Lecture 14. Physical meaning of the divergent series

§ $1+2+3+\cdots = -\frac{1}{12}$? Why?

Ramanujan's letter to Handy

§. Casimir effect:

$$1^3+2^3+3^3+\cdots = \frac{1}{120}$$

① Ramanujan's letter to Hardy

$$S = 1 + 2 + 3 + 4 + 5 + 6 \;.$$

$$4S = \quad 4 \quad + \quad 8 \quad + \quad 12$$

$$-3S = 1 - 2 + 3 - 4 + 5 - 6 = \frac{1}{(1+1)^2} = \frac{1}{4}$$

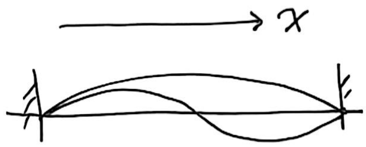
$$S = -\frac{1}{12} \;!$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3$$

$$\frac{1}{(1+x)^2} = \begin{array}{rrrrr} & 1 & -x & +x^2 & -x^3 \\ \times & 1 & -x & +x^2 & -x^3 \\ \hline & 1 & -x & +x^2 & -x^3 \\ & & -x & +x^2 & -x^3 \\ & & & x^2 & -x^3 \\ & & & & -x^3 \\ \hline & 1 & -2x & +3x^2 & -4x^3 \end{array}$$

Can you believe it ?

§3. Simplified version of Casimir effect



standing wave mode

$\psi = \sin k_n x$ with $k_n L = n\pi$

$\Rightarrow$ zero-point motion $E_n = \hbar k_n c = \frac{\hbar c}{L}\pi(n+1/2)$ ← zero point motion

Hence $E_0(L) = \frac{1}{2}\cdot 2\sum_{n=1}^{\infty}\hbar\omega_n = \frac{\hbar c\pi}{L}(1+2+3+\cdots)$

obviously, this summation diverges, Does it really have any significance?

Of course, the absolute value of $E_0(L)$ does not have meaning, we care about the energy changes as $L \to L+\delta L$. But even though

$$F = -\frac{\partial}{\partial L}E(L) = \frac{\hbar c\pi}{L^2}\sum_{n=1}^{\infty} n$$ ← this still does not make sense!

Even the sign is wrong! One would expect the force is repulsive, but as we will see, the force is attractive!

The reason is: that we can not actually sum over n to infinity. For a ~~met~~ metallic plate, there exist a cut off frequency ω_p, (typically the plasmon frequency). Above ω_c, the E & M wave would penetrate the plate, i.e. the plate is transparent

Hence, we should add the cutoff explicitly in the summation:

$$\frac{\hbar c\pi}{L} n < \hbar\omega_p, \quad \text{i.e.} \quad n < N(L) = \frac{\omega_p}{c}\frac{L}{\pi} = L/l, \quad \text{where } l = \frac{c\pi}{\omega_p}.$$

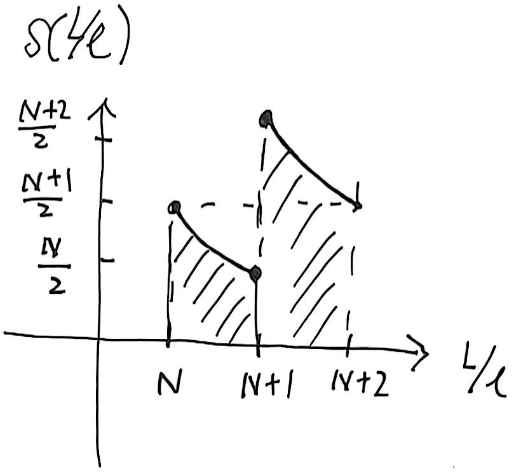
l is a microscopic length scale.

We define $S(\frac{L}{l}) = \left(\sum_{n=1}^{[L/l]} n\right)\frac{1}{L/l}$ $\qquad$ $[L/l]$ is the largest integer $\leq L/l$.

$$\Rightarrow S(\tfrac{L}{l}) = \frac{l}{L}\cdot\frac{N(N+1)}{2} = \frac{l}{L}\,\frac{1}{2}\left[\frac{L}{l}\right]\left(\left[\frac{L}{l}\right]+1\right).$$

consider $N+1 > L/l > N \Rightarrow S[L/l] = \frac{1}{2}N(N+1)/(L/l)$

$N+2 > L/l > N+1 \qquad \frac{1}{2}(N+1)(N+2)/(L/l)$



$S(L/l)$ needs to be smoothed.

