Symmetry and Correlation Aspects of Quantum Dynamics

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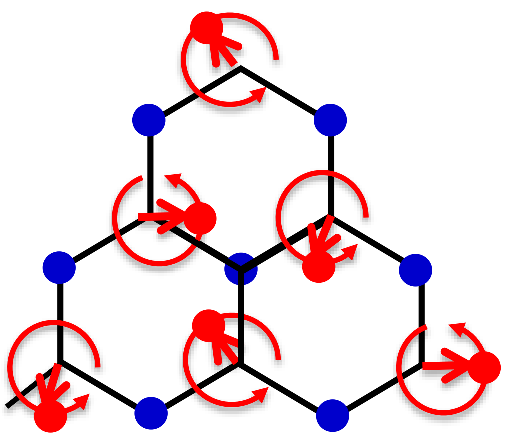

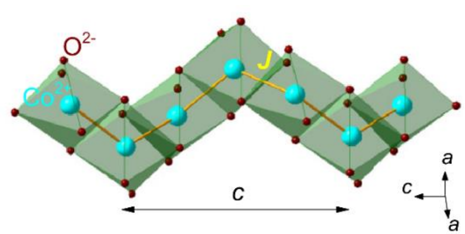

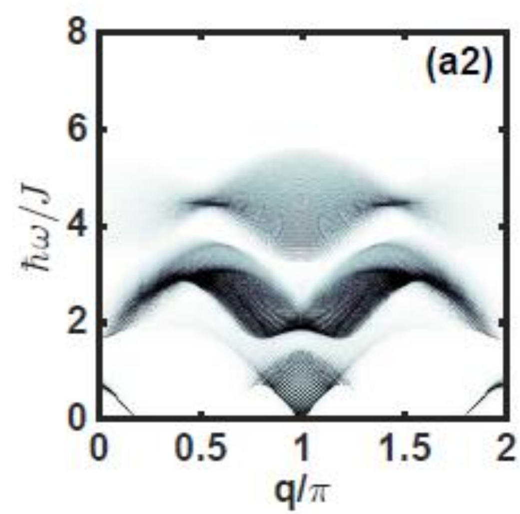

05/28/2019, Univ. of Chicago

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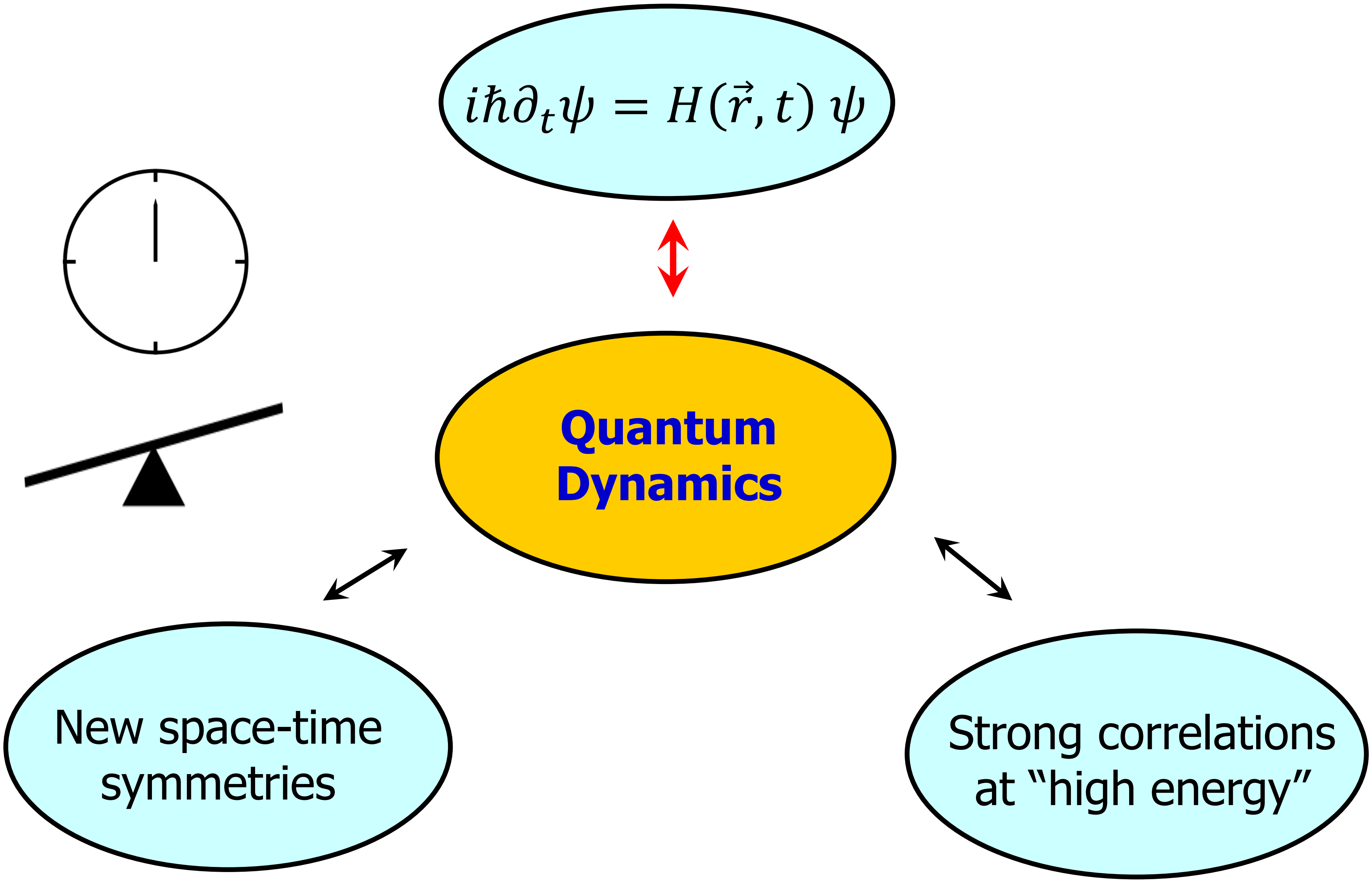
A. Loidl's group (Univ. of Augsburg, Germany)

Supported by AFOSR

Refs.

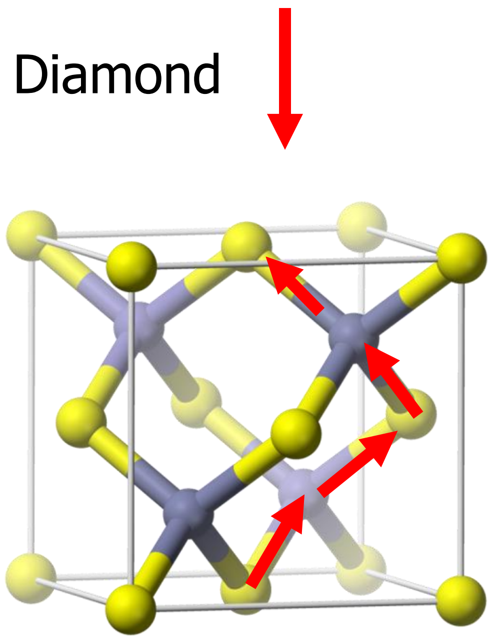
1. Shenglong Xu and Congjun Wu, Phys. Rev. Lett. 120, 096401 (2018) .
2. Wang Yang, Jianda Wu, Shenglong Xu, Zhe Wang, Congjun Wu arXiv:1702.01854.
3. Z. Wang, J. Wu, W. Yang, A. K. Bera, D. Kamenskyi, A.T.M. N. Islam, S. Xu, J. M. Law, B. Lake, C. Wu, A. Loidl, Nature 554, 219 (2018).

Introduction



Crystal – a fundamental of condensed matter

Diamond



- 230 space groups – Fedorov, Schönflies (1891)

Crystal system: **Cubic**

Bravais lattice: **FCC** (face-centered cubic)

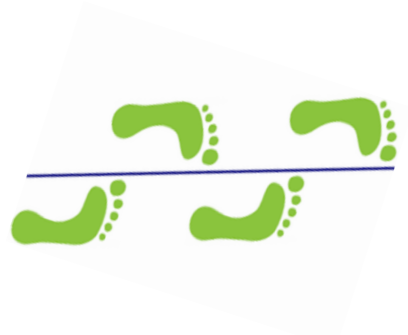
Point group: T_d or $\bar{4}3m$

Space group: O_h^7 or $Fd\bar{3}m$

Non-symmorphic symmetries:

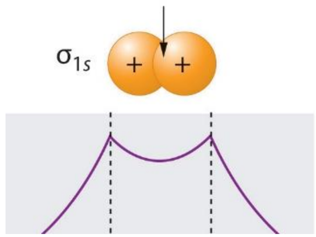
screw rotation

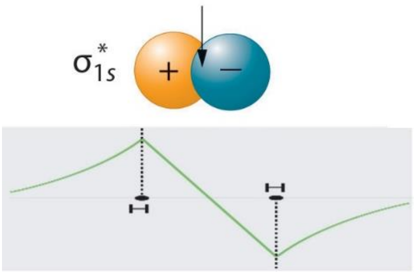
glide reflection



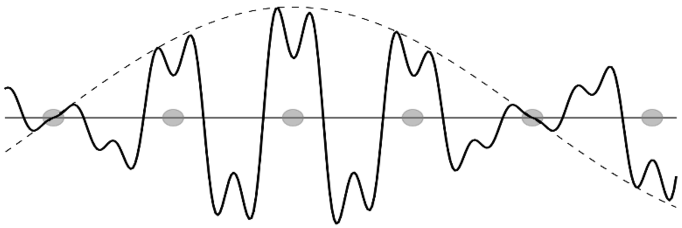
Bloch Theorem (1928)

- Chemical bond (small molecule):

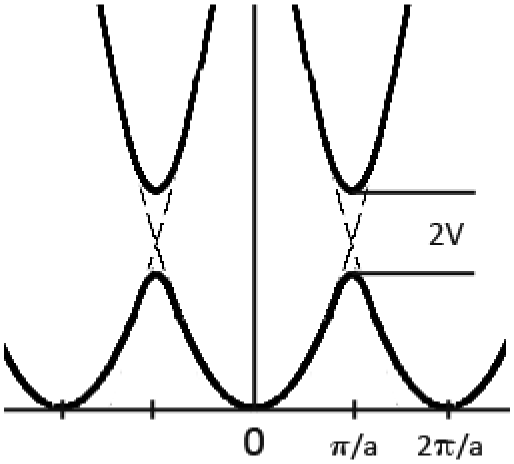




- Bloch band (large crystal)



$$\psi_{k,m}(\vec{r}) = e^{i\vec{k}\cdot\vec{r}} u_m(\vec{r})$$



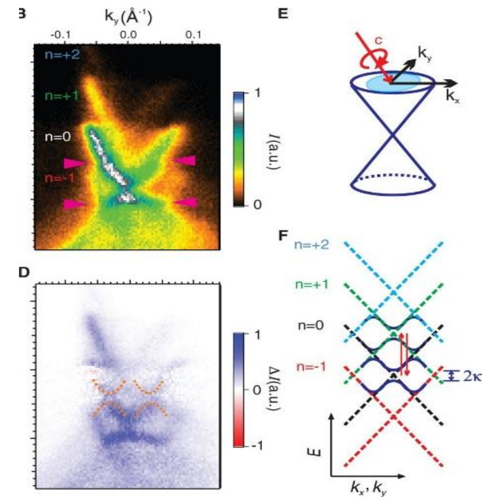
- Origin of (band) insulator is quantum: gap due to the interference of matter wave!

Dynamics under periodic driving

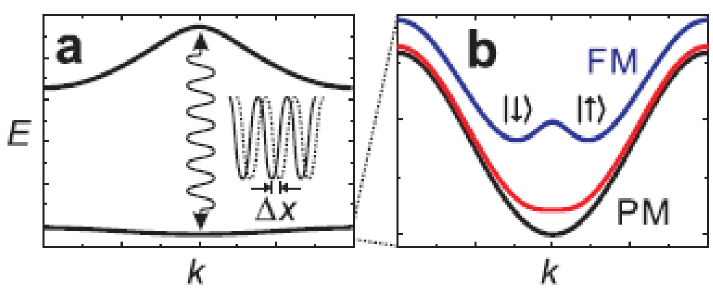
- Floquet Theorem (1883)

$$H(t) = H(t+T) \qquad \Omega = 2\pi/T$$

$$\psi_\omega(t) = e^{-i\omega t} u(t) = e^{-i\omega t} \sum_n a_n e^{-in\Omega T}$$



- Explore many-body physics via Floquet engineering



"ferro"-magnetic domain formation, universal scaling across quantum phase transition

Chin's group, Nature Physics 9, 769-774 (2013), Science 354, 606-610 (2016)

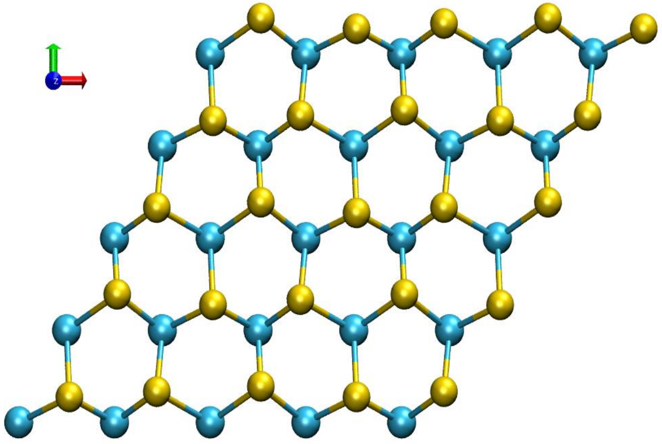
Floquet framework is NOT generic

- Temporal and spatial symmetries decoupled.

  c.f. A 3D crystal is not just a 2D crystal (ab-plane) direct product with a 1D crystal (c-axis)

- Dynamic crystal $\neq$ space crystal $\otimes$ Floquet periodicity!

