Lecture 2. Dimensional analysis (II)

§ Casimir force

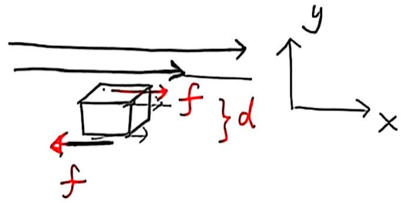
§ Bending a plastic rod $F = Y \frac{r^4}{\ell^2} \cdot \alpha$

§ Fluid mechanics — Reynold number $R = \frac{\rho v d}{\eta} \Rightarrow [\eta] =$ angular momentum dencity

flux - poiseuille formula $Q = \frac{\Delta P}{\Delta l} \frac{d^4}{\eta} \Gamma(R)$

drag coeffient $f = \rho v^2 d^2 C(R)$

Wind tunnel design



unit of η $\frac{f}{A} = \eta \frac{\partial v_x}{\partial y}$

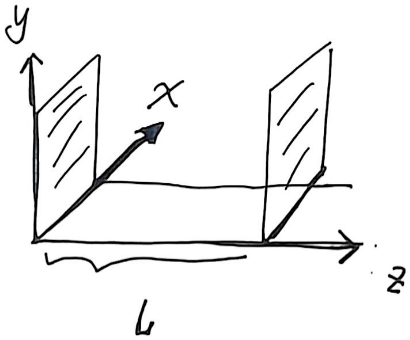
$\tau^{-1} = \frac{\partial v_x}{\partial y}$

time constant

$\Rightarrow \frac{f \tau \cdot d}{A \cdot d} = \eta \Rightarrow \boxed{\frac{\Delta L}{Vol} = \eta}$

Hence $\boxed{\frac{\eta}{\rho v d} = R}$

§1. Casimir force.



Two infinitely large metallic plates put in parallel with distance L in the vaccum. What's the force per area on the plates?

We can set up the Helmholtz Eq for the E&M field with the boundary condition.

$$\boxed{\vec{E}e^{-i\omega t}} \qquad \begin{cases} \nabla^2\vec{E} + k^2\vec{E} = 0 \\ \nabla\cdot\vec{E} = 0 \\ \vec{E}_t\,\big|_{\text{boundary}} = 0 \end{cases}$$

$\vec{E}_t$ is the tangential compnent to the boundary.

$$E_{x,y}(x,y,0) = E_{x,y}(x,y,z=L) = 0 \Rightarrow \boxed{E_{x,y} = E_{0x,y}\, e^{i(k_x x + k_y y)} \sin k_z z} \quad (*)$$

with $k_z = \frac{n\pi}{L}$, $(n=1,2,\cdots)$.

$$\boxed{E_z = E_{0z}\, e^{i(k_x x + k_y y)} \cos k_z z,}$$ such that

$$\nabla\cdot\vec{E} = \left(E_{0x}\, ik_x + E_{0y}\, ik_y - k_z E_{0z}\right) e^{i(k_x x + k_y y)} \sin k_z z = 0$$

$$\Rightarrow \boxed{k_z E_{0z} = i\left(k_x E_{0x} + k_y E_{0y}\right)}$$

Dispersion $\Rightarrow$ $\boxed{k^2 = k_x^2 + k_y^2 + \frac{n^2\pi^2}{L^2}.} \leftarrow \omega = |k|c = \sqrt{k_x^2 + k_y^2 + \left(\frac{n\pi}{L}\right)^2}\cdot c.$

Consider, $k_x = k_y = 0$, $n=1$, $\Rightarrow$ $\begin{cases} E_x = E_{0,x} \sin\frac{\pi}{L} z \\ B_y = B_{0,y} \cos\frac{\pi}{L} z \end{cases}$

a configuration $E_{0,y} = E_{0,z} = 0$.

$$\nabla\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}$$

with $\frac{B_{0,y}}{E_{0,x}} = \frac{c}{i\omega} k_z = -i.$

$$\frac{\partial}{\partial z}E_x - \frac{\partial}{\partial x}E_z = -\frac{1}{c} B_y\,(-i\omega)$$

