Incompressible Quantum Hall Liquid on the Four-Dimensional Sphere

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The quantum Hall effect (QHE) is a cornerstone of topological physics, inspiring extensive explorations of its high-dimensional generalizations such as experimental realizations in synthetic systems including cold atoms, photonic lattices, and metamaterials. However, the many-body effect in the higher dimensional QHE system remains poorly understood. We explore this problem by formulating the microscopic wave functions on a four-dimensional sphere inspired by Laughlin's seminal work. Employing a generalized pseudopotential framework, we derive an exact microscopic Hamiltonian consisting of two-body projection operators that annihilate the microscopic wave functions. Diagonalizations on finite system sizes show that the quasihole states remain zero energy while the quasiparticle states exhibit a finite gap, in consistency with an incompressible state. Furthermore, the pair distribution is calculated to substantiate the liquidlike nature of the wave function. Our Letter provides a preliminary understanding to the fractional quantum Hall states in high dimensions.

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The discovery of the integer quantum Hall effect (IQHE) [1] marked a pivotal milestone in condensed matter physics research, opening up an era for studying topological phenomena. In IQHE systems, two-dimensional electrons subjected to a magnetic field form Landau levels, and the quantization of Hall conductance is intrinsically tied to the Landau level wave function topologies [2]. Moreover, many-body properties become research focus following the discovery of the fractional quantum Hall effect (FQHE) [3]. The groundbreaking insight comes from the celebrated Laughlin wave function [4], which captures the essence of the FQHE, including fractionally quantized Hall conductance and fractionally charged excitations. Various generalizations of the Laughlin wave function have been proposed, including the Read-Moore state [5] and the Haldane-Rezayi state [6], offering new perspectives on the FQHE.

Various investigations have explored the extension of Landau levels to higher-dimensional systems [7–14]. Most notably, Landau levels on the four-dimensional (4D) sphere (S^4) have been constructed [7] by introducing a Yang monopole located at the origin [15], analogous to Haldane's construction of Landau levels on S^2 with a Dirac monopole [16]. The 4D quantum Hall effect (4D QHE) has been studied in the context of noncommutative geometry [17–25]. Experimental advances have also demonstrated the realization of the 4D QHE in cold atom systems [26,27], optical lattices [28], acoustic lattices [29–33], and electric circuits [34–37]. Furthermore, the Yang monopole has been simulated in cold atom experiments and metamaterials [38–40]. Nevertheless, the many-body physics of the 4D FQHE remains poorly understood, constituting an exceptionally difficult and largely unexplored frontier for investigations via the synthetic dimension [41–50] and the hyperbolic lattice [51–56] in future experiments.

In this Letter, we explore the many-body properties based on S^4 augmented by a Yang monopole background. Both the determinant-type and Jastrow-type Laughlin wave functions are constructed, which are the ground states of Hamiltonians with suitably designed pseudopotentials. Exact diagonalizations are performed to the corresponding Hamiltonians, revealing signatures of an incompressible quantum liquid state. These states provide further insights for studying high dimensional interacting topological phases.

We begin with the following single-particle Hamiltonian defined on an S^4 sphere [7,15],

$$H = \frac{(P_\mu + A_\mu)^2}{2M}, \qquad (1)$$

where the radius is R; $A_\mu = -[1/2R(R+x_5)]\eta^a_{\mu\nu} I_a x_\nu$ is the SU(2) gauge field of a Yang monopole, with the t'Hooft

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symbol $\eta^a_{\mu\nu}(\mu,\nu=1,2,3,4;a=1,2,3)$ [57]; I_a are the SU(2) generators with I and $I(I+1)$ the values of the isospin and Casimir, respectively. The single-particle wave functions of Eq. (1) are organized into the SO(5) irreducible representation (IRREP) [58] $(N+2I,N)_{\mathrm{SO}(5)}$ [7,15,59], where $N\geq 0$ is the Landau level index. The eigen energy is given by $E_N=[\hbar^2/2MR^2](\mathcal{C}(N+2I,N)-2I(I+1))$, where $\mathcal{C}(p,q)=(p^2+q^2/2)+2p+q$ is the eigenvalue of the SO(5) Casimir. The lowest Landau level (LLL) states form the IRREP of $(2I,0)_{\mathrm{SO}(5)}$, whose degeneracy is given by

$$d(2I,0)=\frac{1}{3!}(2I+1)(2I+2)(2I+3). \qquad (2)$$

A brief introduction to the so(5) algebra in the presence of a Yang monopole is presented in End Matter. We decompose the LLL states into the IRREPs of the subgroup SO(4), whose six generators can be constructed with the t'Hooft symbol [60],

$$\hat{J}_a=\frac{1}{4}\eta^a_{\mu\nu}L_{\mu\nu}, \qquad \hat{K}_a=\frac{1}{4}\bar{\eta}^a_{\mu\nu}L_{\mu\nu}. \qquad (3)$$

Applying the branching rule to the LLL states, we obtain $(2I,0)_{\mathrm{SO}(5)}=\bigoplus_{j+k=I}(j,k)_{\mathrm{SO}(4)}$. Since $\hat{J}_a$ and $\hat{K}_a$ form two mutually commuting su(2) algebras, $(j,k)_{\mathrm{SO}(4)}$ denotes the direct product of two SU(2) representations characterized by quantum numbers j and k, respectively.

