995. *Abraham Robinson: The Creation of Nonstandard Analysis, a Personal and Mathematical Odyssey*. Princeton, NJ: Princeton Univ. Press

De Coster C, Habets P. 2006. *Two-Point Boundary Value Problems: Lower and Upper Solutions*. Amsterdam: Elsevier

Dryden HL. 1955. Fifty years of boundary layer theory and experiment. *Science* 121:375–80

Eckert M. 2006. *The Dawn of Fluid Dynamics*. Weinheim: Wiley-VCH

Eckhaus W. 1997. *Witus en de jaren van angst*, *Een reconstructie*. Amsterdam: Bas Lubberhuizen

Flügge-Lotz I. Flügge W. 1973. Ludwig Prandtl in the nineteen-thirties: reminiscences. *Annu. Rev. Fluid Mech.* 5:1–9

Friedrichs KO. 1953. *Special Topics in Fluid Mechanics*. Lecture Notes. New York: New York Univ.

Friedrichs KO. 1955. Asymptotic phenomena in mathematical physics. *Bull. Am. Math. Soc.* 61:367–81

Friedrichs KO, Stoker JJ. 1941. The nonlinear boundary value problem of the buckled plate. *Am. J. Math.* 63:839–88

Friedrichs KO, Wasow W. 1946. Singular perturbations of nonlinear oscillations. *Duke Math. J.* 13:367–81

Germain P. 2000. The 'new' mechanics of fluids of Ludwig Prandtl. In *Ludwig Prandtl, ein Führer in der Strömungslehre*, ed. GEA Meier, pp. 31–40. Braunschweig: Vieweg

Gol'denveizer AL. 1960. Some mathematical problems in the linear theory of thin elastic shells. *Russ. Math. Surv.* 15(5):1–73

Goldstein S. 1930. Concerning some solutions of the boundary layer equations in hydrodynamics. *Proc. Camb. Philos. Soc.* 26:1–30

Goldstein S, ed. 1938. *Modern Development in Fluid Dynamics*. Oxford: Oxford Univ. Press

Goldstein S. 1963. Richard von Mises 1883–1953. In *Selected Papers of Richard von Mises*, Vol. 1, ed. P Frank et al., pp. ix–xiv. Providence, RI: Am. Math. Soc.

Goldstein S. 1969. Fluid mechanics in the first half of this century. *Annu. Rev. Fluid Mech.* 1:1–28

Gorn MH. 1992. *The Universal Man: Theodore von Kármán's Life in Aeronautics*. Washington, DC: Smithsonian Inst. Press

Greenberg JL, Goldstein JR. 1983. Theodore von Kármán and applied mathematics in America. *Science* 222:1300–4

Hanle PA. 1982. *Bringing Aeronautics to America*. Cambridge, MA: MIT Press

Hirschel EH, Prem H, Madelung G, eds. 2004. *Aeronautical Research in Germany: From Lilienthal until Today*. Berlin: Springer

Holton G. 1988. *Thematic Origins of Scientific Thought: Kepler to Einstein*. Cambridge, MA: Harvard Univ. Press. Rev. ed.

Kaplun S. 1967. *Fluid Mechanics and Singular Perturbations*, ed. PA Lagerstrom, LN Howard, C-S Liu. New York: Academic

Krylov NM, Bogoliubov NN. 1943. *Introduction to Nonlinear Mechanics*. Princeton, NJ: Princeton Univ. Press

Lagerstrom PA. 1988. *Matched Asymptotic Expansions*. New York: Springer-Verlag

Lamb H. 1924. *Hydrodynamics*. Cambridge, UK: Cambridge Univ. Press. 5th ed.

Lienhard JH. 1970. Ludwig Prandtl. In *Dictionary of Scientific Biography*, ed. CC Gillispie, 11:123–25. New York: Scribner

Mac Lane S. 1994. Jobs in the 1930s and the views of George D. Birkhoff. *Math. Intell.* 16(3):6–7

Mac Lane S. 2005. *Saunders Mac Lane: A Mathematical Autobiography*. Wellesley: A.K. Peters

Math. Res. Center. 1970. *Studies and Essays Presented to Yu-Why Chen*. Taipei: Math. Res. Center, Natl. Taiwan Univ.

Mehrtens H, Kingsbury VM. 1989. The Gleichschaltung of mathematical societies in Nazi Germany. *Math. Intell.* 11:48–60

Meier GEA, ed. 2000. *Ludwig Prandtl, ein Führer in der Strömungslehre*. Vieweg: Braunschweig

Meier GEA. 2006. Prandtl's boundary layer concept and the work in Göttingen. See Meier et al. 2006, pp. 1–18

Meier GEA, Sreenivasan KR, Heinemann H-J, eds. 2006. *IUTAM Symposium on One Hundred Years of Boundary Layer Research*. Dordrecht: Springer

Morawetz CS, ed. 1968. *Kurt Otto Friedrichs Selecta*. Boston: Birkhäuser

Nohel JA. 1998. Commentary. In *Selected Papers of Norman Levinson*, Vol. 1, ed. JA Nohel, DH Sattinger, pp. 267–87. Boston: Birkhäuser

O'Malley RE Jr. 1991. *Singular Perturbation Methods for Ordinary Differential Equations*. New York: Springer-Verlag

O'Malley RE Jr. 1993. Obituary: Wolfgang R. Wasow. *SIAM News*, Dec., pp. 2–3

Pinl M, Furtmüller L. 1973. Mathematicians under Hitler. In *Year Book of the Leo Baeck Institute*, 18:129–82. London: Leo Baeck Inst.