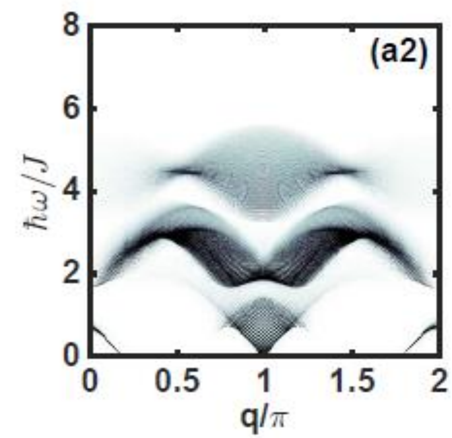
- A general framework for space-time coupled symmetries – space-time group!



Introduction

$$i\hbar\partial_t\psi = H(\vec{r},t)\,\psi$$

Quantum Dynamics

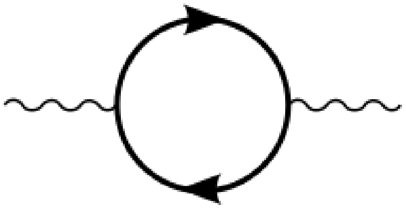


New space-time symmetries

Strong correlations at “high energy”

Strong correlation physics

- Central theme: spectra functions based on the Kubo formula.



1. Emphasis on **low energy** physics
2. **Imaginary** (Matsubara) frequency
3. **Imaginary** time evolution – quantum Monte Carlo

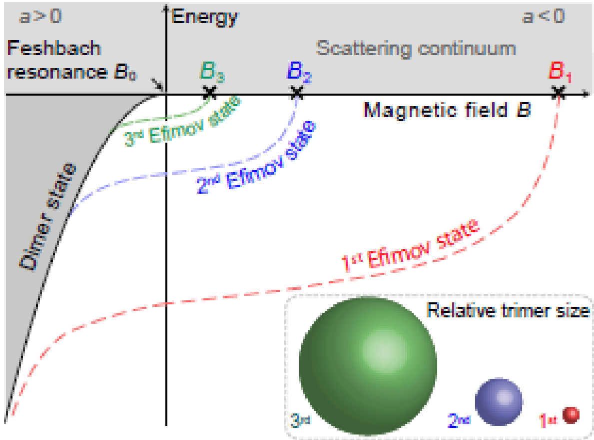
- How can integrable models help?

High **real-frequency** spectra beyond effective low energy theory.

Multi-particle (anti)-bound states

- Resonance states in high energy physics.

- Efimov states in nuclear physics and in cold atom physics.

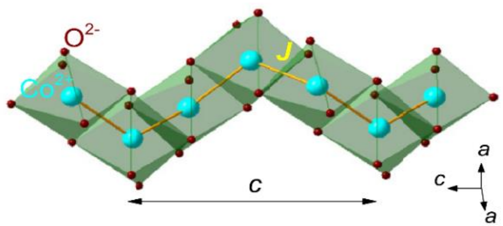


Chin's group, Phys. Rev. Lett. 113, 240402 (2014)

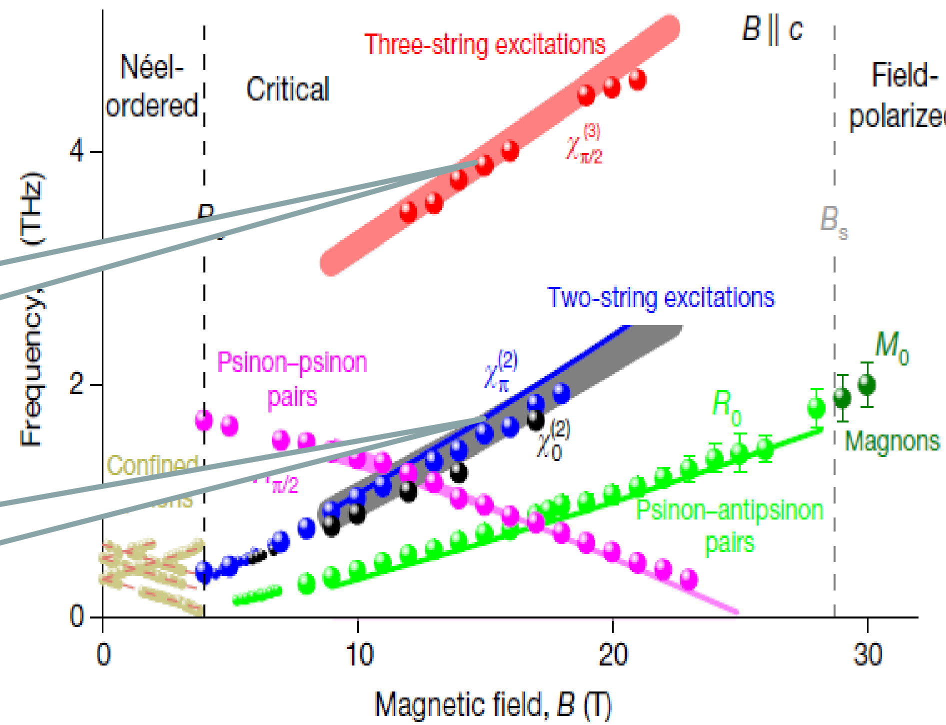
Spin dynamics in antiferromagnet

- Bethe string states (magnon anti-bond states)

$SrCo_2V_2O_8$



Electron spin resonance (ESR)



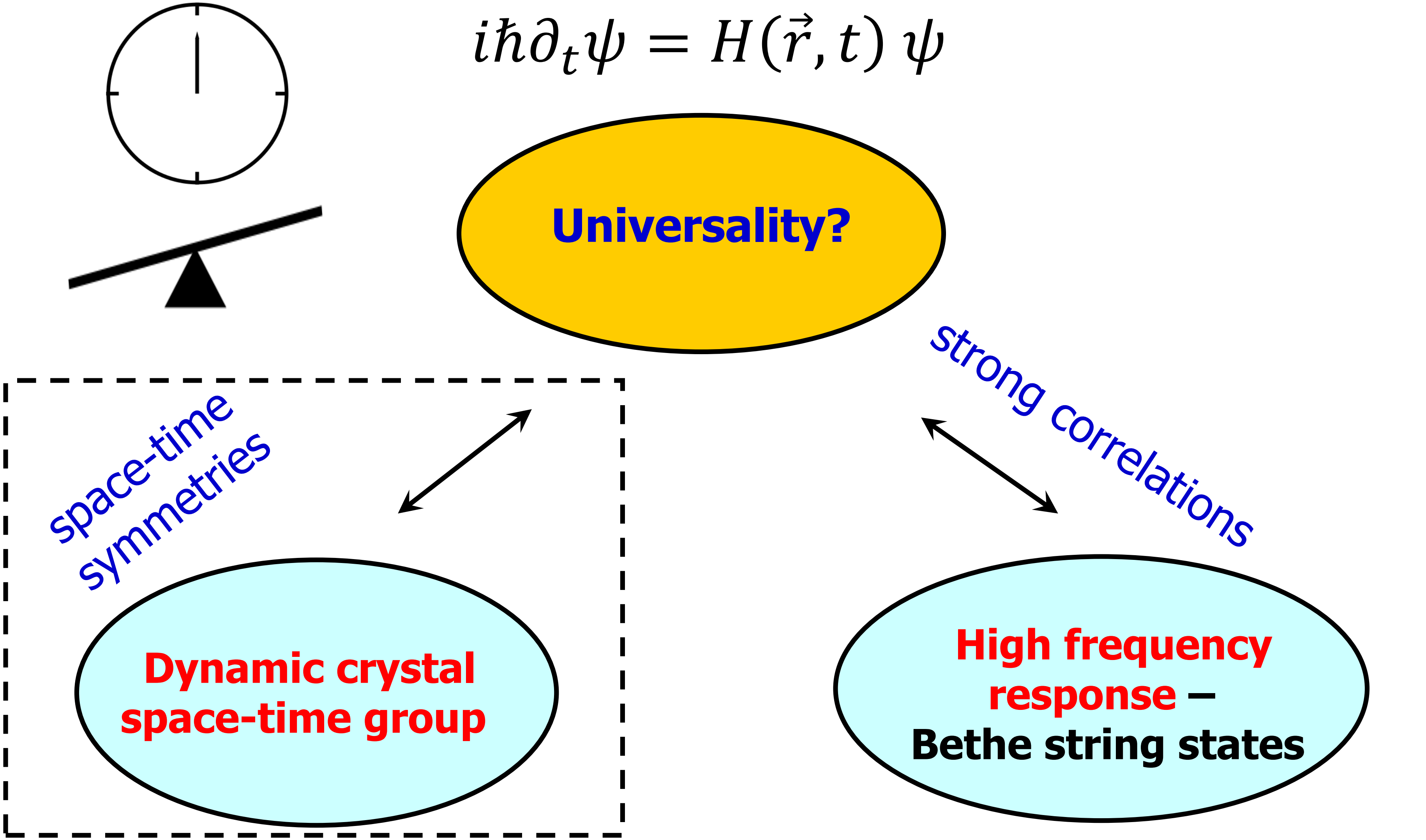
3 (body)- string states

2 (body) - string states

Loidl's group, Wu's group, et al, *Nature 554, 219 (2018).*

Quantum dynamic systems

$$i\hbar\partial_t\psi = H(\vec{r},t)\,\psi$$



S. L. Xu and C. Wu, Phys. Rev. Lett. 120, 096401 (2018) .

Dynamic "crystal" – space-time symmetries

- Space-time unit cell ≠ space domain ⊗ time domain.

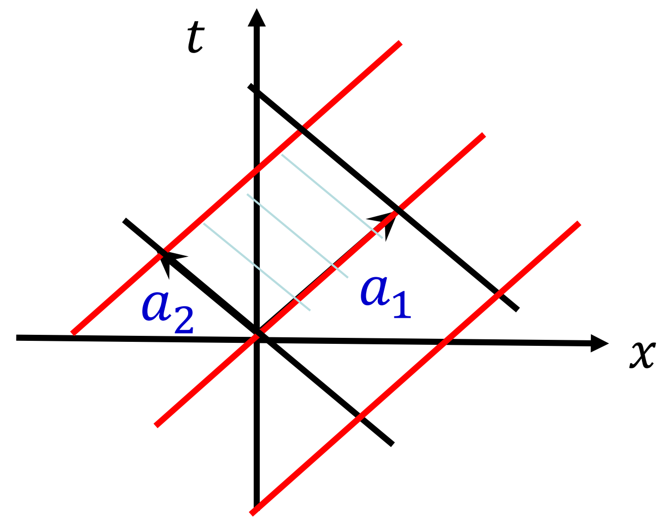
temporal periodicity unnecessary

$$V(0,t) = \cos\omega_1 t + \cos\omega_2 t$$

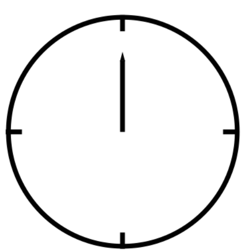
spacial periodicity unnecessary

$$V(x,0) = \cos k_1 x + \cos k_2 x$$

$$V(x,t) = \cos(k_1 x - \omega_1 t) + \cos(k_2 x - \omega_2 t)$$



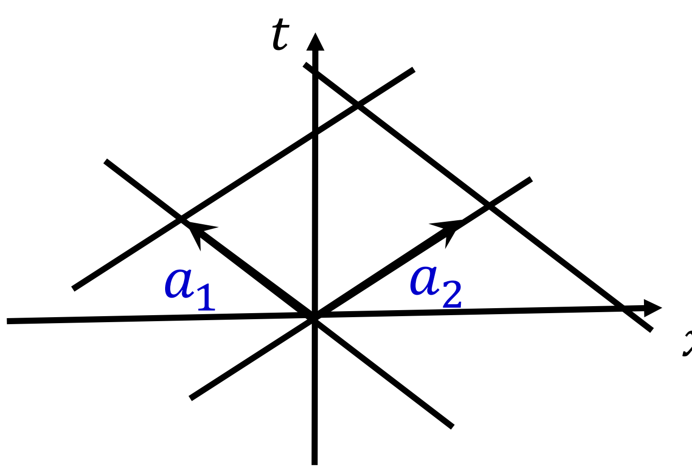
- New framework → space-time group.

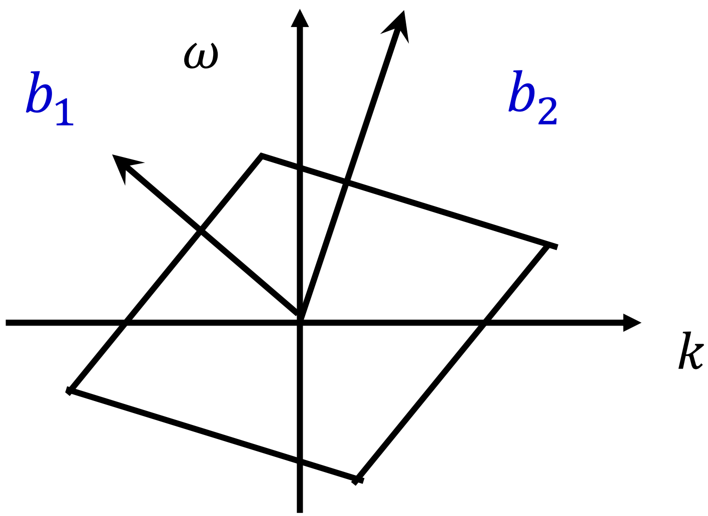


S. L. Xu and C. Wu, Phys. Rev. Lett. 120, 096401 (2018) .

Reciprocal lattice (momentum-energy)

$$V(\vec{r},t)=V(\vec{r}+\vec{u}_i,t+\tau_i),\quad i=1\ldots,D+1$$





$$a_i=(\vec{u}_i,\tau_i)$$

$$b_i=(\vec{G}_i,\Omega_i), i=1,2,\ldots,D+1$$

$$b_i\cdot a_j=\vec{G}_i\cdot\vec{u}_j-\Omega_i\tau_j=2\pi\delta_{ij}$$

- Time quasi-crystal with D+1 frequencies (beyond Floquet).