Next we explain how to express the LLL wave functions by the homogeneous polynomials of the four-component fundamental SO(5) spinors ψ_α with $\alpha=1\sim 4$. Through the second Hopf map, one defines a unit vector $x_a=\psi^\dagger_\alpha\Gamma^a_{\alpha\beta}\psi_\beta$, $(a=1,\ldots,5)$ on S^4. The 4×4 Clifford algebra matrices Γ^a mutually anticommute and their explicit representation is presented in Sec. II of Supplemental Material (SM) [61]. Conversely, each ψ_α can be expressed through x_a and a two-component spinor $u=(u_1,u_2)^T$ with the isospin coordinates on S^2 defined via the 1st Hopf map $n_i=u^\dagger\sigma_i u$. The LLL wave functions can be organized into a $(2I+1)$-component spinor denoted by $f_{I_3}(x_a)|II_3\rangle$, with each component being an eigenstate of I_3 satisfying $I\geq I_3\geq -I$. For compactness, $|II_3\rangle$ is represented on S^2 by $u_1^{I+I_3}u_2^{I-I_3}/\sqrt{(I+I_3)!(I-I_3)!}$. Then the LLL wave functions are expressed as

$$f^{(2I,0)}_{j,m_1;k,m_2}(x_a,n_i)=\mathcal{N}\psi_1^{j+m_1}\psi_2^{j-m_1}\psi_3^{k+m_2}\psi_4^{k-m_2}, \qquad (4)$$

where $\mathcal{N}=1/\sqrt{(j+m_1)!(j-m_1)!(k+m_2)!(k-m_2)!}$ is the normalization factor.

For later convenience in exploring many-body wave functions, we first construct the two-body states under the SO(5) rotational symmetry on S^4, which are significantly more complex than the S^2 case under the SU(2) symmetry [16]. The product of two single-particle LLL states can be decomposed into different SO(5) channels. According to the decomposition of the direct product of SO(5) IRREPs [62], we have

$$(2I,0)_{\mathrm{SO}(5)}\otimes(2I,0)_{\mathrm{SO}(5)}=\bigoplus_{n_1,n_2}(n_1+n_2,n_1-n_2)_{\mathrm{SO}(5)}, \qquad (5)$$

where $0\leq n_1\leq 2I$, and $0\leq n_2\leq n_1$. The simplest set of the two-body states lie in the channel of $(4I-2n,0)_{\mathrm{SO}(5)}$ with $0\leq n\leq 2I$. An SO(5) invariant is constructed as $S(x_a,n_i;x'_a,n'_i)=\psi_\alpha(x_a,n_i)\mathcal{R}_{\alpha\beta}\psi_\beta(x'_a,n'_i)$ with $\mathcal{R}$ being the charge conjugation matrix for the SO(5) fundamental spinor, satisfying $\mathcal{R}^2=-1$, $\mathcal{R}^{\mathbf{T}}=\mathcal{R}^{-1}=\mathcal{R}^\dagger=-\mathcal{R}$ [63]. With the help of $\mathcal{R}$, we obtain the two-body states in the form of

$$\Phi^{(4I-2n,0)}_{J,M_1;K,M_2}(x_a,n_i;x'_a,n'_i)=\mathcal{N}^{(4I-2n,0)}_{J,M_1;K,M_2}S^n \times f^{4I-2n}_{J,M_1;K,M_2}(x_a,n_i;x'_a,n'_i), \qquad (6)$$

where $\mathcal{N}^{(4I-2n,0)}_{J,M_1;K,M_2}$ is the normalization coefficient. The center-of-mass part $f^{4I-2n}_{J,M_1;K,M_2}$ is a symmetric homogeneous polynomial with the degree $4I-2n$ defined as

$$f^{4I-2n}_{J,M_1;K,M_2}(x_a,n_i;x'_a,n'_i) =C^{J,M_1}_{j,m_1;j',m'_1}C^{K,M_2}_{k,m_2;k',m'_2}f^{(2I-n,0)}_{j,m_1;k,m_2}(x_a,n_i)f^{(2I-n,0)}_{j',m'_1;k',m'_2}(x'_a,n'_i), \qquad (7)$$

where $C^{J,M_1}_{j,m_1;j',m'_1}$ is the SU(2) group Celebsh-Gordon (CG) coefficient. The statistics of the two-body wave function is captured by the power of the relative part S^n: Odd and even values of n correspond to the fermionic and bosonic statistics, respectively.