Prandtl L. 1905. Über Flüssigkeits bewegung bei kleiner Reibung. *Verh. III Int. Math. Kongr.*, pp. 484–91. Leipzig: Tuebner

Prandtl L. 1948. Mein Weg zu Hydrodynamischen Theorien. *Phys. Bl.* 4:89–92

Prandtl L. 1961. *Gesammelte Abhandlungen zur angewandten Mechanik, Hydro- und Aerodynamik*, ed. W Tollmien, H Schlichting, H Görtler. Berlin: Springer-Verlag

Reid C. 1976. *Courant in Göttingen and New York: The Story of an Improbable Mathematician*. New York: Springer-Verlag

Reingold N. 1981. Refugee mathematicians in the United States of America, 1933–1941: reception and reaction. *Ann. Sci.* 38:313–38

Reissner E. 1949. On the theory of thin elastic shells. In *Reissner Anniversary Volume: Contributions to Applied Mechanics*, ed. Dep. Aeronaut. Eng. Appl. Mech. Polytech. Inst. Brooklyn, pp. 231–47. Ann Arbor, MI: J.W. Edwards

Reissner H. 1912. Spannungen in Kugelschalen (Kuppeln). In *Festschrift H. Müller-Breslau*, pp, 181–193. Leipzig

Rosenhead L, ed. 1963. *Laminar Boundary Layers*. Oxford: Clarendon Press

Rothe E. 1933a. Über asymptotische Entwicklungen bei Randwertaufgaben der Gleichung $\Delta\Delta u + \lambda^k u = \lambda^k \psi$. *Math. Ann.* 109:267–72

Rothe E. 1933b. Zur Theorie des Skin-effekts. *Z. Phys.* 83:184–86

Rothe E. 1936. Über asymptotische Entwicklungen bei gewissen nicht-linearen Randwertaufgaben. *Comp. Math.* 3:310–27

Rothe E. 1939. Asymptotic solution of a boundary value problem. *Iowa State College J. Sci.* 13:369–72

Schlichting H, Gersten K. 2000. *Boundary Layer Theory*. Berlin: Springer. 8th ed.

Segal SL. 2003. *Mathematicians Under the Nazis*. Princeton, NJ: Princeton Univ. Press

Siegmund-Schultze R. 1997. The emancipation of mathematical research publishing in the United States from German dominance (1878–1945). *Hist. Math.* 24:135–66

Siegmund-Schultze R. 2001. *Rockefeller and the Internationalization of Mathematics Between the Two World Wars*. Basel: Birkhäuser

Siegmund-Schultze R. 2009. *Mathematicians Fleeing from Nazi Germany*. Princeton, NJ: Princeton Univ. Press

Tani I. 1977. History of boundary-layer theory. *Annu. Rev. Fluid Mech.* 9:87–111

Taylor GI. 1973. Memories of von Kármán. *SIAM Rev.* 15:447–52

Ting L. 2000. Boundary layer theory to matched asymptotics. *Z. Angew. Math. Mech.* 80:845–55

Tschen Y-W. 1935. Über das Verhalten der Lösungen einer Folge von Differential gleichungen welche im Limes ausarten. *Comp. Math.* 2:378–401

Van Dyke M. 1994. Nineteenth-century roots of the boundary layer idea. *SIAM Rev.* 36:415–24

Verhulst F. 2005. *Methods and Applications of Singular Perturbations: Boundary Layers and Multiple Timescale Dynamics*. New York: Springer

Vogel-Prandtl J. 1993. *Ludwig Prandtl, Ein Lebensbild Erinnerungen*. Mitterluugen aus dem Max-Plank-Institut für Ström. No. 107. Göttingen

von Kármán T. 1954. *Aerodynamics: Selected Topics in Light of Their Historical Development*. Ithaca, NY: Cornell Univ. Press

von Kármán T, Edson L. 1967. *The Wind and Beyond: Theodore von Kármán, Pioneer in Aviation and Pathfinder in Space*. Boston: Little-Brown

von Mises R, Friedrichs KO. 1971. *Fluid Dynamics*. New York: Springer-Verlag

Wasow W. 1942. *On boundary layer problems in the theory of ordinary differential equations*. PhD diss., New York Univ.

Wasow WR. 1965. *Asymptotic Expansions for Ordinary Differential Equations*. New York: Wiley-Interscience

Wasow WR. 1985. *Linear Turning Point Theory*. New York: Springer-Verlag

Wasow WR. 1986. *Memories of Seventy Years, 1909–1979*. Private printing

Yamaguti M, Nirenberg L, Mizohata S, Sibuya Y, eds. 1993. *Mitio Nagumo Collected Papers*. Tokyo: Springer-Verlag

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