$$\tilde{S}(L/l) = \int_N^{N+1} d(L/l)\,\frac{1}{L/l}\cdot\frac{N(N+1)}{2} = \frac{N(N+1)}{2}\ln\frac{N+1}{N} \quad \leftarrow \ln(1+\tfrac{1}{N})$$

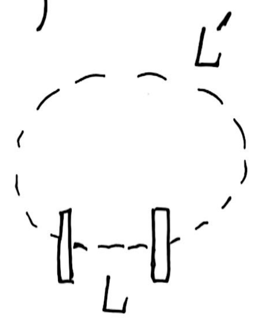
$$= \frac{N(N+1)}{2}\left(\frac{1}{N} - \frac{1}{2N^2} + \frac{1}{3N^3} + \cdots\right)$$

$$= \frac{N+1}{2} - \frac{N+1}{4N} + \frac{N+1}{6N^2} + \cdots = \frac{N}{2} + \frac{1}{4} - \frac{1}{12N} + \cdots$$

$$= \frac{1}{2}\frac{L}{l} + \frac{1}{4} - \frac{1}{12}\frac{l}{L}$$

$$E(L) = \frac{\hbar c \pi}{l} \cdot \underbrace{\frac{1}{4l} \sum_{n=1}^{[L/l]} n}_{\text{Smoothed}} = \frac{\hbar c \pi}{l} \tilde{S}(L/l)$$

$$= \frac{\hbar c \pi}{l} \left(\frac{1}{2} \frac{L}{l} + \frac{1}{4} - \frac{1}{12} \frac{l}{L} + \cdots \right)$$



The EM wave also exists ~~in~~ outside the plates. We think it a 1D ring

$$E(L') = \frac{\hbar c \pi}{l} \left(\frac{1}{2} \frac{L'}{l} + \frac{1}{4} - \frac{1}{12} \frac{l}{L'} + \cdots \right) \quad \text{with } L + L' = L_{tot}$$

vanish

$$\Rightarrow E_{tot} = \frac{\hbar c \pi}{l} \left(\frac{1}{2} \frac{L_{tot}}{l} + \frac{1}{2} - \frac{l}{12} \left(\frac{1}{L} + \frac{1}{L'} \right) + \cdots \right)$$

Then $\boxed{F = -\frac{\partial E_{tot}}{\partial L} = -\frac{\hbar c \pi}{12} \frac{\partial}{\partial L} \left(\frac{-1}{L} \right) = -\frac{\hbar c \pi}{12} \frac{1}{L^2}}$

(*) A more sophisticated regularization method

we introduce the convegence factor $t = \frac{1}{N(L)} = \frac{\cancel{\hbar c} \, l}{L}$

$$S = \sum_{n=1}^{\infty} n e^{-tn} \quad \leftarrow \text{effectively}$$

$$= -\frac{d}{dt} \sum_{n=1}^{\infty} e^{-tn} = -\frac{d}{dt} \frac{1}{1 - e^{-t}}$$

$$\frac{1}{1-e^{-t}} = \left(t - \frac{t^2}{2} + \frac{t^3}{3!} - \frac{t^4}{4!}\right)^{-1} = t^{-1}\underbrace{\left(1 - \frac{t}{2} + \frac{t^2}{6} - \frac{t^3}{24} + \frac{t^4}{120}\right)^{-1}}$$

$$\begin{aligned}
&1 + \frac{t}{2} - \frac{t^2}{6} + \frac{t^3}{24} - \frac{t^4}{120} \\
&\qquad\quad \frac{t^2}{4} - \frac{t^3}{6} + \left(\frac{1}{36} + \frac{1}{24}\right)t^4 \\
&\qquad\qquad\quad + \frac{t^3}{8} \quad 3 \cdot \frac{1}{2^2} \cdot \frac{1}{6} t^4 \\
&\qquad\qquad\qquad\quad + \frac{1}{24} t^4
\end{aligned}$$

$$1 + \frac{t}{2} + \frac{t^2}{12} + 0 - \frac{1}{720}t^4$$

$$S = -\frac{d}{dt}\left(\frac{1}{t} + \frac{1}{2} + \frac{t}{12} - \frac{t^3}{720} + \cdots\right)\bigg|_{t\to 0}$$

$$= \frac{1}{t^2} - \frac{1}{12} + O(t^2)$$

$$\Rightarrow E_0(L) = \frac{\hbar c\pi}{L}\left(\cdots \left(\frac{L}{l}\right)^2 - \frac{1}{12} + \cdots\right)$$

$$= \frac{\hbar c\pi}{l^2} L - \underbrace{\frac{1}{12}\frac{\hbar c\pi}{L}}$$

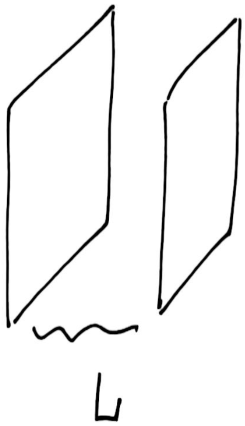
but the UV part ~~is~~ depends on regularization