We move forward to the two-body states in more complex channels with $n_1>n_2$. Take the case of $n_1-n_2=1$ as an example which involves the regular SO(5) harmonics in the absence of the Yang monopole. In analogy to the SO(5) invariant S, an SO(5) vector, defined as $X_a(x_a,n_i;x'_a,n'_i)=\psi_\alpha(x_a,n_i)(\mathcal{R}\Gamma_a)_{\alpha\beta}\psi_\beta(x'_a,n'_i)$, transforms under the IRREP $(1,1)_{\mathrm{SO}(5)}$. X_a can be decomposed into an SO(4) scalar X_5 and a 4-vector $X_{1\sim 4}$ transforming as $(\frac{1}{2};\frac{1}{2})_{\mathrm{SO}(4)}$ (see Table S5 in SM [61]). S and X_5 is orthogonal to each other, since the former is an SO(5) invariant and the latter is a component of the 5 vector. According to group theoretical analysis, the SO(5) of IRREP $(4I-2n+1,1)_{\mathrm{SO}(5)}$ with $0\leq n\leq 2I$ can be decomposed into two branches of SO(4) IRREPs as $(4I-2n+1,1)_{\mathrm{SO}(5)}=\bigoplus_{j_1+j_2=2I-n}[(j_1,j_2)_{\mathrm{SO}(4)}\oplus(j_1+\frac{1}{2},j_2+\frac{1}{2})_{\mathrm{SO}(4)}]$. Hence, we have two sets of two-body states as

$$\Phi^{(4I-2n+1,1)}_{J,M_1;K,M_2}(x_a,n_i;x'_a,n'_i)=\mathcal{N}^{(4I-2n+1,1)}_{J,M_1;K,M_2}X_5S^{n-1}f^{4I-2n}_{J,M_1,K,M_2}(x_a,n_i;x'_a,n'_i),$$
$$\Phi^{(4I-2n+1,1)}_{J+\frac{1}{2},M_1;K+\frac{1}{2},M_2}(x_a,n_i;x'_a,n'_i)=\mathcal{N}^{(4I-2n+1,1)}_{J+\frac{1}{2},M_1;K+\frac{1}{2},M_2}C^{J+\frac{1}{2},M_1}_{\frac{1}{2},\alpha;J,m_1}C^{K+\frac{1}{2},M_2}_{\frac{1}{2},\beta;K,m_2}\Psi^{(1,1)}_{\frac{1}{2},\alpha;\frac{1}{2},\beta}S^{n-1}f^{4I-2n}_{J,m_1,K,m_2}(x_a,n_i;x'_a,n'_i), \tag{8}$$

in which $\Psi^{(1,1)}_{\frac{1}{2},\pm\frac{1}{2};\frac{1}{2},\pm\frac{1}{2}}=\frac{1}{\sqrt{2}}(X_1\pm iX_2)$, and $\Psi^{(1,1)}_{\frac{1}{2},\pm\frac{1}{2};\frac{1}{2},\mp\frac{1}{2}}=\frac{1}{\sqrt{2}}(X_3\pm iX_4)$. More generally, for $(n_1+n_2,n_1-n_2)_{\mathrm{SO}(5)}$, the two-body states in these IRREPs are constructed from the higher-rank SO(5) harmonics, as detailed in Sec. IV of SM [61].

To implement numerical calculations, we generalize the pseudopotential formalism [16,64] from S^2 to S^4. The pseudopotential, based on the two-body states constructed above, is represented by a series of projection operators as follows:

$$H=\sum_{n_1,n_2}V_{n_1,n_2}P_{n_1,n_2}, \tag{9}$$

where $P_{n_1,n_2}=\sum_\lambda|n_1,n_2,\lambda\rangle\langle n_1,n_2,\lambda|$ is the projection operator into the SO(5) IRREP $(n_1+n_2,n_1-n_2)_{\mathrm{SO}(5)}$ and λ denotes the SO(4) group quantum numbers associated with the state. The interaction strength V_{n_1,n_2} is $\langle n_1,n_2||V||n_1,n_2\rangle$, where $||$ means that the reduced matrix element only depends on the SO(5) IRREP.

We delve into the many-body microscopic wavefunctions. There are two different approaches to construct Laughlin wave functions, namely the determinant-type and Jastrow-type Laughlin wave functions, respectively [4,7,16]. It is noteworthy that the determinant type [Eq. (10) below] is not identical to the Jastrow type [Eq. (15) below]. The difference has been highlighted in the study of the FQHE states on the $\mathbb{CP}^2$ manifold [65,66]. Nevertheless, both types of wave functions coincide on S^2.