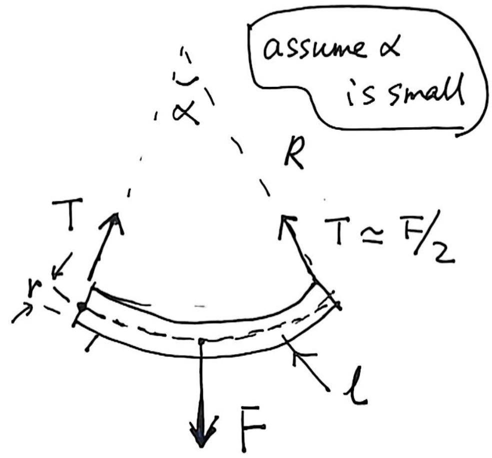
Even in the vaccum, there exist zero point energy of the E & M modes. Nevertheless, the naive calculation results in infinity. Renormalization is needed to extract meaningful finite results. Using dimensional analysis, we arrive at:

$P = F/A$, depends on $L, \hbar, c$. Only one dimensionless quantity can be constructed among $P, L, \hbar, c$.

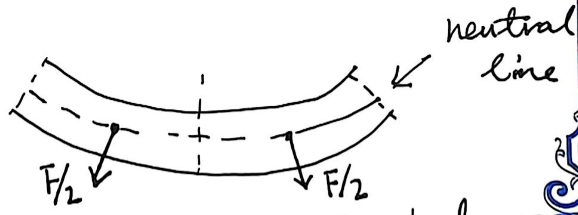
$$\boxed{P = \frac{[E]}{[L^3]} = \frac{\hbar/[T]}{[L^3]} = \frac{\hbar c/[L]}{[L^3]} \sim \frac{\hbar c}{L^4}}$$

Hower, dimensional analysis can not determine P is repulsive or attractive. (Neverthess, recently there appears new understanding. The current and charge fluctuations between two plates coordinate together to ~~al~~ achive an attraction.)

§2. Bending of a plastic ~~rod~~.



A rod of length "l". It's bent under the push of the force F at the center and the forces T at the ends. The bending angle is α. The radius of the rod is r. The rod's Young's modulus is Y.



There exists a neutral line which is kept at the original length "l".

Above the neutral line, it's compressed; Below it is streched.

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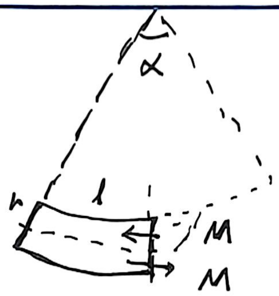
Consider the distortion of the left-hand side.



$$\left.\begin{array}{l} R\cdot\alpha = l \\ (R\pm r)\alpha = l \pm \Delta l \end{array}\right\} \Rightarrow \quad r\cdot\alpha = \Delta l$$

If we fix α, $\boxed{\text{the strain } \frac{\Delta l}{l} = \frac{r}{l}\cdot\alpha}$.

On the other hand, the bending effect $\propto$ the torque $\sim \boxed{F\cdot l}$

Hence, if fix other parameters. $F \propto l^{-2}$.

Then dimensional analysis yield

$$[F] = [P][A] = [Y][L^2] = [Y][r^4][l^{-2}]$$

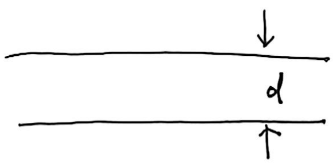
$$P = Y\,\Delta l/l$$

F should proportional to the bending angle $\Rightarrow \boxed{F \simeq Y\frac{r^4}{l^2}\cdot\alpha}$

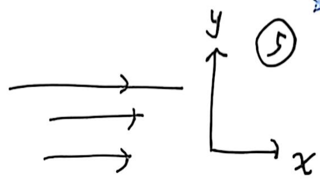
If we apply it to a bow, it means a thicker bow is much more difficult to pull the bow.

§ Fluid mechanics example

Conside a pipe with the diameter d.



Fluid with density ρ, viscocity η, flow velocity v, pressure gradient $\Delta P/\Delta l$. We can construct $5-3=2$ two dimensionless quantities.