The generalized Bloch-Floquet theorem

$$i\hbar\partial_t\psi(\vec{r},t) = \left(-\frac{\hbar^2}{2m}\nabla^2 + V(\vec{r},t)\right)\psi(\vec{r},t)$$

$$\psi_{\kappa,m}(\vec{r},t) = e^{i(\vec{k}\cdot\vec{r}-\omega t)}u_m(\vec{r},t)$$

$\kappa = \left(\vec{k},\omega\right)$: the (lattice) momentum-energy vector (mod B)

$u_m(\vec{r},t)$: the same space-time periodicity of $V(\vec{r},t)$

S. L. Xu and CW, Phys. Rev. Lett. 120, 096401 (2018)

$$u_m(\vec{r},t) = \sum_B c_{m,B}\, e^{i(\vec{G}\cdot\vec{r}-\Omega t)}$$

$$\sum_{B'}\{[-\Omega + \epsilon_0(k+G)]\delta_{B,B'} + V_{B-B'}\}\; c_{m,B'} = \omega_m\, c_{m,B}$$

$B = \left(\vec{G},\Omega\right)$ take all D+1 dim. reciprocal lattice vectors

"Space-time" group

$$(\vec{r}, t) \xrightarrow{\Gamma} (R\vec{r} + \vec{u}, st + \tau)$$

point group — space group — $s = \pm 1$ magnetic group — space-time group

Representations:

- $M_\Gamma \psi_\kappa = \psi_\kappa(\Gamma^{-1}(\vec{r}, t))$ for s=1
- $M_\Gamma \psi_\kappa = \psi_\kappa^* (\Gamma^{-1}(\vec{r}, t))$ for s=-1 (anti-unitary)

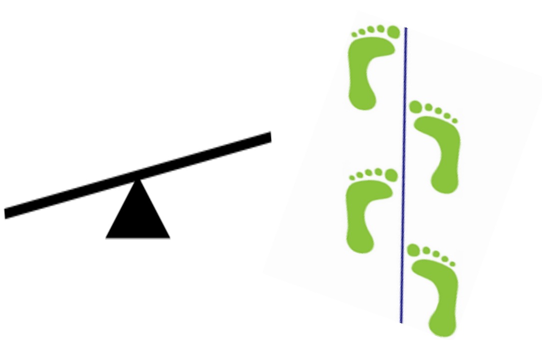
"Space-time" non-symmorphic symm.

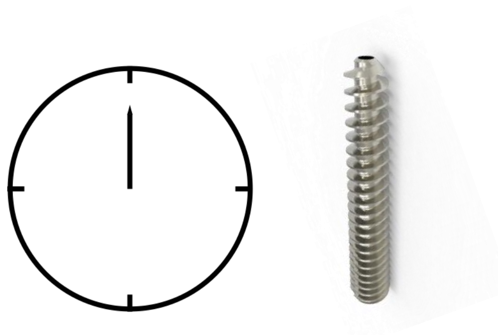
- If τ itself is not a symmetry → space-time nonsymmorphic symm.

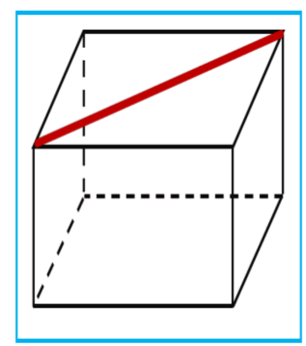
1+1D: time-glide reflection ($\det R = -1$).

2+1D: time-screw rotation ($\det R = 1$)

3+1D: time-screw rotary reflection ($\det R = -1$).



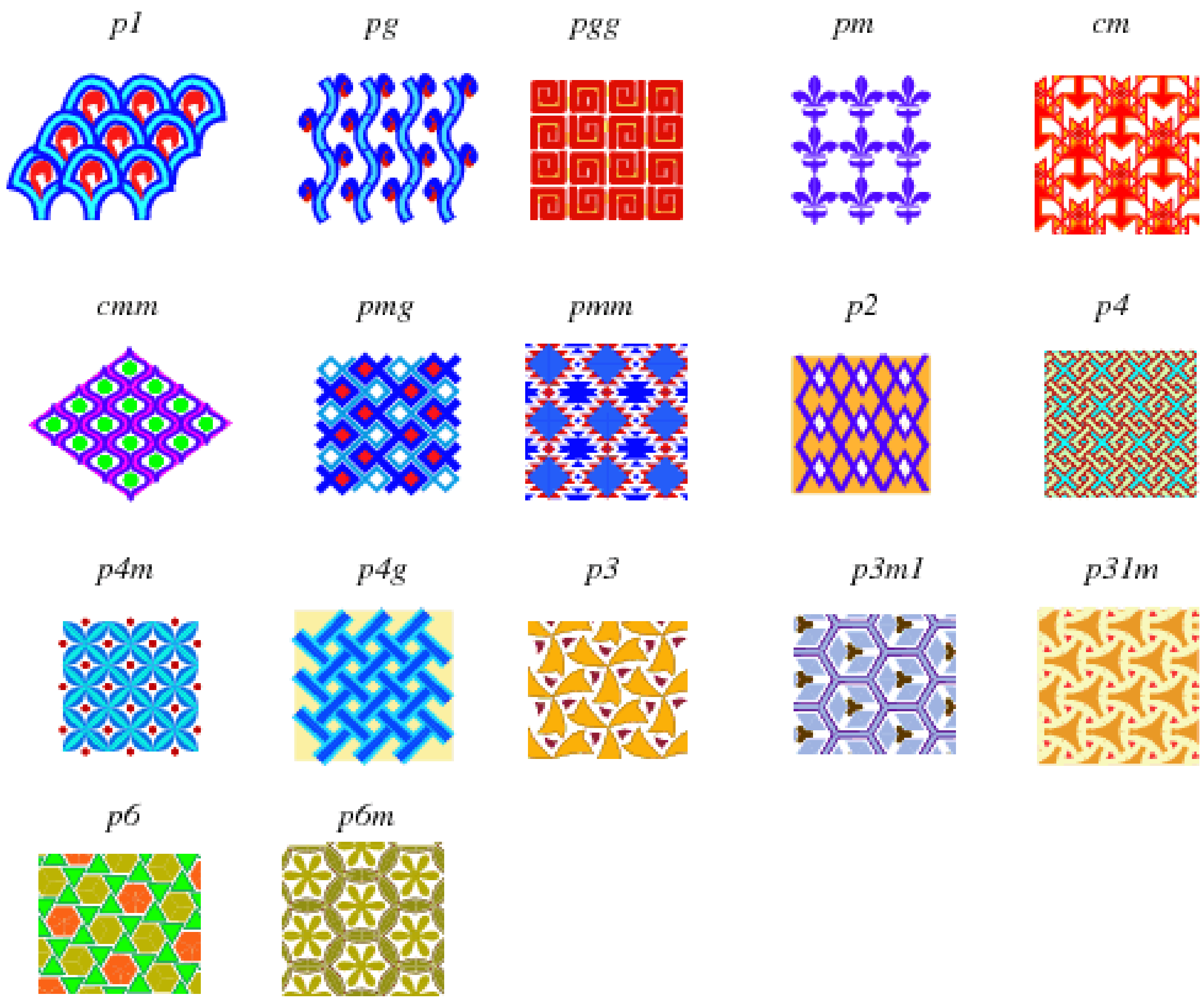




S. L. Xu and C. Wu, Phys. Rev. Lett. 120, 096401 (2018) .

T. Morimoto, et al, PRB (2017)

c.f. 17 wallpaper groups in 2D



1+1 D space-time group

Only 2-fold axis allowed.

3,4,6-fold ones are not.

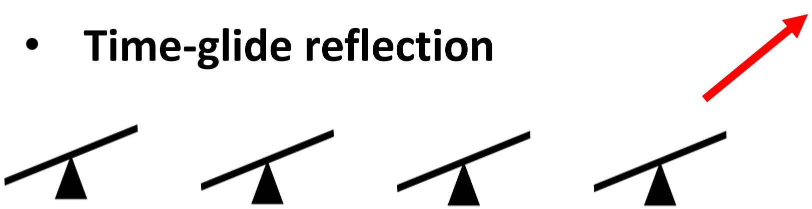
- **Reflection**

$$m_x: (x, t) \to (-x, t)$$

- **Time-reversal**

$$m_t: (x, t) \to (x, -t)$$

- **Time-glide reflection**

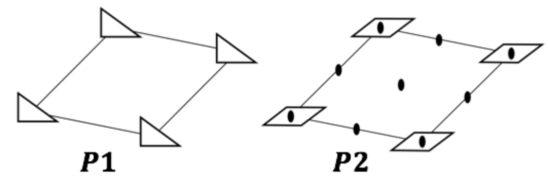


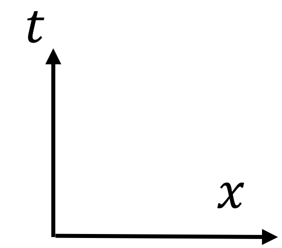
$$g_x: (x, t) \to \left(-x, t + \frac{T}{2}\right)$$

- **glide time-reversal**

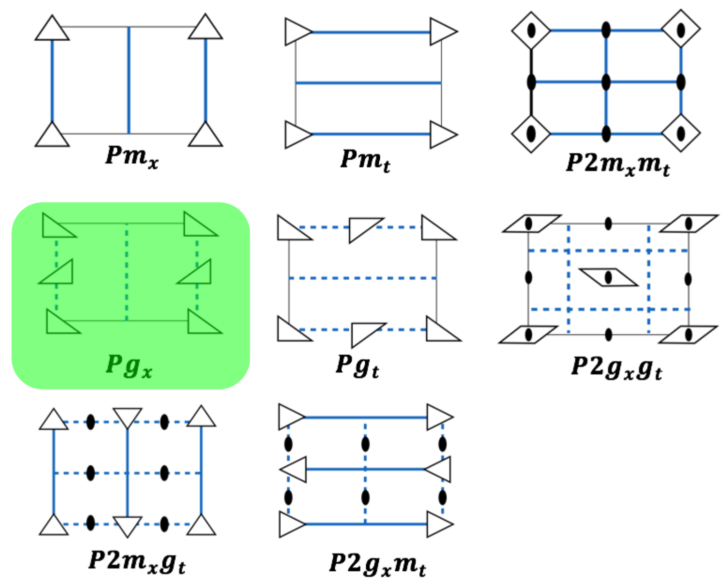
$$g_t: (x, t) \to \left(x + \frac{a}{2}, -t\right)$$

Oblique:

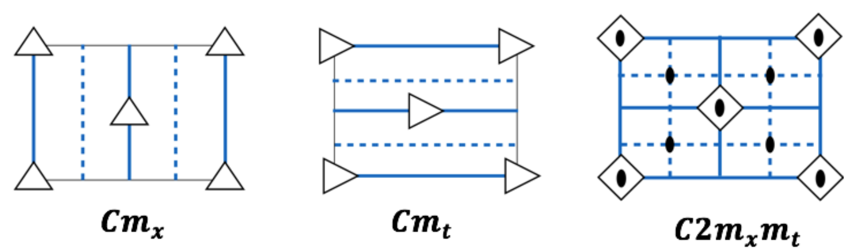




Orthorhombic :

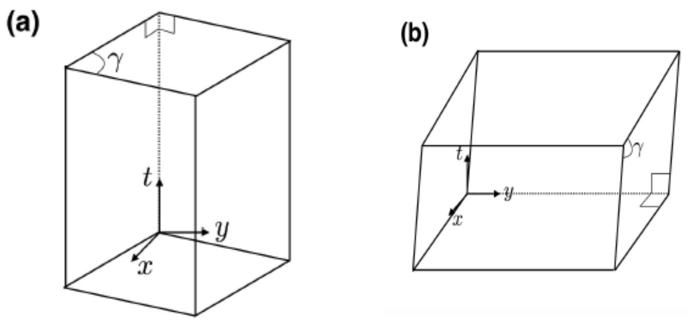


Centered orthorhombic:



2+1 D space-time group

- No cubic crystal system.
- Two different monoclinic crystal systems.