The determinant-type Laughlin wave function takes the form of the Slater determinant raised to the m th power,

$$\begin{aligned}&\Psi^m_{\mathrm{det}}(x_{1,a},n_{1,i};\ldots x_{N,a},n_{N,i})\\&=\Big[\varepsilon_{P_1\ldots P_N}f^{(2I,0)}_{A_1}(x_{P_1,a},n_{P_1,i})\\&\quad\times f^{(2I,0)}_{A_2}(x_{P_2,a},n_{P_2,i})\cdots\times f^{(2I,0)}_{A_N}(x_{P_N,a},n_{P_N,i})\Big]^m,\end{aligned} \tag{10}$$

where N is the particle number, A_h is the abbreviation of the SO(4) IRREP $(j_h,m_{h,1};k_h,m_{h,2})$ for the h th particle, and P is permutation of 1 to N. The filling of the state of Eq. (10) is $\nu=[d(2I,0)/d(2mI,0)]$, which approaches $1/m^3$ in the large-m limit.

We focus on the determinant Laughlin wave function with $m=3$ in the case of the smallest Hilbert space. The wave function of a full shell with $I=\frac{1}{2}$, i.e., $N=4$, is $\Psi_{\mathrm{det}}=\varepsilon_{P_1P_2P_3P_4}\Pi^4_{k=1}\psi_k(x_{P_k,a},n_{P_k,i})$, and the corresponding determinant Laughlin state is

$$\Psi^{(m=3)}_{\mathrm{det},N=4}=\left(\varepsilon_{P_1P_2P_3P_4}\prod_{j=1}^4\psi_j(x_{P_j,a},n_{P_j,i})\right)^3. \tag{11}$$

By counting the power of SO(5) spinors, the single-particle LLL orbitals involved in Eq. (11) correspond to the case of $mI=\frac{3}{2}$ with the degeneracy $d(3,0)=20$. The full-shell determinant state is an SO(5) invariant. Its power, the Laughlin determinant state, is likewise an SO(5) singlet. Upon exchanging two particles, the wave function acquires a phase factor $(-1)^3=-1$, ensuring the Fermi statistics. Hence, when expanding Eq. (11), the relative wave function between a pair of particles k and l only contains

$$(\mathbf{\Psi}(x_{k,a},n_{k,i})\mathcal{M}\mathbf{\Psi}(x_{l,a},n_{l,i}))^3, \tag{12}$$

where the antisymmetry matrix kernel $\mathcal{M}$ takes either $\mathcal{R}$ or $\mathcal{R}\Gamma^a$. Each antisymmetric combination between two particles is denoted as one contraction. For example, the structure of Eq. (12) exhibits 3 contractions.

Now we construct the pseudopotential Hamiltonian for the many-body wave function [Eq. (10)]. The pseudopotential Hamiltonian, constructed from two-body projection operators, is positive definite and annihilates the wave function [Eq. (10)]. Consequently, the annihilated wavefunction is a zero-energy ground state. For concreteness, the special case of Eq. (11) with $m=3$, $N=4$ is employed as an example. Consider two fermions in the LLL orbits with $I'=mI=\frac{3}{2}$. The two-body states satisfying the fermionic statistics are decomposed into SO(5) IRREPs of $(4I'-2r-1-s,2r+1-s)_{\mathrm{SO}(5)}$ with $0\le r<I'$ and $0\le s\le 2r+1$. According to the two-body wave functions given in Eq. (VI.7) in Sec. VI of SM [61], the contraction number equals $2r+1$. Hence, the IRREPs with $r=0$, $(4,0)_{\mathrm{SO}(5)}$ and $(5,1)_{\mathrm{SO}(5)}$, are orthogonal to the two-body sectors of the wave function in Eq. (11), as revealed by the contraction numbers. Consequently, these two pseudopotential channels annihilate the wave function, which is the zero-energy ground state. The above analysis can be easily generalize to an arbitary value of N at $m=3$. Then the pseudopotential Hamiltonian is constructed as

$$\begin{aligned}H=\sum_{1\le k<l\le N}\Big\{&V_{(4mI-1,1)}P_{4mI-1,1}(k,l)\\&+V_{(4mI-2,0)}P_{(4mI-2,0)}(k,l)\Big\},\end{aligned} \tag{13}$$

where $P_{m,n}(k,l)$ is the projection operator of two-body states of the k th and l th particles.

