Lect 1. Dimensional analysis (I)

1. What does "Physics at the back of envelop" mean?

2. The first atomic bomb explosion – yield estimation

   G. I. Taylor, Proc. Roy. Soc. A 201, 175, (1950).

3. Π - theorem

4: Water wave velocity

§1. Estimate the energy released in the first atomic bomb explosion

- The photos show the radius of the fire ball as a function of time. $R(t)$.
- $R(t)$ should depend on the physical quantites of

  the energy released in the explosion E

  the air density ρ.
- We neglect the initial radius of fire ball, i.e. the radius of the bomb.

Every physical quantity has a unit, which can be expressed in terms of the product of fundemental units. The fundemental quantities are length (L), time (T), mass (M). If we use the Gaussian unit, no additional unit needs to be introduc

For example, $[e^2/L^2] = [\text{force}] = [M][L][T]^{-2}$

hence $\boxed{[e^2] = [M][L]^3[T]^{-2}}$

Temperature does not need a unit either, since k_BT is alw combined together. (We often neglect k_B).

$$\boxed{[k_BT] = [E] = [M][L]^2[T]^{-2}}$$

Now, we try $E \sim mv^2$, $m \sim \rho R^3$, $v \sim R/t$ $\Rightarrow$ $E \sim \rho R^5/t^2$.

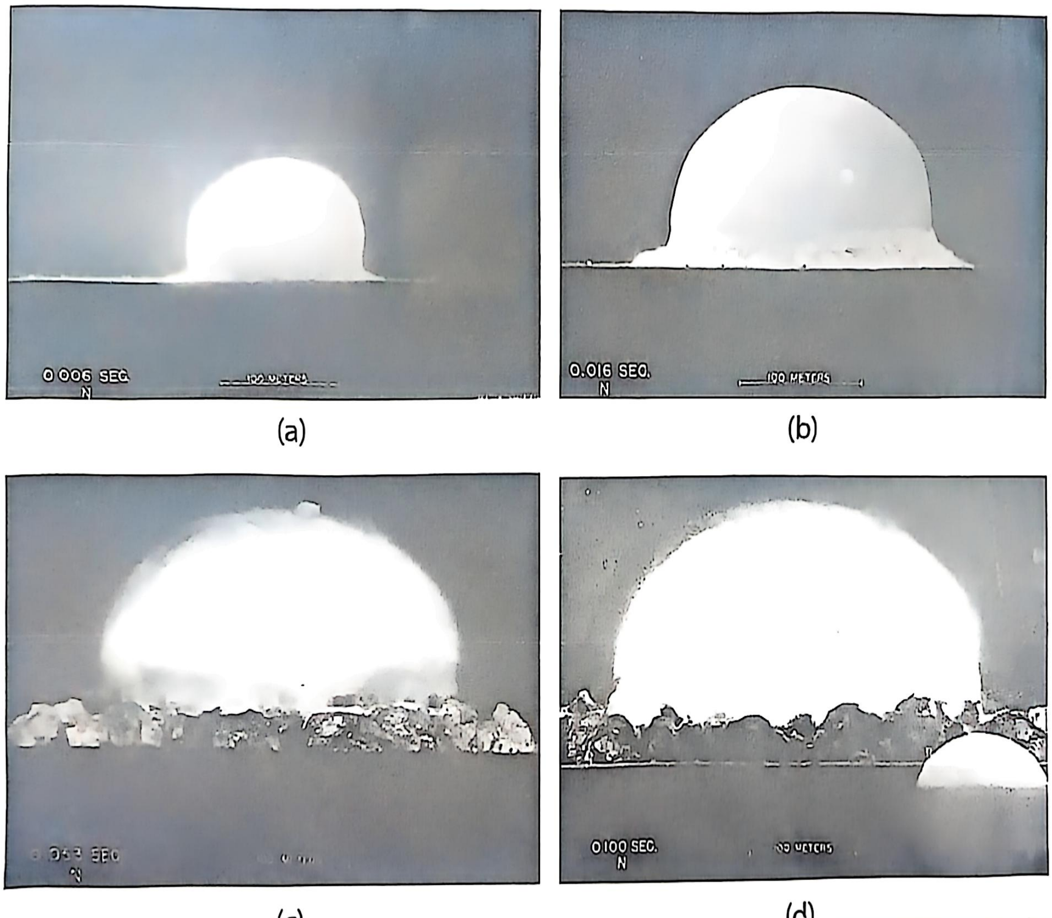
or $R \sim \left(\frac{Et^2}{\rho}\right)^{1/5}$. $\leftarrow$ Does it make sense or not?