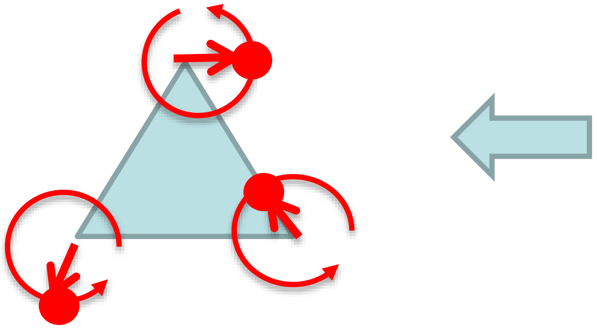


- Classification: 275 space-time groups in 2+1 D

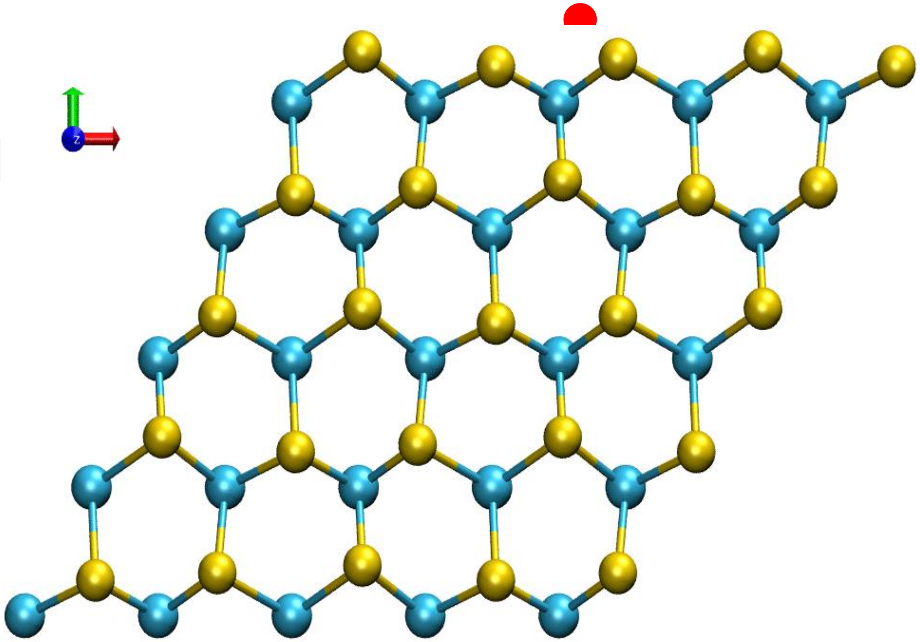
Crystal System	MP Group	Bravais Lattice	$G(2,1)$
Triclinic	$1, 2'$	Primitive	2
T-Monoclinic	$11', 2, 21'$	Primitive	8
		Centered	5
R-Monoclinic	$m, m', m'm2'$	Primitive	8
		Centered	5
Orthorhombic	$mm2, m'm'2$ $mm21', m1'$	Primitive	68
		T-Base-Centered	15
		R-Base-Centered	22
		Face-Centered	7
		Body-Centered	15
Tetragonal	$4, 41', 4'$ $4mm, 4mm1'$ $4'm'm, 4m'm'$	Primitive	49
		Body-Centered	19
Trigonal	$3, 6', 3m$ $3m', 6'm'm$	Primitive	18
		Rhombohedral	7
Hexagonal	$6, 61', 31'$ $6mm, 6m'm'$ $6mm1', 3m1'$	Primitive	27

Space-time symmetry in 2D materials (in progress)

- Coherent lattice dynamics: chiral phonon → BN, MoS_2,WSe_2



Q. Niu et al, PRL 115, 115502 (2015).

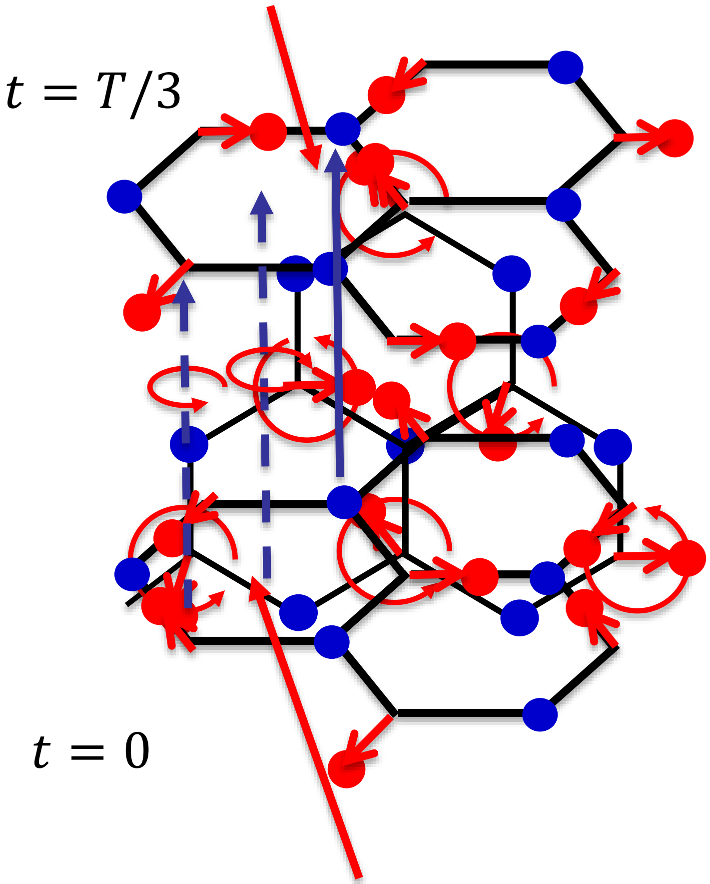


- Realized in WSe2 by inter-valley transfer of holes through hole-phonon interaction

Xiang Zhang's group, Science 359， 579 (2018).

Space-time symmetry in 2D materials (in progress)

R: 3-fold rotation



S: time-screw rotation

- Blue site → 3-fold axis

$$R:\ (x, y, t) \to \left(-\frac{1}{2}x - \frac{\sqrt{3}}{2}y + \sqrt{3}, \frac{\sqrt{3}}{2}x - \frac{1}{2}y, t\right)$$

- Plaquette center: time-screw axis

$$S:\ (x, y, t) \to \left(-\frac{1}{2}x + \frac{\sqrt{3}}{2}y, -\frac{\sqrt{3}}{2}x - \frac{1}{2}y, t + \frac{T}{3}\right)$$

- Central position of a red site → time-screw axis

Hu, Wu et al, in progress.

Degeneracy from space-time symmetry

- Theorem: operations for the wavevector group of $\vec{k}$, satisfying

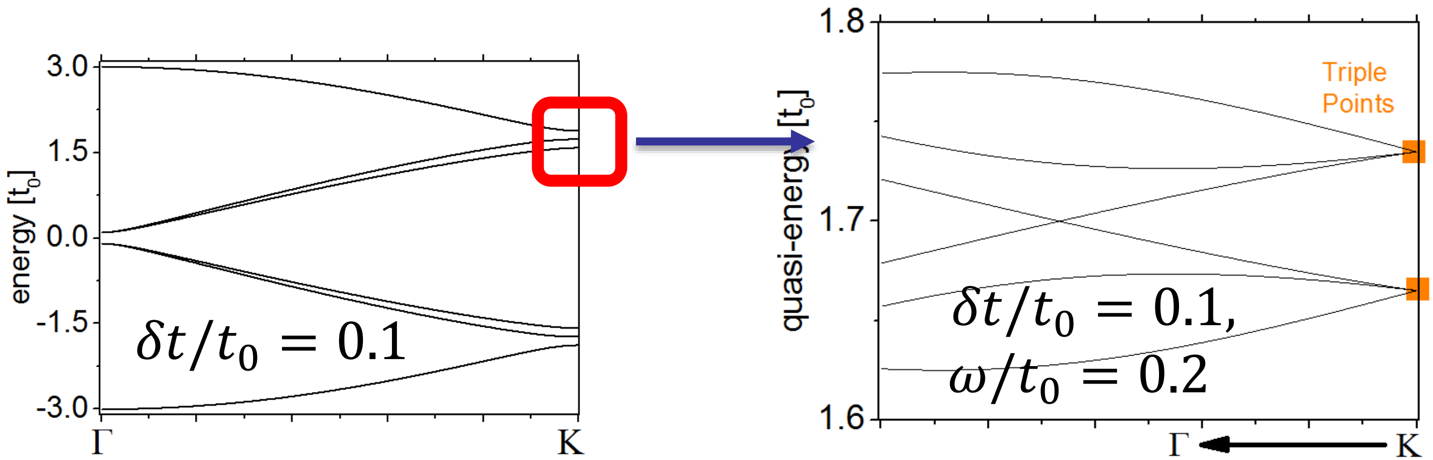
$$g_1 g_2 = T(\vec{u}) g_2 g_1, \text{ with } \vec{k}\cdot\vec{u} = 2\pi p/q \quad (p/q \text{ co-prime})$$

→ q-fold degeneracy at $\kappa = (\vec{k}, \omega)$

S and R both leave K invariant: $(S \cdot R)|_K = \exp\left(i\frac{2\pi}{3}\right)(R \cdot S)|_K$

Nondegeracy with static distortion (only R)

Triple degeneracy at K (R and S)



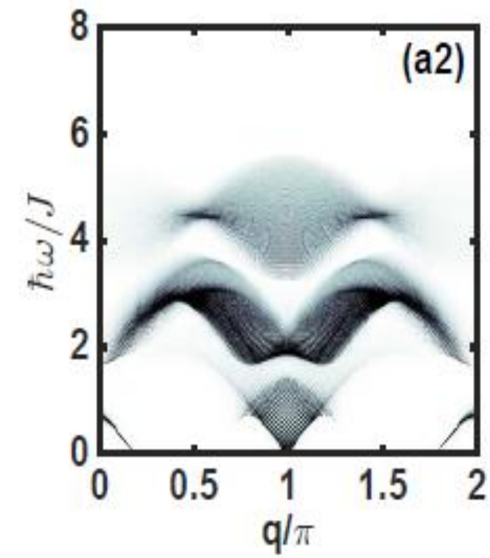
Further developments in speculation

- Time dependent potential for phononic and photonic crystals, optical lattices for cold atoms …

- Semi-classic transport – non-adiabatic treatment ……

- Time crystal -- Spontaneous discrete time translation symmetry breaking. (Nayak, Wilczek)

$$i\hbar\partial_t\psi = H(x,t)\,\psi$$



Quantum Dynamics

Space time symmetries

Strong correlations

Dynamic crystal
space-time group

High frequency response –
Bethe string states

W. Yang, J. Wu, S. L. Xu, Z. Wang, C. Wu arXiv:1702.01854.

Z. Wang, J. Wu, W. Yang, A. K. Bera, D. Kamenskyi, A.T.M. N. Islam, S. Xu, J. M. Law, B. Lake, C. Wu, A. Loidl , Nature 554, 219 (2018).

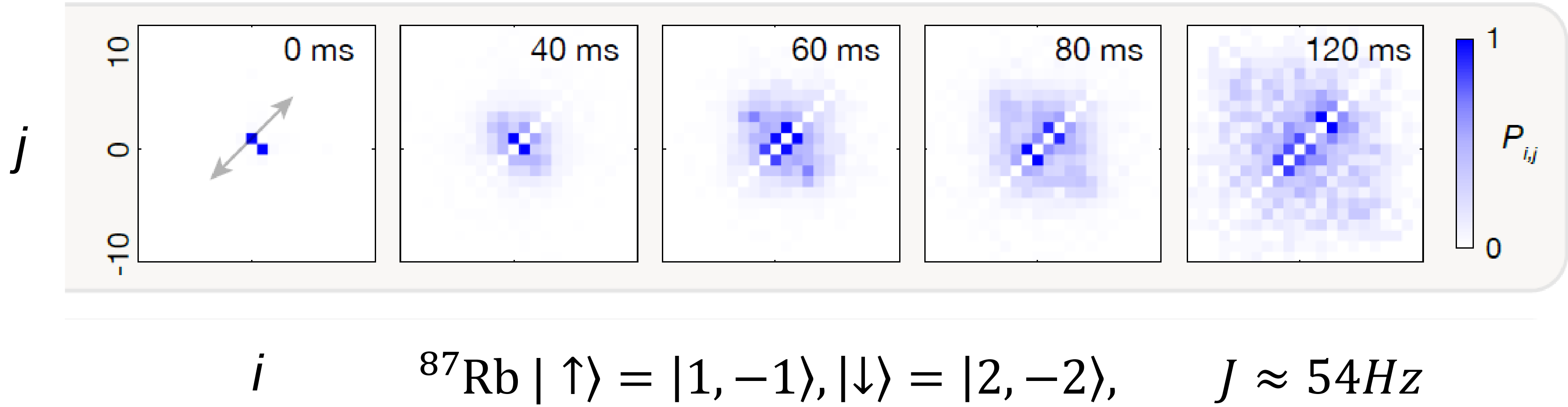
Magnon (anti)-bound states – Bethe string states

- 1D ferromagnet: spin-flip → magnon attraction → bound state

- Cold boson Mott insulators - bound state propagation.

I. Bloch's group, Nature 502 (2013).

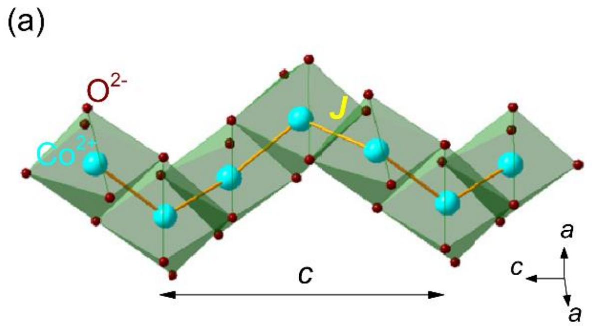
Joint probability P_{ij} peaks at $j = i \pm 1$

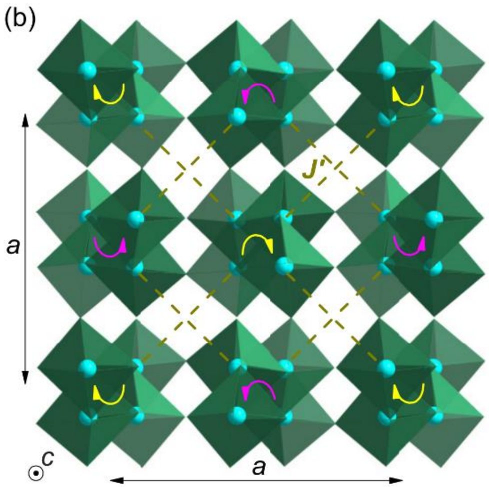


$^{87}\mathrm{Rb}\ |\uparrow\rangle = |1,-1\rangle, |\downarrow\rangle = |2,-2\rangle, \qquad J \approx 54Hz$

Quasi-1D antiferromagnet $SrCo_2V_2O_8$

Co^{2+} spin-1/2





Screw chain consisting of CoO_6 octahedra running along the crystalline c-axis

$$H = J\sum_{n=1}^{N}\left\{S_n^x S_{n+1}^x + S_n^y S_{n+1}^y + \Delta(S_n^z S_{n+1}^z - \frac{1}{4})\right\} - g\mu_B h\sum_{n=1}^{N} S_n^z$$

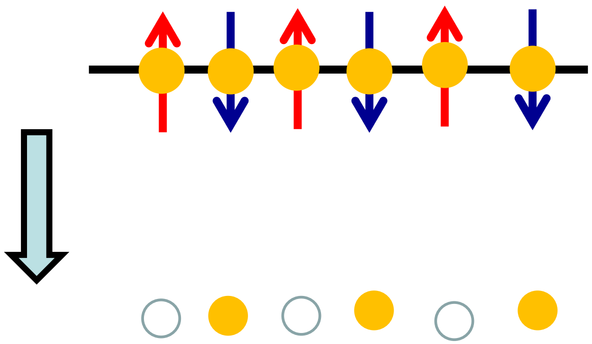
$$J \approx 3.55meV,\ \Delta \approx 2.04,\ \ g \approx 5.85$$

fitted by thermodynamic property measurements: spin gap, critical field, and saturation field.