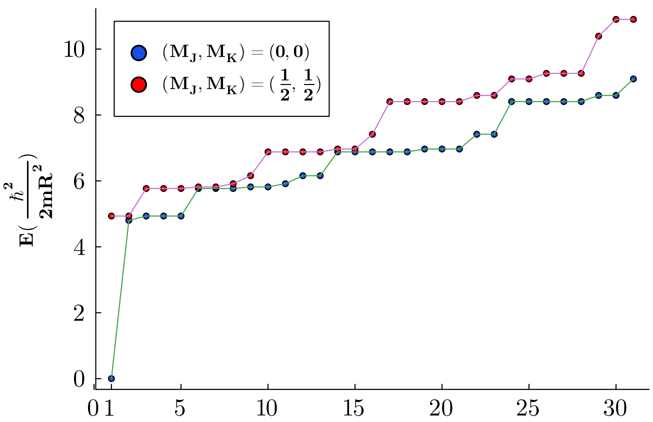

FIG. 1. The energy spectra are shown in blocks marked by the quantum numbers (M_J, M_K) for the Hamiltonian (13) with $N = 4$ particles occupying the LLL orbitals in the SO(5) IRREP $(3,0)_{\mathrm{SO(5)}}$. The two datasets with blue and red spots represent two branches of states with integer and half-integer SO(4) quantum numbers. The number of orbitals N_o is 20. The energy unit is taken as $\hbar^2/(2MR^2)$, and the horizontal axis shows the index of states in the ascending order in terms of energy. The zero energy ground state is an SO(5) singlet and unique. The lowest excitation is another SO(5) single state with a gap $\Delta = 4.8$.

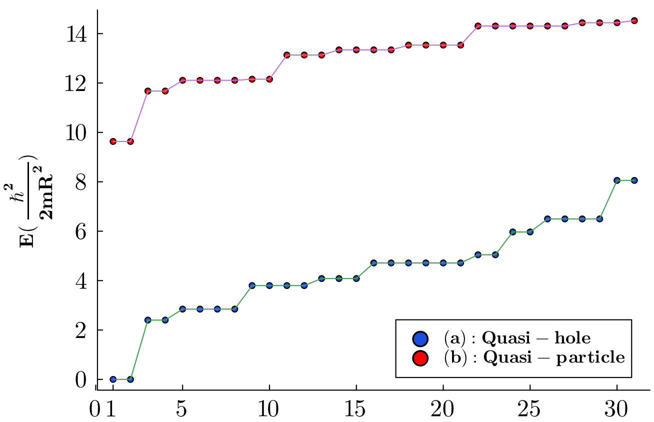

FIG. 2. The energy spectra of the same system as in Fig. 1: (a) quasihole states with $N = 3$, (b) quasiparticle states with $N = 5$. Only the block of $(\frac{1}{2}, 0)$ is represented. The ground states of the quasihole case remain at zero energy, while those of the quasiparticle case exhibit a finite gap. The ground states for both the quasihole and quasiparticle cases belong to the 20-dimensional SO(5) IRREP $(3,0)_{\mathrm{SO(5)}}$.

Next we present the exact digonalization (ED) results for the case of $m = 3$ and $N = 4$. The SO(5) is a rank-2 Lie group with two commutable generators, which can be taken as the sum over all particles of the diagonal operators of two inter-commutable SU(2) subgroups defined in Eq. (3): $J_3 = \sum_{l=1}^{N} J_3(l)$ and $K_3 = \sum_{l=1}^{N} K_3(l)$. The pseudopotential Hamiltonian is diagonalized in blocks labeled by different values of (M_J, M_K), which are J_3 and K_3 eigenvalues, respectively. Previously analysis shows that the wave function of Eq. (11) is an SO(5) singlet and ground state of the Hamiltonian (13). Diagonalizations are performed and the ground state is found nondegenerate, hence, the wave function of Eq. (11) is the unique ground state. Energy spectra in the blocks of (0,0) and $(\frac{1}{2}, \frac{1}{2})$ are shown in Fig. 1. The energies of the lowest and next lowest excited states are very close at 4.80 and 4.93 in the unit of $(\hbar^2/2MR^2)$. The lowest excited state is also an SO(5) scalar and the next lowest excited states are 35-fold degenerate, forming the SO(5) IRREP of $(4,0)_{\mathrm{SO(5)}}$. Such an SO(5) IRREP is decomposed into different SO (4) IRREPs carrying integer quantum numbers of $(2,0)_{\mathrm{SO(4)}} \oplus (1,1)_{\mathrm{SO(4)}} \oplus (0,2)_{\mathrm{SO(4)}}$ with dimensions of 5, 9, and 5, respectively, and those carrying half-integer quantum numbers of $(\frac{3}{2}, \frac{1}{2})_{\mathrm{SO(4)}} \oplus (\frac{1}{2}, \frac{3}{2})_{\mathrm{SO(4)}}$, both of dimension 8, as shown in Table S9 of SM [61].

We now present the spectra of excitations of the quasihole ($N = 3$) and quasiparticle ($N = 5$) sectors as shown in Figs. 2(a) and 2(b), respectively. Since each SO(5) IRREP contains SO(4) IRREPS $(j_1, j_2)_{\mathrm{SO(4)}} \oplus (j_2, j_1)_{\mathrm{SO(4)}}$ symmetrically, we only present the state in the sector $(\frac{1}{2}, 0)$ in terms of (M_J, M_K) as a representative. As for the quasihole states of $N = 3$, its ground states remain at zero energy with the 20-fold degeneracy, lying in the SO(5) IRREP of $(3,0)_{\mathrm{SO(5)}}$, which is decomposed to the SO(4) IRREPs of $(\frac{3}{2}, 0)_{\mathrm{SO(4)}} \oplus (0, \frac{3}{2})_{\mathrm{SO(4)}} \oplus (1, \frac{1}{2})_{\mathrm{SO(4)}} \oplus (\frac{1}{2}, 1)_{\mathrm{SO(4)}}$. In contrast, as for the quasiparticle states with $N = 5$, there exist a finite excitation gap. The ground states form the SO(5) IRREP of $(3,0)_{\mathrm{SO(5)}}$, which is the same as the quasihole case.