$\boxed{\text{G. I. Taylor 1950, Proc. Roy. Soc, A201 (1065) 175, 1950.}}$

Yes! R increases as E increases and t increases; if air is denser, then R decreases.

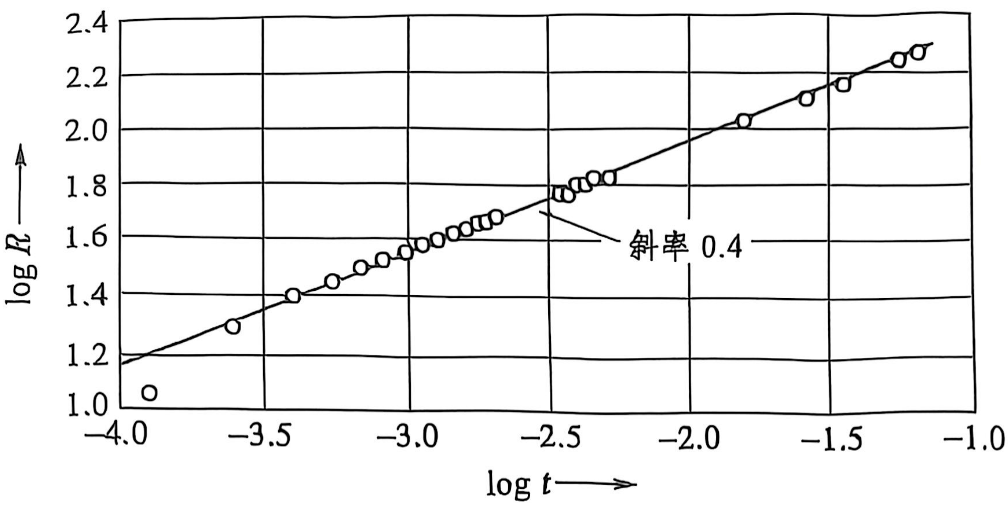
Although we do not understand any fluid mechnics, we arrive at the correct answer up to a numeric coefficient!

* Test, plot $\ln R \simeq \frac{2}{5}\ln t$ + const, check the slope of the ln-ln plot.



(a) (b) (c) (d)

t/ms	R/m	R^5/t^2 (m^5/ms^2)	t/ms	R/m	R^5/t^2 (m^5/ms^2)
0.38	25.4	7.32×10^7	1.5	44.4	7.67×10^7
0.66	31.9	7.58×10^7	4.61	4.61	6.49×10^7
1.08	38.9	7.66×10^7			



To be more precise, we can express

$$R = \left(\frac{E t^2}{\rho}\right)^{1/5} \Phi(\gamma),$$

where γ is a dimensionless quantity. γ rely on the effective degress of freedom of air moleculan/atom. In fact $\gamma = C_p/C_v = 1+\frac{2}{d}$. If the temperature of the fire ball is defferent, such that γ is not a constant, then ~~take~~ we can not use the above scaling. Fortunataly, he found the slope is about 0.4. If we further assume $\Phi(\gamma) \simeq 1$ for an order estimation up to,

$$E \sim \rho \frac{R^5}{t^2} = 0.9\times10^{13}\,J \approx 10^{14}\,J$$

As an estimation, we take

$$\begin{cases} R^5/t^2 \simeq 7\times10^7\ m^5/ms^2 = 7\times10^{13}\ m^5/s^2 \\ \rho = 1.29\times \cancel{10}\ kg/m^3 \end{cases}$$