Wang, Zhe, M. Schmidt, A. K. Bera, A. T. M. N. Islam, B. Lake, A. Loidl, and J. Deisenhofer, *PRB* 91, no. 14 140404 (2015).

Many-body physics of repulsive magnons (spin- ↓)

- 1D spin-1/2 antiferromagnet (Ising anisotropy)



spin-↑ →

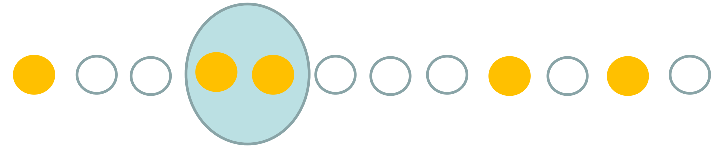
spin-↓ → hard-core boson → spinless fermion (Jordan-Wigner)

- $H > H_{c_1}$: magnetization → dope vacancies

a snapshot of ground state

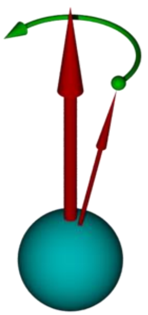
- Anti-bond states at high energies

2-string state
energy cost ~ J



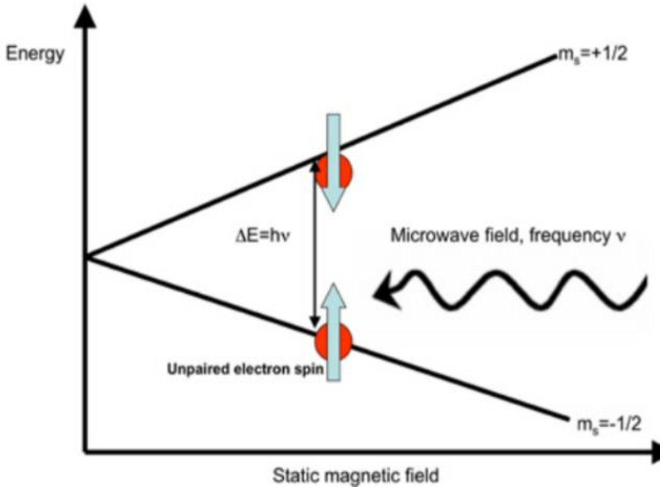
Measure excitations - electron spin resonance (ESR)

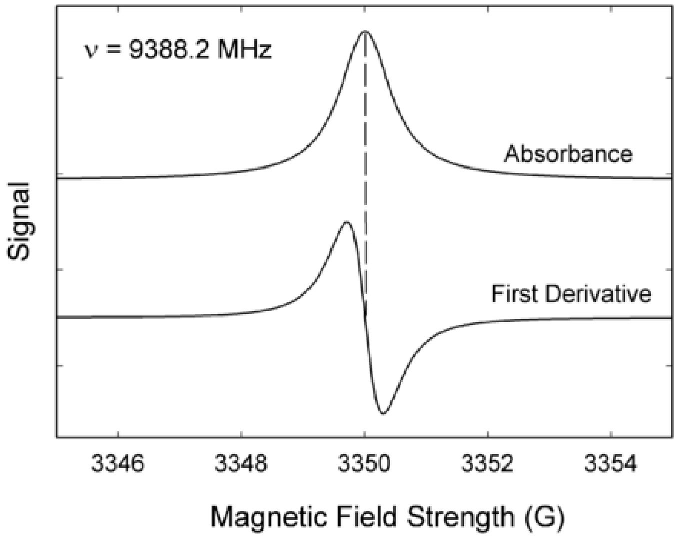
- Larmor precession:



$$H = -g_e \mu_B h S_z$$

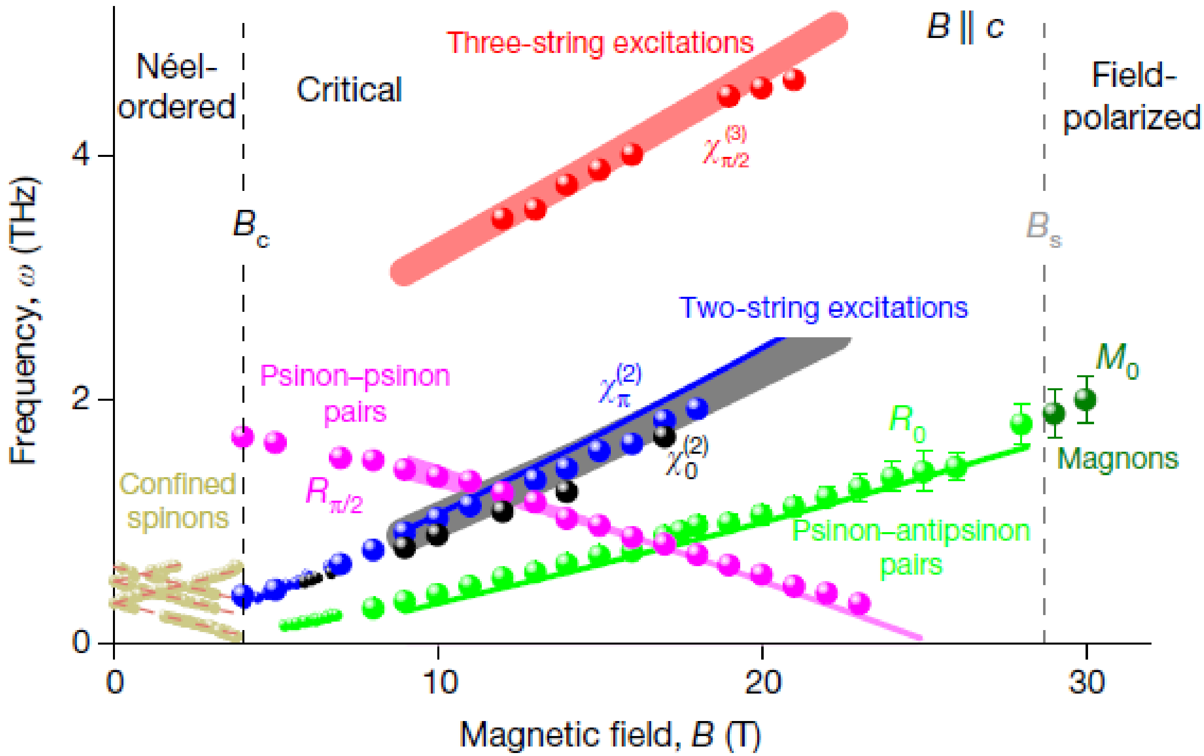
$$H\text{: 3500G},\ \nu = \frac{\omega}{2\pi} = 9 - 10GHz$$





High real-frequency spin excitation spectra

- ESR in the longitudinal B-field
- THz light along *c*-axis: $S^{+-}(q,\omega)$ and $S^{-+}(q,\omega)$ at $q = 0, \pm\frac{\pi}{2}, \pi$.



Loidl's group, Wu's group, et al, *Nature 554, 219 (2018).*

Dynamic spin structure factor

- Observable: ESR and neutron spectroscopy

Fourier spectra of real-time correlation: $\langle G|S_i^a(t)S_j^{\bar{a}}(t')|G\rangle$

$$S^{a\bar{a}}(q,\omega) = 2\pi \sum_{\mu} \left|\langle \mu | S_q^{\bar{a}} \ |G\rangle\right|^2 \delta(\omega - E_\mu + E_{GS})$$

Transverse: $S^{+-}(q,\omega)$, $S^{-+}(q,\omega)$

- Each matrix element → Summation over excitations → Check saturation with sum rules.

Why Bethe Ansatz?

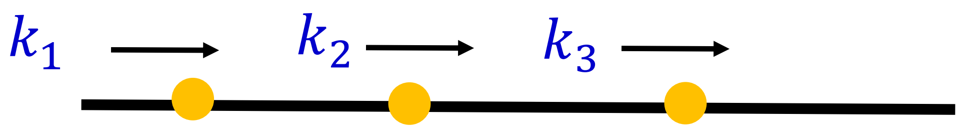
- All eigenstates are known not just the ground state → Spin dynamics at intermediate and high energies.

  Nature of excitations manifest – good Bethe quantum numbers

- **Exact diagonalization:** very small size.
- **TEBD:** difficult to handle gapless systems.
- **QMC**: difficult to handle real frequency.
- **Luttinger liquid:** only applies at low energy.

Correlation functions via Bethe Ansatz (BA)

- Coordinate BA inapplicable for correlation function calculations



m! terms

$$\psi = \sum_P \; A_{p_1 p_2 .. p_m} \, e^{ik_{p1}x_1 + k_{p2}x_2 + \cdots k_{pm}x_m}$$

- Algebraic Bethe ansatz – Form factor

L. A. Takhtadzhan and L. D. Faddeev *Russ. Math. Sur.* 34,11 (1979)

Many-body matrix elements → determinants;

Dynamic spin structure factor not done before for the XXZ model via BA

N. Kitanine, J. M. Maillet and V. Terras *Nucl. Phys. B* 554, 647 (1999)

Spectra of $S^{+-}(q,\omega)$

$$S^{+-}(q,\omega): \quad \sum_{\mu} \langle G|S_i^{+}(t)|\boldsymbol{\mu}\rangle\langle\boldsymbol{\mu}|S_j^{-}(t')|G\rangle$$

$|\mu\rangle$: Add a spin down ( ) to the ground state

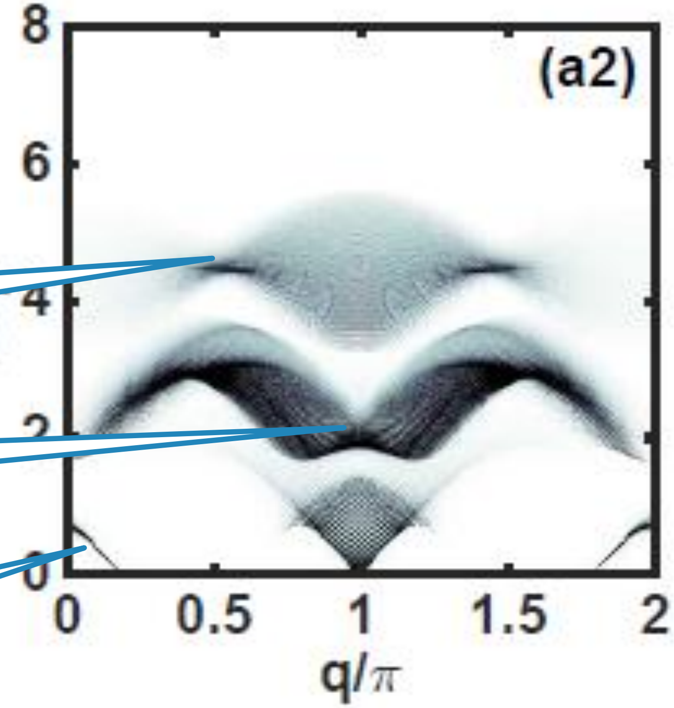
$$\langle\boldsymbol{\mu}|\boldsymbol{S_z}|\boldsymbol{\mu}\rangle = \langle\boldsymbol{G}|\boldsymbol{S_z}|\boldsymbol{G}\rangle - \mathbf{1}:$$

$2m = 0.2, \ \Delta = 2$

3 (body)- string states

2 (body)- string states

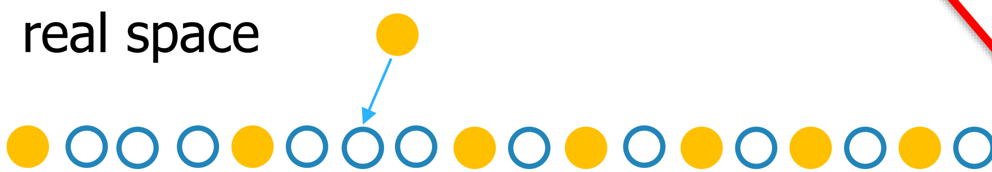
Larmor mode

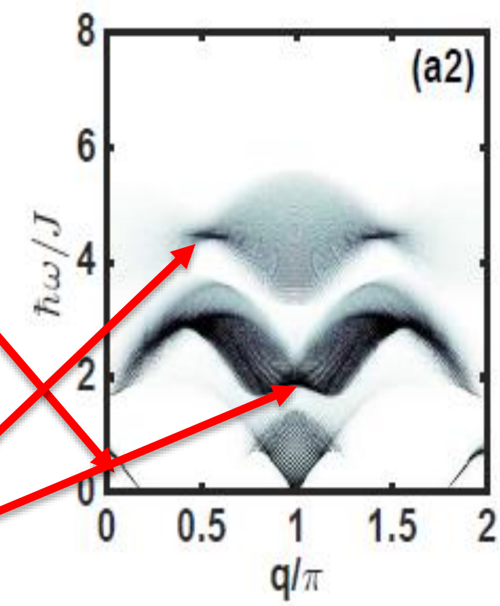


String states (anti-bound states)

- Low energy – gapless →Larmor mode

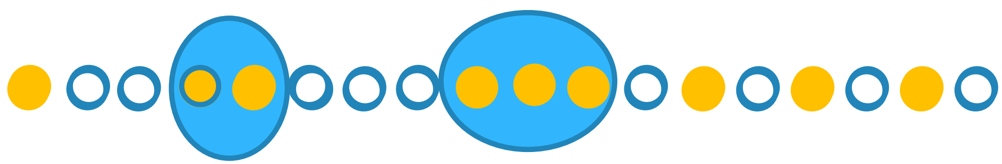
real space





- Gapped excitations at intermediate and high energies.