↑ this part is the same

Similarly, we have $F = -\frac{\hbar c\pi}{12}\frac{\partial}{\partial L}\left(-\frac{1}{L}\right) = -\frac{\hbar c\pi}{12}\frac{1}{L^2}$

① 3D Casimir force

$$\psi_n(x,y,z,t) = e^{-i\omega_n t}\, e^{i(k_x x + k_y y)} \sin k_n z$$



$$\omega_n = c\sqrt{k_x^2 + k_y^2 + \frac{n^2\pi^2}{L^2}}$$

$$E = 2\cdot\frac{\hbar}{2} A \int \frac{dk_x dk_y}{(2\pi)^2} \sum_{n=1}^{\infty} \omega_n$$

$$\frac{\omega_p^2}{c^2} - \frac{n^2\pi^2}{L^2}$$

$$\frac{E}{A} = \hbar c \int \frac{dk_x dk_y}{(2\pi)^2} \underbrace{\sum_{n=1}^{\infty} \left(k_x^2 + k_y^2 + \frac{n^2\pi^2}{L^2}\right)} = \frac{\hbar c}{4\pi}\underbrace{\int_0 dq^2 \sum_{n=1}^{\infty}} \left(q^2 + \frac{n^2\pi^2}{L^2}\right)^{1/2}$$

$$k_x^2 + k_y^2 + \frac{n^2\pi^2}{L^2} < \frac{\omega_p^2}{c^2}$$

$$\sum_{n=1}^{\infty} \int_0^{\frac{\omega_p^2}{c^2} - \frac{n^2\pi^2}{L^2}} dy \left(y + \frac{n^2\pi^2}{L^2}\right)^{1/2}$$

$$= \frac{2}{3}\left[\frac{\omega_p^3}{c^3} - \left(\frac{n\pi}{L}\right)^3\right]$$

$$\Rightarrow \frac{E}{A} = \text{const} - \frac{2}{3}\frac{\hbar c \pi^3}{4\pi L^3} \sum_{n=1}^{\infty} n^3$$

$$= -\frac{\hbar c}{6}\frac{\pi^2}{L^3} \sum_{n=1}^{\infty} n^3 \quad + \text{const}$$

$$\cancel{n < \frac{\omega_p}{c}\frac{L}{\pi} =}$$

~~Again we define~~ $\cancel{S(L/\ell) = \left[\sum_{n=1}^{[L/\ell]} n^3\right] \Big/ (L/\ell)^3}$

$$\cancel{= \left(\frac{\ell}{L}\right)^3 \frac{N^2(N+1)^2}{4} = \left(\frac{\ell}{L}\right)^3 \cdot \frac{1}{4}[L/\ell]^2[L/\ell + 1]^2}$$

Define

$$S = \sum_{n=1}^{\infty} n^3 e^{-tn} = -\frac{d^3}{dt^3} \sum_{n=1}^{\infty} e^{-tn}$$

$$= -\frac{d^3}{dt^3}\left(\frac{1}{t} + \frac{1}{2} + \frac{t}{12} - \frac{t^3}{720} + \cdots\right)\Big|_{t\to 0}.$$

$$= \frac{6}{t^4} + \frac{1}{120} + \cdots$$

$$\Rightarrow \frac{E_0(L)}{A} = -\frac{\hbar c}{6}\frac{\pi^2}{L^3}\left[\left(\frac{L}{\ell}\right)^4 + \frac{1}{120}\right]$$

$$= -\frac{\hbar c}{6}\frac{\pi^2}{\ell^4} L - \frac{\hbar c}{720}\frac{\pi^2}{L^3}$$

$$\Rightarrow \frac{F}{A} = -\frac{\partial E}{\partial L} = -\frac{\hbar c \pi^2}{720}\frac{\partial}{\partial L}\left(\frac{-1}{L^3}\right)$$

$$\frac{F}{A} = -\frac{\hbar c \pi^2}{\boxed{240}}\frac{1}{L^4}$$