Early studies of FQHE debated whether the ground state should be an incompressible quantum liquid or a crystalline phase such as a Wigner crystal [67–71]. The pair distribution function is frequently employed to reveal the incompressible nature of quantum liquids [64]. We calculate the probability $h(\theta)$ of two particles separated by the chord distance, which depends only on θ due to the rotation symmetry. It is defined as

$$h(\theta) = \frac{\int d^3n_1 d^3n_2 \langle \Psi_0 | \rho(x_a, n_1) \rho(x'_a, n_2) | \Psi_0 \rangle}{\rho(x_a)\rho(x'_a)} - 1, \quad (14)$$

where $|\Psi_0\rangle$ is the ground state of Hamiltonian (13), $\cos\theta = x_a x'_a$; $\rho(x_a, n_1)$ is the density operator, and $\rho(x_a) = \int d^3n_1 \langle \Psi_0 | \rho(x_a, n_1) | \Psi_0 \rangle$. The calculated behavior of $h(\theta)$ is shown in Fig. 3, which is analogous to the case of the FQHE on the S^2 sphere [72,73]. As θ goes to 0, $h(\theta)$ approaches -1 due to the Fermi statistics. Its amplitude features a smooth decay to zero as $\theta \to \pi$ without exhibiting pronounced peaks. This result implies the absence of crystalline order, and is consistent with the quantum-liquid-like ground state in 4D.

Now we discuss the construction of the Jastrow-type Laughlin wave function on S^4 based on the SO(5) singlet

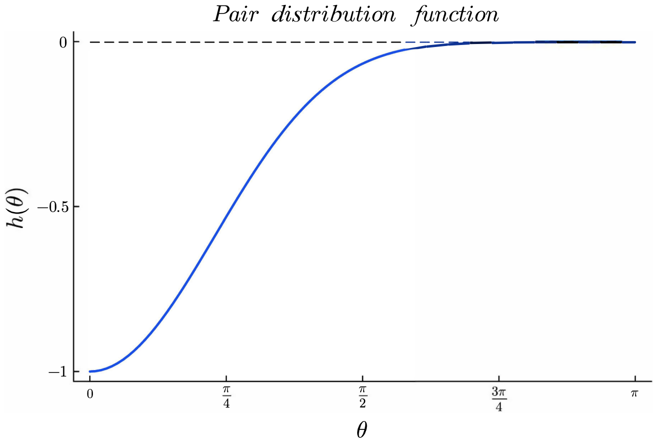

FIG. 3. The pair distribution function of the 4D determinant-type Laughlin wave function lacks a peak at long distance, indicating liquidlike behavior.

$\psi_\alpha(x_1, n_1)\mathcal{R}_{\alpha\beta}\psi_\beta(x_2, n_2)$, which is a generalization of the SU(2) singlet function $u_i v_j - u_j v_i$ [16]. The explicit wave function is expressed with the assistance of $\mathcal{R}$,

$$\Psi^m_{\text{Jastrow}} = \prod_{1\le k<l\le N} (\psi_\alpha(x_{k,a}, n_{k,i})\mathcal{R}_{\alpha\beta}\psi_\beta(x_{l,a}, n_{l,i}))^m. \tag{15}$$

The power of the spinor ψ of each particle appearing in Eq. (15) is equal to $l = m(N-1)$, hence, the filling should be $\nu = N/d(l, 0)$ with $d(l, 0)$ defined in Eq. (2) approaching $1/(m^3 N^2) \to 0$ as $N \to \infty$. The parent Hamiltonian to this wave function is constrained in the channels of $\bigoplus_{n_1+n_2\ge l-m+1} (n_1 + n_2, n_1 - n_2)_{\mathrm{SO}(5)}$. There is no well-defined thermodynamic limit for the Jastrow-type wave function.