2-string states
3-string states



- No 4-string state contribution

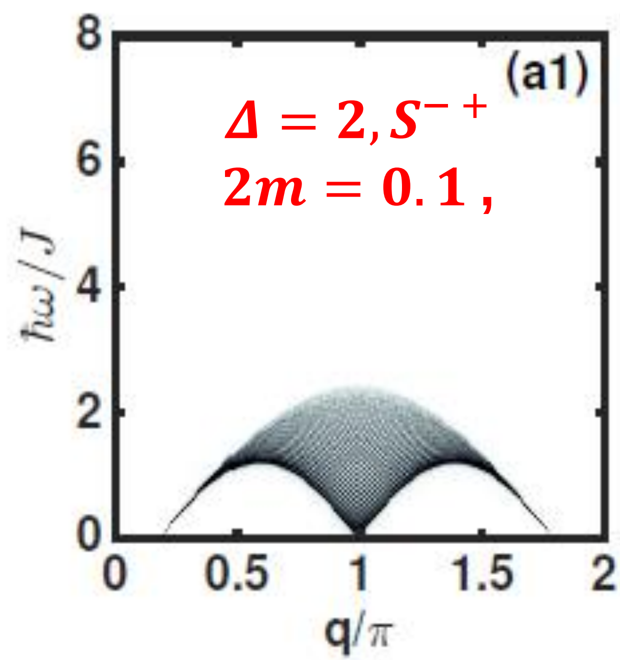
Dynamic spin-structure factor - $\boldsymbol{S^{-+}(q,\omega)}$

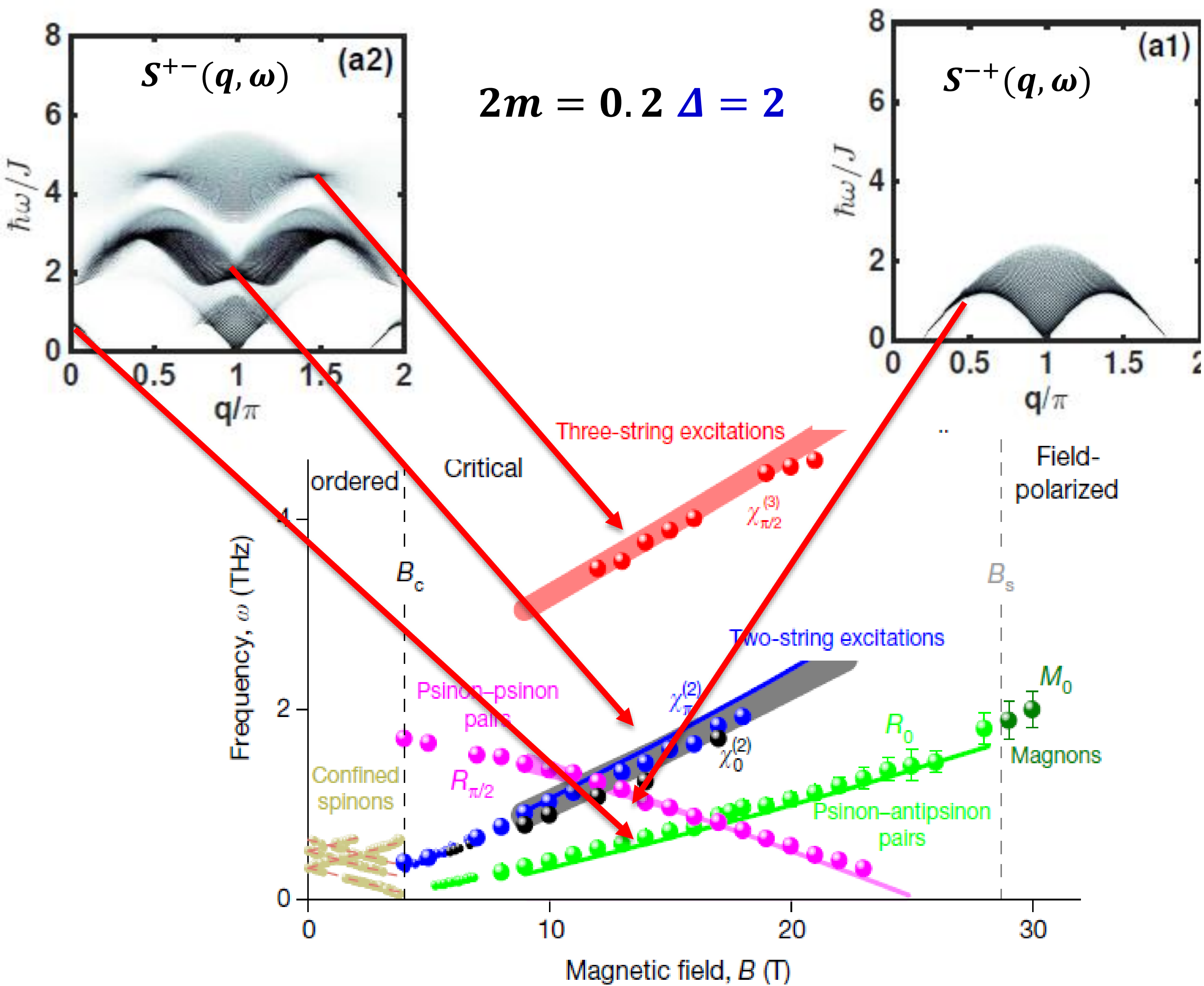
$$S^{-+}(q,\omega): \quad \sum_{\mu} \langle G|S_i^-(t)|\boldsymbol{\mu}\rangle\langle\boldsymbol{\mu}|S_j^+(t')|G\rangle$$

$|\mu\rangle$: remove a spin down (●) from the ground state

$$\langle\boldsymbol{\mu}|\boldsymbol{S_z}|\boldsymbol{\mu}\rangle = \langle\boldsymbol{G}|\boldsymbol{S_z}|\boldsymbol{G}\rangle + \mathbf{1}:$$

No string-state contribution

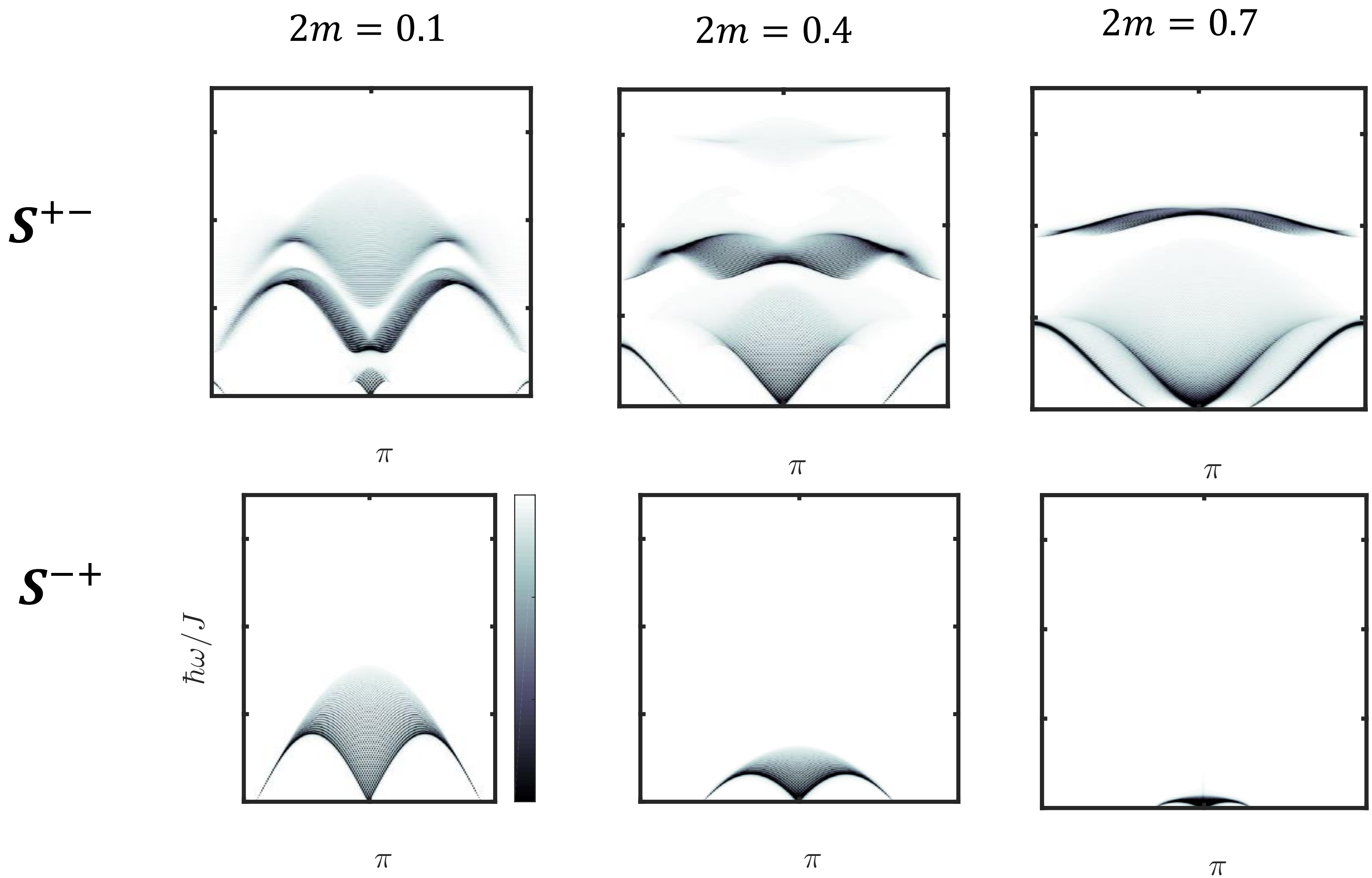




$S^{+-}(q,\omega)$
(a2)
$2m = 0.2$ $\Delta = 2$
$S^{-+}(q,\omega)$
(a1)
$\hbar\omega/J$
q/π
Three-string excitations
$\chi^{(3)}_{\pi/2}$
ordered
Critical
Field-polarized
B_c
B_s
Two-string excitations
$\chi^{(2)}_{\pi}$
$\chi^{(2)}_{0}$
Psinon–psinon pairs
$R_{\pi/2}$
Confined spinons
R_0
M_0
Magnons
Psinon–antipsinon pairs
Frequency, ω (THz)
Magnetic field, B (T)

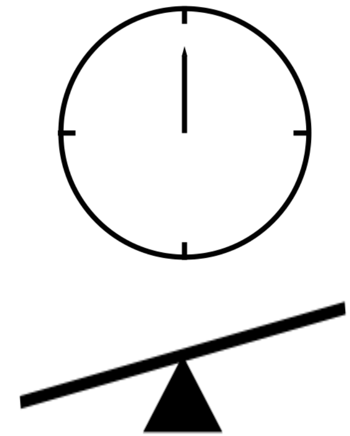
Transverse DSF – Evolution with magnetization

N=200, $\Delta = 2$



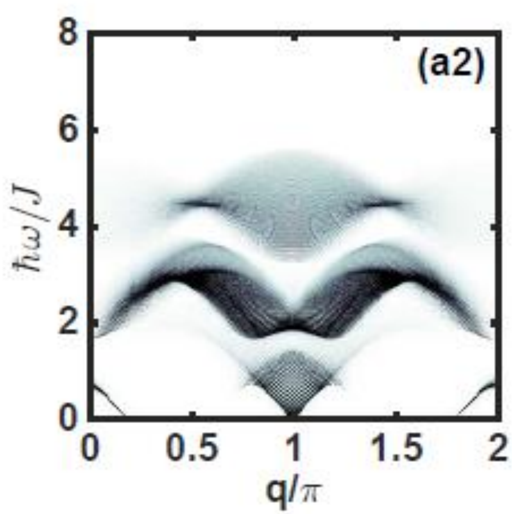
Summary

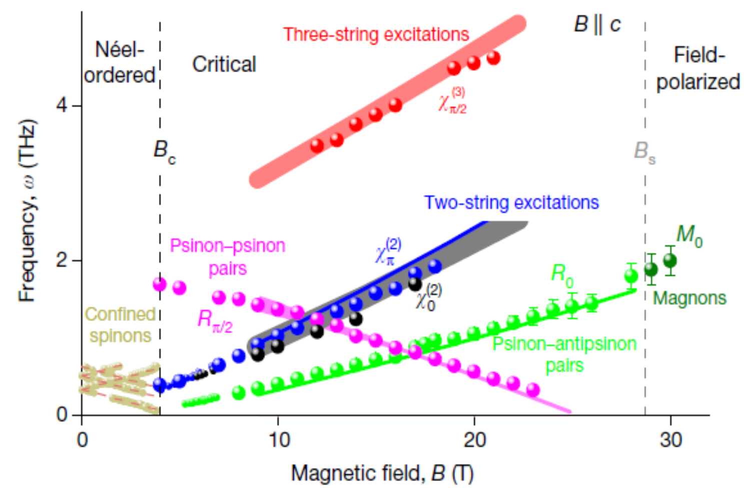
• A platform for (periodical) dynamic systems for everyone.



space-time group, Bloch-Floquet theorem

• High real-frequency – identification of 3-string states – hint for high dimensional states….