$\boxed{1\,g\ \text{TNT} \sim 1\,\text{Kcal} \simeq 4.2\times10^{3}\,J}$

$1\ \text{ton TNT} \sim 4.2\times10^{9}\,J$

$$\Rightarrow E \sim \frac{1}{4.2}\times10^{5}\ \text{ton TNT}$$

$$\sim 2.2\times10^{4}\ \text{ton TNT.}$$

published 25 ± 2 kton.

certainly, since $\Phi(\gamma)$ cannot be calculated precisely here, the above result should be precise. Nevertheless, it's a pretty good estimation.

② How about the pressure insider the fireball ?

we assume P is uniform inside.

$$PV \sim E \quad \Rightarrow \quad p \sim E/R^3 = \frac{E}{(Et^2/\rho)^{3/5}} = \left(\frac{E^2\rho^3}{t^6}\right)^{1/5}$$

For example, at $R = 10\,km$ for the center of explosion

$$P \sim \frac{E}{R^3} \simeq \frac{8\times10^{14}\,J}{10^{4\times3}\ m^3} = 200\ Pa \sim \underline{20\,kg/m^2}.$$

§2. Π Theorem (Buckingham 1914)

Suppose the unit system we use has m fundemental quantities, such that we use them as basis of unit: $P_1, P_2, \cdots P_m$. Then we use them to express the unit of n-physical quantities $Q_1, \cdots Q_n$.

$$\begin{pmatrix} \ln[Q_1] \\ \ln[Q_2] \\ \vdots \\ \ln[Q_n] \end{pmatrix} = \underbrace{\begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nm} \end{pmatrix}}_{A} \begin{pmatrix} \ln[P_1] \\ \vdots \\ \ln[P_m] \end{pmatrix} \leftarrow [Q_1] = [P_1]^{a_{11}} \cdots [P_m]^{a_{1m}}$$

$\cdots$ and so on.

if the rank of A, i.e, the number of linearly independent row vector of A, is k, i.e there're k independent dimensinless quanties.

Then there exist $n-k$ independent linear relations

$$\sum_{j=1}^{n} b_{ij} \ln[Q_j] = 0, \quad i = 1, \cdots n-k.$$

In ~~orde~~ other words, we have $n-k$ independent dimensinless quantitie

$$\boxed{[Q_1]^{b_{i1}} \cdots [Q_n]^{b_{in}} \quad \text{for } i = 1, 2, \cdots n-k.}$$

§ wave velocity (water wave) → (gravity wave) v, g, λ, h

1. Shallow water wave $\lambda \gg h$, such that we take the limit $\lambda \to \infty$.

Then v, g, h

	M	L	T
v	0	1	-1
g	0	1	-2
h	0	1	0

$\Rightarrow v \sim \sqrt{g \cdot h}$

(Tsunami is actually a shallow water wave $\lambda \sim 100\,km$, $h \sim 5\,km$).

Question: why tsunami does have huge height as approaching coast?

2. Deep water wave $\lambda \ll h$, $h \to \infty$ (deep water wave could be the water wave in a pond)

$v \sim \sqrt{g \cdot \lambda}$.

Combine 1 and 2, we can form two dimensionless quantities.

$$v/\sqrt{g\lambda} \quad \text{and} \quad h/\lambda.$$

Hence we have $v = \sqrt{g\lambda}\, \Phi(h/\lambda)$, $\rightarrow v = \sqrt{g\lambda} \tanh\left(\sqrt{h/\lambda}\right)$

Φ satisfies, $x \to \infty$, $\Phi(x) \to$ const
$x \to 0$, $\Phi(x) \to \sqrt{x}$
$\Rightarrow$ try $\Phi(x) = \tan \sqrt{x}$

1° shallow, $h \to 0$. v is independent on λ,
2 deep $h \to \infty$, $v \propto \lambda^{1/2}$.