In conclusion, we have studied the 4D incompressible quantum liquid based on the Landau level under the SU(2) Yang monopole, specifically using the determinant-type and Jastrow-type Laughlin wave functions. The parent Hamiltonian for these wave functions was constructed within the pseudopotential formalism. The ED calculations show that the unique ground state is an SO(5) singlet. In addition, quasihole states remain at zero energy, while quasiparticle excitations are separated by a finite gap. The liquidlike behavior is further supported by the pair distribution function. However, a direct numerical analysis of both determinant-type Laughlin and Jastrow wave functions for larger particle numbers is prohibitive due to the enormous size of the Hilbert space. For the next accessible system size, N equals 10 corresponding to the IRREP of $(2, 0)_{\mathrm{SO}(5)}$, and the LLL orbitals at $m = 3$ correspond to the IRREP of $(6, 0)_{\mathrm{SO}(5)}$ which is 84 dimensional, then the Hilbert space dimension already exceeds 10^{11}. Nonetheless, the recent experimental demonstration of the few-electron Coulomb liquid [74–77] provides a promising platform for the experimental validation of our theoretical framework. Another intriguing avenue is to explore the statistical properties of excitations in the 4D fractional topological states, particularly investigating the loop or membrane statistics in higher dimensions [78–85]. Furthermore, studying the entanglement spectra [86–88] offers valuable insights into these states and warrants further investigations.

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Data availability—The data that support the findings of this article are openly available [89], embargo periods may apply.

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End Matter

Yang monopole—Consider a quantum mechanical particle with an electric charge e is constrained to S^2 enclosing a U(1) Dirac monopole with the monopole charge g. The Dirac quantization condition leads to $eg/c = n\hbar/2$, where n is equivalent to the first Chern number [90]. It gives to wave functions described by two spinors: $u = \cos(\theta/2)e^{-(\mathrm{i}\phi/2)}$ and $v = \cos(\theta/2)e^{(\mathrm{i}\phi/2)}$. When the S^2 sphere is replaced by S^4, the U(1) monopole should be augmented by the SU(2) one. Correspondingly, the Dirac quantization condition is also generalized [15], which is equivalent to the second Chern class:

$$\mathcal{C}_2 = \frac{1}{8\pi^2}\int_{S^4} \mathrm{Tr}(F \wedge F). \tag{A1}$$

Here, $\mathcal{C}_2$ is the 2nd Chern number, and $F = \frac{1}{2}\eta^a_{\mu\nu} I_a e^\mu \wedge e^\nu$, which in the vielbein is the curvature of the Yang monopole. The SU(2) generators I_a are in the spin-I representation. $\mathcal{C}_2$ can be evaluated as

$$\mathcal{C}_2 = \frac{1}{32\pi^2}\mathrm{Tr}(I_a I_b)\int_{S^4} \eta^a_{\mu\nu}\eta^b_{\rho\sigma} e^\mu \wedge e^\nu \wedge e^\rho \wedge e^\sigma = \frac{2I(2I+1)(2I+2)}{6}. \tag{A2}$$

We use the identity of ’t Hooft symbols: $\eta^a_{\mu\nu}\eta^a_{\rho\sigma} = \delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho} + \varepsilon_{\mu\nu\rho\sigma}$ [57]. The volume of S^4 is defined as $\mathrm{Vol}(S^4) = \int_{S^4}(1/4!)\varepsilon_{\mu\nu\rho\sigma} e^\mu \wedge e^\nu \wedge e^\rho \wedge e^\sigma$. Similar to the case on S^2, wave functions of 4D QHE require description by four Hopf spinor components as shown in Table I, accounting for the half spin carried by the SU(2) gauge field.

so(5) Lie algebra—We define the mechanical angular momentum $\Lambda_{\mu\nu} = -\mathrm{i}(x_\mu D_\nu - x_\nu D_\mu)$, where $D_\mu = \partial_\mu - \mathrm{i}A_\mu$. The curvature $F_{\mu\nu}$ is $F_{\mu\nu} = \mathrm{i}[D_\mu, D_\nu]$. Now examine the commutation relation of $\Lambda_{\mu\nu}$:

$$[\Lambda_{\mu\nu}, \Lambda_{\rho\sigma}] = \mathrm{i}(\Lambda_{\mu\rho}\delta_{\nu\sigma} - \Lambda_{\mu\sigma}\delta_{\nu\rho} - \Lambda_{\nu\rho}\delta_{\mu\sigma} + \Lambda_{\nu\sigma}\delta_{\mu\rho}) + \mathrm{i}(x_\mu x_\rho F_{\nu\sigma} - x_\mu x_\sigma F_{\nu\rho} - x_\nu x_\rho F_{\mu\sigma} + x_\nu x_\sigma F_{\mu\rho}). \tag{B1}$$

TABLE I. Comparison between S^2 and S^4 Quantum Hall Systems.