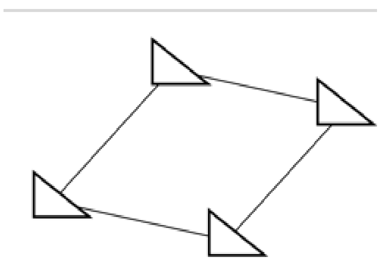


Back up

Symmetry consequences on dispersions

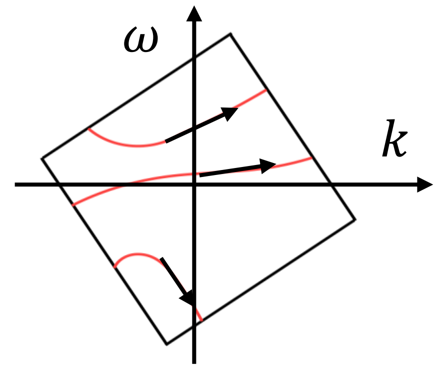
Dispersion relation $f(k,\omega)=0$ → generally multi-valued

- Winding in the Brillouin zone torus: (w_1, w_2)



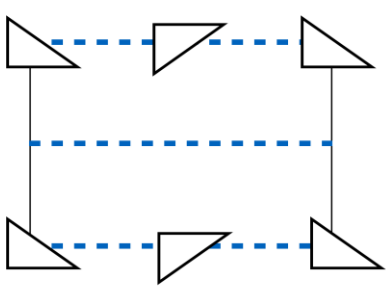
***P*1**

w=(2, 1)

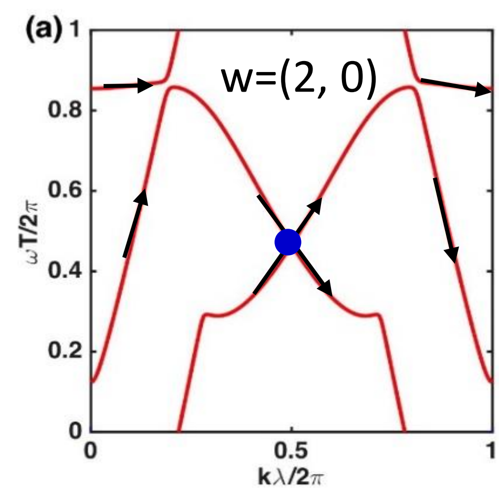


- **Non-spinor** Kramers degeneracy by $\boldsymbol{g_t}: (\boldsymbol{x}, \boldsymbol{t}) \to \left(\boldsymbol{x} + \frac{\boldsymbol{a}}{\mathbf{2}}, -\boldsymbol{t}\right)$

$M_{g_t}^{\;2}\psi_\kappa = \psi_\kappa(x-a,t) = -\psi_\kappa$ for $\kappa = (\pi, \omega)$



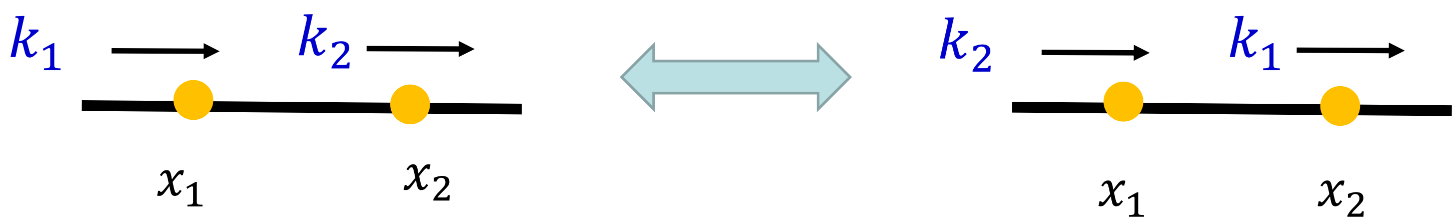
Winding #: (even, 0)



Bethe Ansatz (BA)

- Many-body scattering amplitude = a product of two-particle ones.

$$\psi = A_{12}\, e^{ik_1x_1+k_2x_2} + A_{21}e^{ik_1x_1+k_2x_2}$$



$$A_{21}/A_{12} = -e^{i\Theta(k_2,k_1)}$$

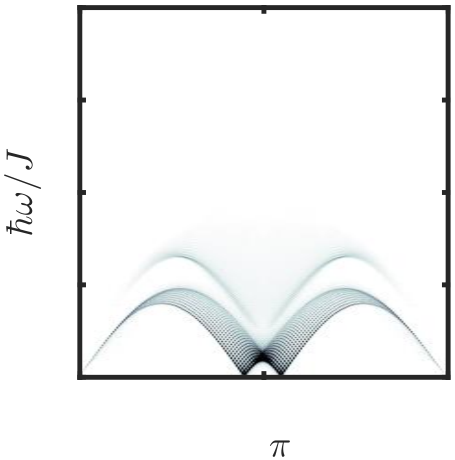
periodical boundary condition:

$$k_iN + \sum_{j\neq i} \pi + \Theta\left(k_j, k_i\right) = 2\pi I_i$$

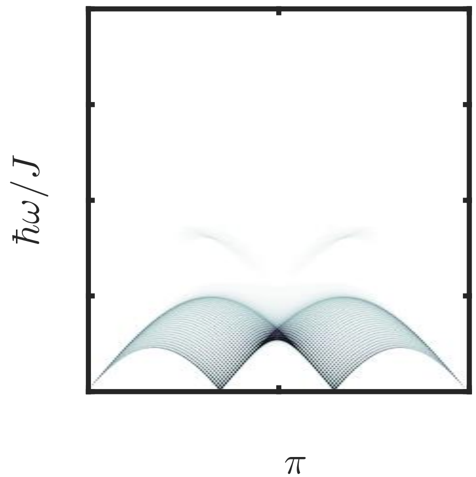
Bethe quantum number

- Ground state energy (Heisenberg chain): $\frac{E_G}{NJ} = \frac{1}{4} - ln2$

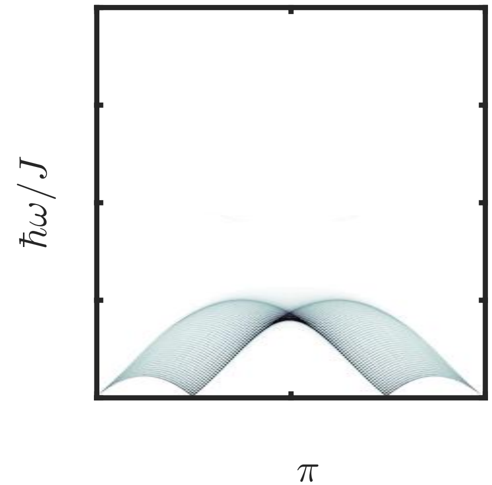
Longitudinal DSF $S^{zz}(q,\omega)$- intensity plot



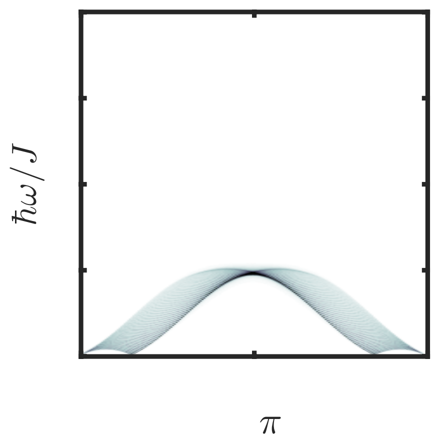
2m=0.1



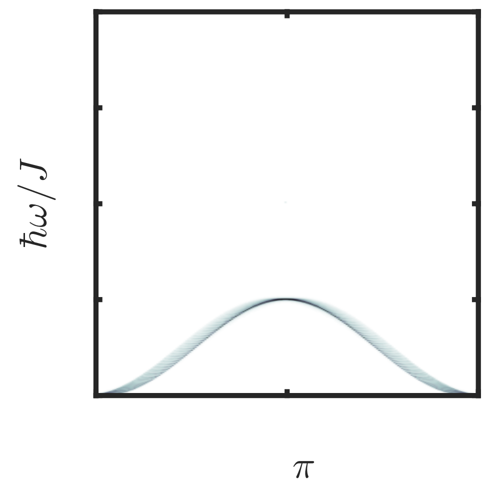
2m=0.3



2m=0.5



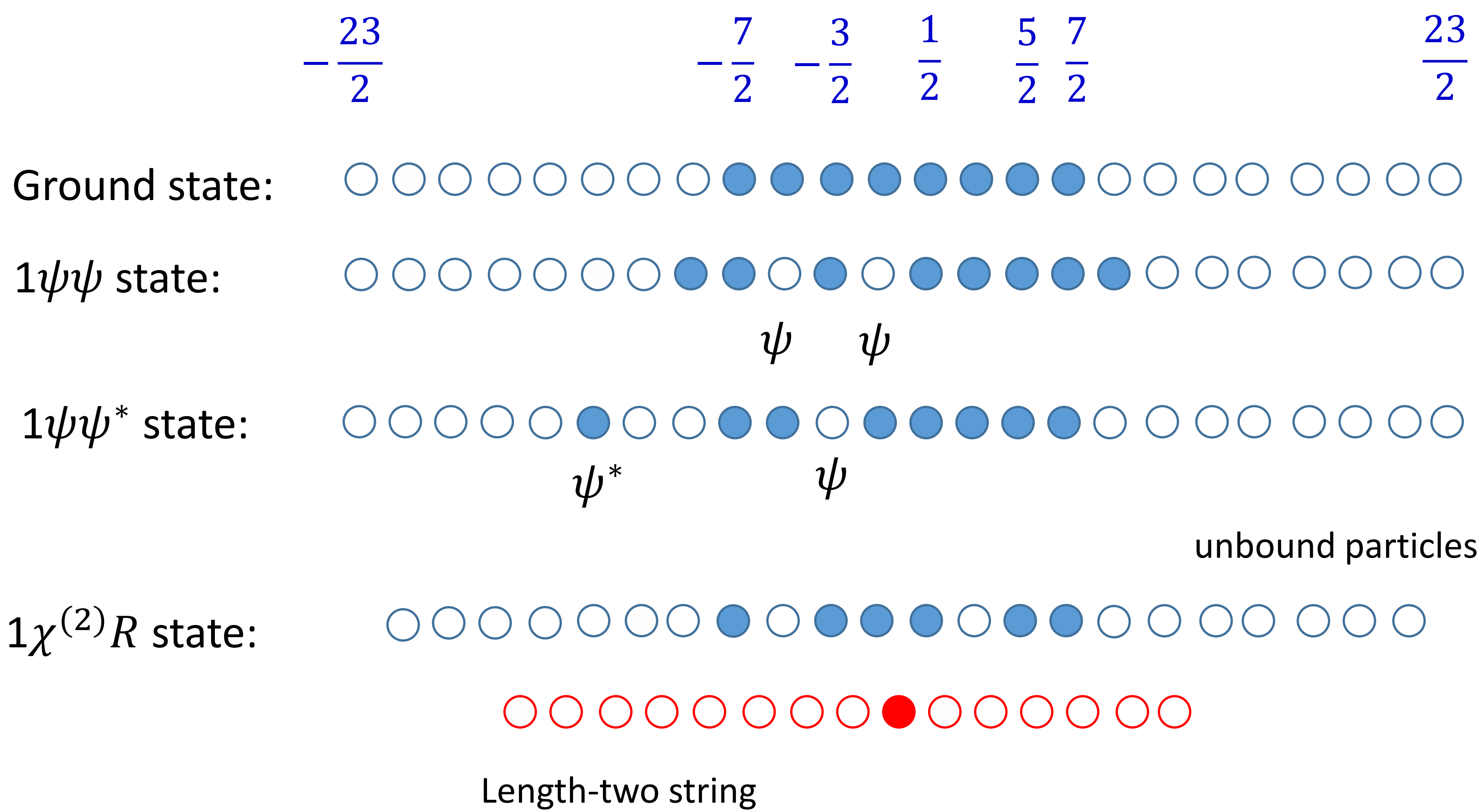
2m=0.7



2m=0.9

Bethe quantum numbers

$-\frac{M-1}{2} - S^z < \boldsymbol{I_\alpha^{(n)}} < \frac{M-1}{2} + S^z$: N=32, M=8 (spin-down).



Algebraic Bethe Ansatz

Yang-Baxter Equation:

$$R_{12}(\lambda_1,\lambda_2)R_{13}(\lambda_1,\lambda_3)R_{23}(\lambda_2,\lambda_3) = R_{23}(\lambda_2,\lambda_3)R_{13}(\lambda_1,\lambda_3)R_{12}(\lambda_1,\lambda_2)$$

Monodromy matrix:

$$\mathcal{T}(\lambda) = R_{0n}(\lambda, i\frac{\eta}{2})...R_{02}(\lambda, i\frac{\eta}{2})R_{01}(\lambda, i\frac{\eta}{2}) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}_{[0]}$$

Transfer matrix and XXZ Hamiltonian:

$$T(\lambda) = \mathrm{Tr}\mathcal{T}(\lambda) \qquad H = \sin(i\eta)\frac{d}{d\lambda}\ln T(\lambda)|_{\lambda=i\eta/2} + \mathrm{const.}$$

Magnon creation operator:

$$\Psi(\lambda_1,\lambda_2,...,\lambda_r) = B(\lambda_1)B(\lambda_2)...B(\lambda_r)|\uparrow\uparrow...\uparrow\rangle$$

L. A. Takhtadzhan and L. D. Faddeev *Russ. Math. Sur.* 34,11 (1979)

Algebraic Bethe ansatz and quantum inverse problem

Three key ingredients:

Monodromy matrix:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

B: magnon creation

C: magnon annihilation

A+D: conserved quantity

- Magnon creation operators

$$|\Psi(\lambda)\rangle = B(\lambda_1)B(\lambda_2)\ldots B(\lambda_M)|\uparrow\uparrow\uparrow\cdots\uparrow\rangle$$

- Quantum inverse problem

  (Relate local spin operators with quasi-particle operators.)

$$\sigma_i^- = \prod_{\alpha=1}^{i-1}(A+D)(\xi_\alpha)\cdot B(\xi_i)\cdot\prod_{\alpha=i+1}^{N}(A+D)(\xi_\alpha)$$

- F-basis

  (Simplifies quasi-particle operators.)

$$FB(\lambda)F^{-1} = \sum_{i=1}^{N}\sigma_i^- \otimes_{j\neq i} diagonal\ matrix\ at\ site\ j$$

Form factors can be evaluated.