Manifold	S^2	S^4
Gauge field	U(1)	SU(2)
Quantization	$(1/2\pi)\int_{S^2} F \in \mathbb{Z}$	$(1/8\pi^2)\int_{S^4} \mathrm{Tr}(F \wedge F) \in \mathbb{Z}$
Symmetry	SO(3)	SO(5)
Generator	$\vec{L} = \vec{r} \times \vec{\pi} - (eg/c)\hat{r}$	$L_{\mu\nu} = \Lambda_{\mu\nu} - R^2 F_{\mu\nu}$
Algebra	$[\hat{L}_i, \hat{L}_j] = \mathrm{i}\varepsilon_{ijk}\hat{L}_k$	$[L_{\mu\nu}, L_{\rho\sigma}] = \mathrm{i}(L_{\mu\rho}\delta_{\nu\sigma} - L_{\mu\sigma}\delta_{\nu\rho} - L_{\nu\rho}\delta_{\mu\sigma} + L_{\nu\sigma}\delta_{\mu\rho})$
Hamiltonian	$H = [L^2 - (eg/c)^2)]/2mR^2$	$H = [\sum_{\mu<\nu}(L_{\mu\nu}^2 - F_{\mu\nu}^2)/2mR^2]$
Fundamental spinor	1st Hopf spinor [16]	2nd Hopf spinor [7,61]

The mechanical angular momentum does not satisfy the so (5) algebraic structure. We introduce the canonical angular momentum $L_{\mu\nu} = \Lambda_{\mu\nu} - R^2 F_{\mu\nu}$, which obeys the so (5) algebra:

$$[L_{\mu\nu}, L_{\rho\sigma}] = \mathrm{i}(L_{\mu\rho}\delta_{\nu\sigma} - L_{\mu\sigma}\delta_{\nu\rho} - L_{\nu\rho}\delta_{\mu\sigma} + L_{\nu\sigma}\delta_{\mu\rho}). \quad \text{(B2)}$$

We note that $\sum_{\mu<\nu}\Lambda_{\mu\nu}^2 = R^2\pi^2 - (\vec{R}\cdot\vec{\pi})^2 + 3\mathrm{i}(\vec{R}\cdot\vec{\pi})$, where $\pi_\mu = -\mathrm{i}D_\mu$. The particle motion is restricted to the sphere S^4 of radius R, so $\vec{R}\cdot\vec{\pi} = 0$. Hence, the Hamiltonian is expressed in terms of $L_{\mu\nu}$,

$$H = \frac{1}{2mR^2}\vec{\pi}^2 = \frac{1}{2mR^2}\sum_{\mu<\nu}\Lambda_{\mu\nu}^2 = \frac{1}{2mR^2}\sum_{\mu<\nu}(L_{\mu\nu}^2 - F_{\mu\nu}^2). \quad \text{(B3)}$$

For the SU(2) gauge field components defined as $F_{\mu\nu} = \eta_{\mu\nu}^a I_a$, the sum of the squared field strength yields $\sum_{\mu<\nu} F_{\mu\nu}^2 = 2I(I+1)$. This result follows from the contraction $\eta_{\mu\nu}^a \eta_{\mu\nu}^b = 4\delta^{ab}$. The eigenvalues of $\sum_{\mu<\nu} L_{\mu\nu}^2$ are given by the SO(5) Casimir $\mathcal{C}(N+2I, N)$ [59].

Experimental aspect—Topological phases and the associated integer quantum Hall effects in four dimensions have been realized in various platforms, including cold atom systems, optical lattices, acoustic lattices, and electric circuits [26–37]. The internal degrees of freedom of atoms, typically the hyperfine spin components, are often employed as synthetic dimensions to simulate high dimensional systems, though the lengths of synthetic dimensions are finite [26,27]. Another method is via the 2D Thouless pumping, where a 2D system modulated by two additional adiabatic parameters (ϕ_x, ϕ_y) is topologically equivalent to a 4D integer topological system [91].

Strong repulsive interactions are necessary to achieve fractional topological states. In ultracold atom systems, Feshbach resonances are typically employed to tune interatomic interactions [92]. Significant progress has been achieved in realizing interaction effects in Thouless pumping processes [93–96]. Alternatively, the plasmonium photon box array [76] offers a compelling photonic platform, in which preliminary fractional quantum Hall signatures have been observed. The key mechanism is the blockade effect derived from the substantial anharmonicity of plasmonium modes [97]. This effect prohibits double occupancy on a single site, thereby naturally generating the strong repulsive interactions essential for fractional quantum Hall states.

Hence, it is desired to integrate synthetic dimensions with interaction control (e.g., Feshbach resonances) to experimentally explore fractional topological states in four dimensions. In principle, Feshbach resonances could be designed between a series of carefully chosen pairs of internal states, for example, between every two neighboring sites along synthetic dimensions. In this case, the interaction would be short-ranged not only in three-dimensional real space but also in synthetic dimensions with internal states.

Certainly, designing feasible proposals to engineer strong interactions in desired forms to realize four-dimensional fractional topological states would be highly nontrivial, which is certainly beyond the scope of this Letter. It would naturally be a potential focus for ultracold atom studies to bridge the gap between high-dimensional fractional topological states and realistic experimental platforms. Our Letter serves a starting point to motivate and stimulate studies both experimental and theoretical. While the current Letter focuses on characterizing the ground state, future investigations into the dynamical properties and transport signatures of 4D quantum incompressible fluids are also desired. We hope our findings serve as a timely guidance to these ongoing efforts in quantum simulations.