N. Kitanine, J. M. Maillet and V. Terras *Nucl. Phys. B* 554, 647 (1999)

Derivation of determinant formulae

$$\langle \mu | S_n^a | \lambda \rangle = \frac{\langle \Psi(\{\mu_i\}) | S_j^a | \Psi(\{\lambda_j\}) \rangle}{\sqrt{\langle \Psi(\{\mu_i\}) | \Psi(\{\mu_i\}) \rangle} \cdot \sqrt{\langle \Psi(\{\lambda_j\}) | \Psi(\{\lambda_j\}) \rangle}}$$

Quantum inverse problem:

$$\sigma_i^- = \prod_{\alpha=1}^{i-1} (A+D)(\xi_\alpha) \cdot B(\xi_i) \cdot \prod_{\alpha=i+1}^{N} (A+D)(\xi_\alpha),$$

$$\sigma_i^+ = \prod_{\alpha=1}^{i-1} (A+D)(\xi_\alpha) \cdot C(\xi_i) \cdot \prod_{\alpha=i+1}^{N} (A+D)(\xi_\alpha),$$

$$\sigma_i^z = \prod_{\alpha=1}^{i-1} (A+D)(\xi_\alpha) \cdot (A-D)(\xi_i) \cdot \prod_{\alpha=i+1}^{N} (A+D)(\xi_\alpha)$$

F-basis:

$$\widetilde{D}_{1\ldots N}(\lambda;\xi_1,\ldots,\xi_N) \equiv F_{1\ldots N}(\xi_1,\ldots,\xi_N)\; D_{1\ldots N}(\lambda;\xi_1,\ldots,\xi_N)\; F^{-1}_{1\ldots N}(\xi_1,\ldots,\xi_N)$$

$$= \bigotimes_{i=1}^{N} \begin{pmatrix} b(\lambda,\xi_i) & 0 \\ 0 & 1 \end{pmatrix}_{[i]}.$$

$$\widetilde{B}_{1\ldots N}(\lambda) = \sum_{i=1}^{N} \sigma_i^- \; c(\lambda,\xi_i) \bigotimes_{j\neq i} \begin{pmatrix} b(\lambda,\xi_j) & 0 \\ 0 & b^{-1}(\xi_j,\xi_i) \end{pmatrix}$$

Determinant Formulas for Form Factors

$$
\begin{aligned}
|\langle\mu|S_q^-|\lambda\rangle|^2 &= N\delta_{q,q\{\lambda\}-q\{\mu\}}|\sin(i\eta)|\frac{\Pi_{j=1}^{M+1}|\sin(\mu_j - i\eta/2)|^2}{\Pi_{j=1}^{M}|\sin(\lambda_j - i\eta/2)|^2} \\
&\quad \Pi_{j>k=1}^{M+1}|\sin^2(\mu_j-\mu_k)-\sin^2(i\eta)|^{-1}\Pi_{j>k=1}^{M}|\sin^2(\lambda_j-\lambda_k)-\sin^2(i\eta)|^{-1} \\
&\quad \frac{|\mathrm{det}H^-|^2}{|\mathrm{det}\Phi(\{\mu\})|\ |\mathrm{det}\Phi(\{\lambda\})|}
\end{aligned}
$$

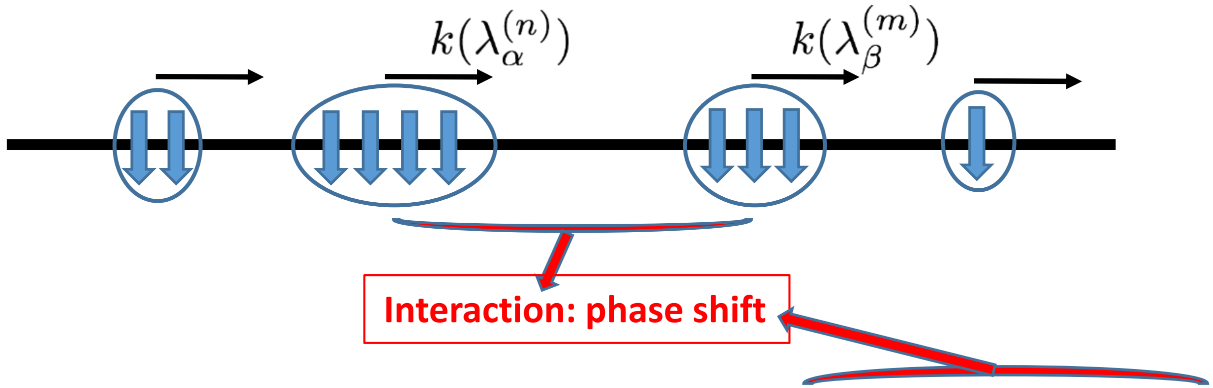
V. E. Korepin *Commun. Math. Phys.* 86, 391 (1982)
J. M. Maillet and J. Sanchez De Santos *arXiv: q-alg/9612012* (1996)
N. Kitanine, J. M. Maillet and V. Terras *Nucl. Phys. B* 554, 647 (1999)

- For string states, the formulas need to be regularized.

J. Mossel, and J-S Caux *New J. Phys.,* 12.5 (2010)

Bethe-Gaudin-Takahashi (BGT) equations

- Reference state: all spins up. Spin-down particles act as particles.
- String states: multi-particle bound states with complex rapidities.



Non-interacting:
$Nk = 2\pi I$

$$N\theta_n(\lambda_\alpha^{(n)}) = 2\pi I_\alpha^{(n)} + \sum_{(m,\beta)\neq(n,\alpha)} \Theta_{nm}(\lambda_\alpha^{(n)} - \lambda_\beta^{(m)})$$

$\lambda_\alpha^{(n)}$: rapidity $I_\alpha^{(n)}$: Bethe quantum number

Θ_{nm} : phase shift due to interaction

Sum rules

- Integrated intensity: $c_a = \pm 1, 0$, for $a = \pm, z$.

$$R_{a\bar{a}} = \frac{1}{N}\sum_q \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} S^{a,\bar{a}}(q,\omega) = \frac{1}{4} + \frac{m}{2}c_a$$

- Transverse first frequency moment (FFM).

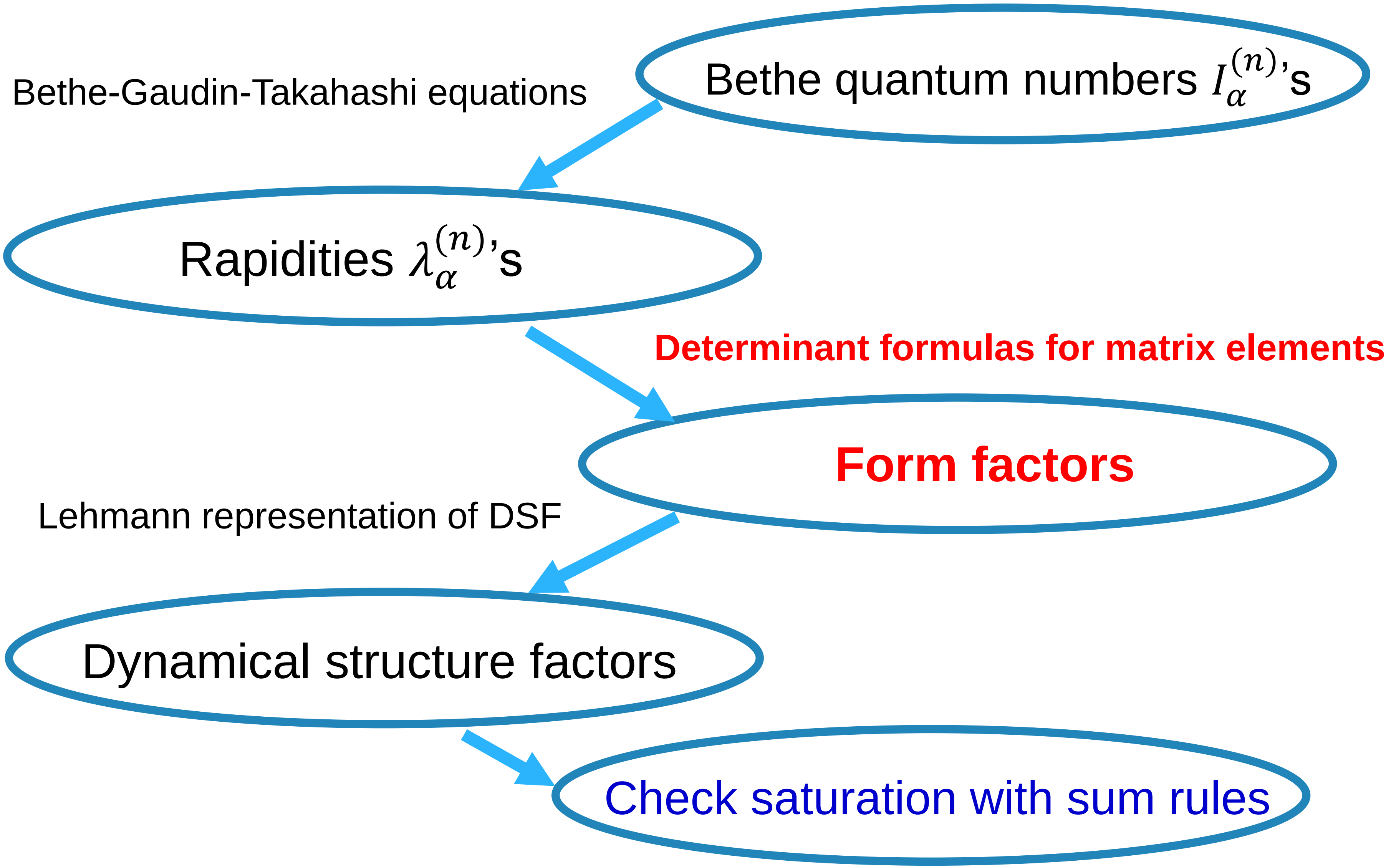
$$W_\perp(q) = \frac{1}{2}\int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, \omega \left(S^{+-}(q,\omega) + S^{-+}(q,\omega)\right) = \alpha_\perp + \beta_\perp \cos q$$

$$\alpha_\perp = -e_0 - \Delta \frac{\partial e_0}{\partial_\Delta} + mh \qquad \beta_\perp = (2-\Delta^2)\frac{\partial e_0}{\partial_\Delta} + \Delta e_0$$

- Longitudinal first frequency moment (FFM).

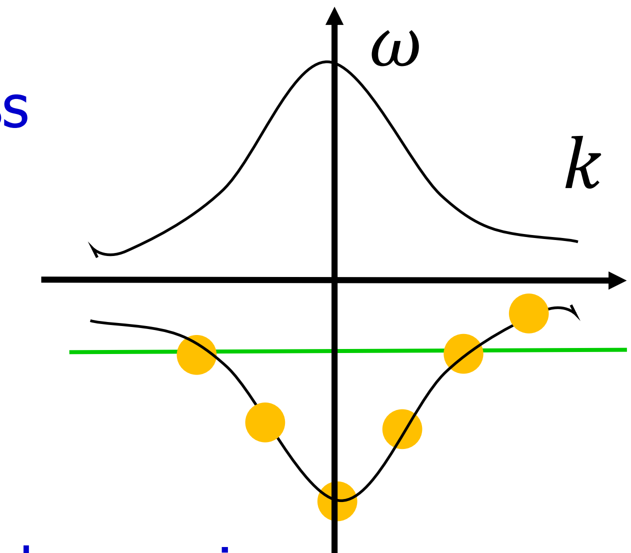
$$W_\parallel(q) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, \omega S^{zz}(q,\omega) = (1-\cos q)\alpha_\parallel \qquad \alpha_2 = -e_0 + \Delta \frac{\partial}{\partial_\Delta} e_0$$

Algorithm: dynamics for integrable systems

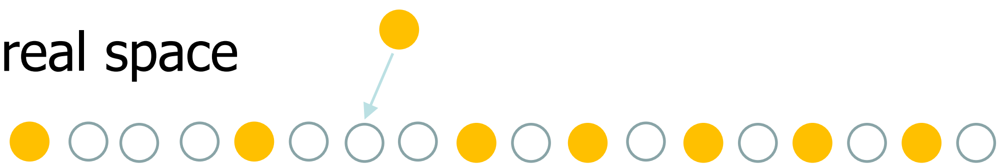


String states (anti-bound states)

- Low energy – intra-band transition, gapless
  →Larmor modes

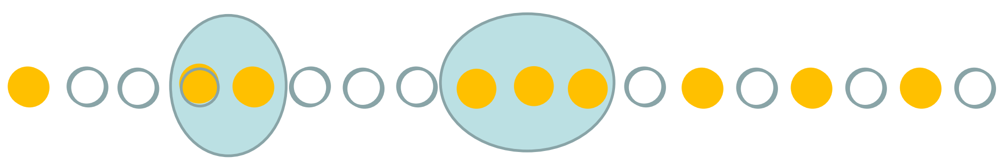


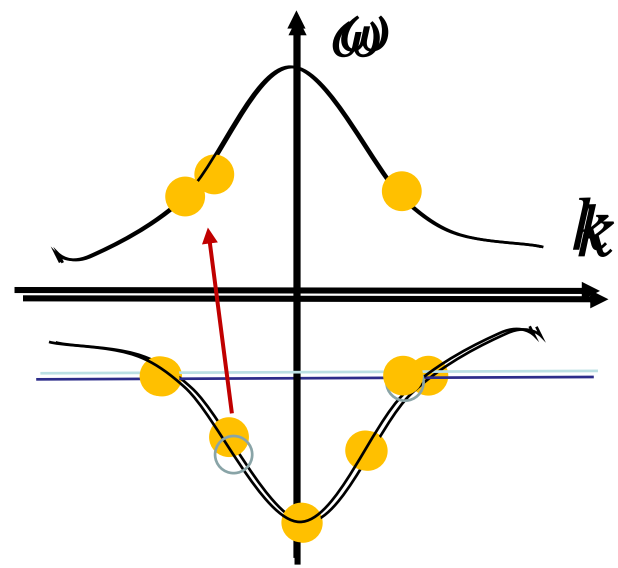
real space



- Gapped excitations at intermediate and high energies.

2-string states





- No 4-string state contribution.