Check η's dimenssion: $\dfrac{f}{A} = \eta \cdot \dfrac{\partial v_x}{\partial y}$

$$[\eta] = [F][L]/[L^2]\cdot[v] = [F][T]/[L^2] = \frac{[\text{angular momentum}]}{[L^3]}$$

$$= [M][L]^{-1}[T]^{-1}$$

$$\Delta P/\Delta l = [F]/[L^3] = [M][L]^{-2}[T]^{-2}$$

	M	L	T
ρ	1	-3	0
d	0	1	0
v	0	1	-1
η	1	-1	-1
$\Delta P/\Delta l$	1	-2	-2

$$\Rightarrow \left.\begin{array}{l}[\rho/\eta] = [L]^{-2}[T] \\ [dv] = [L][T]^{-1}\end{array}\right\} \Rightarrow \frac{\rho d v}{\eta} = \Pi_1$$

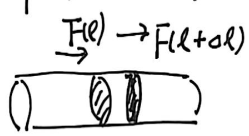
(Reynolds number)

what's the physical meaning of Π_1: consider a ball of diameter d, It falls in the air. The resistance has two contributions of linear drag and quadratic drag. The quddratic drag is due to pushing the air to move with ball. $F_2 = \rho v^2 d^2$. The linear drag is due to viscocity $F_1 \simeq \eta d v$ $\Rightarrow$ $\boxed{\dfrac{F_2}{F_1} = \dfrac{\rho d v}{\eta} = \Pi_1 =} R.$

$\Delta P/\Delta l$ is the force density, $[\Delta P/\Delta l \cdot d] \sim$ energy density

hence $\boxed{\dfrac{(\Delta P/\Delta l)\, d}{\rho v^2} = \Pi_2}$

The Reynolds number has important meaning. For example, the pressure gradient ~~across~~ at a cross section



$$\frac{\Delta F}{\Delta l} = \frac{\Delta P}{l} \cdot S$$

From Π_2, we can express $\Delta P/\Delta l \simeq \frac{\rho v^2}{d} \cdot \Phi(R)$ ← R is another dimensionless quantity

$$\Rightarrow \frac{\Delta F}{\Delta l} \simeq \frac{\rho v^2}{d} \cdot \underbrace{\frac{\pi}{4} d^2}_{S} \Phi(R) \sim \rho v^2 d \, \Phi(R)$$

flux

$$\Rightarrow Q = \bar{v} S = \frac{d}{\rho v \Phi(R)} \frac{\Delta F}{\Delta l} \sim \frac{d^3}{\rho v \Phi(R)} \frac{\Delta P}{\Delta l} = \frac{d^4}{R \Phi(R) \cdot \eta} \frac{\Delta P}{\Delta l}$$

$$\Rightarrow \boxed{Q = \frac{\Delta P}{\Delta l} \frac{d^4}{\eta} P(R)}$$

only rely on $R = \frac{\rho d v}{\eta}$

$$P(R) = \frac{1}{R \Phi(R)}$$

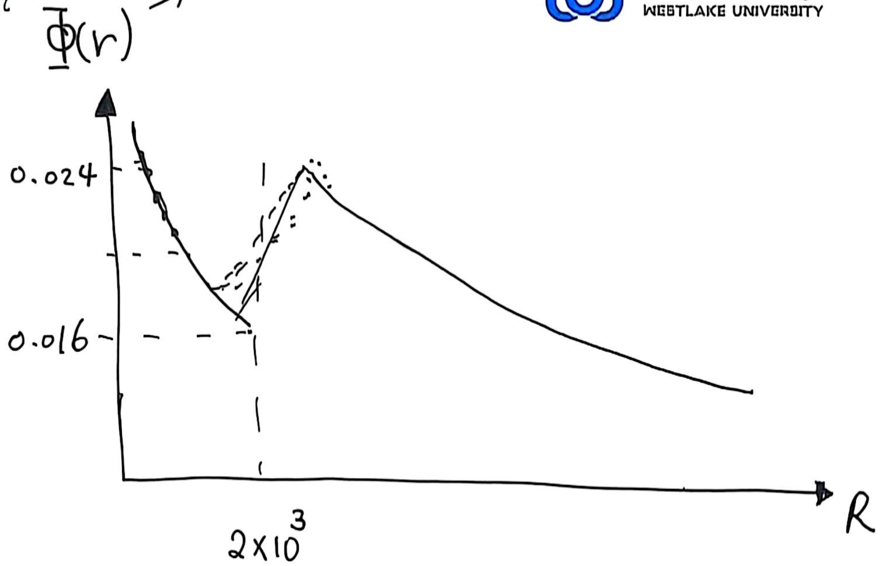
In principle, we could use water as a example to meansure $\Phi(R)$ or, $P(R)$, and apply it to other liquids, even air (when we neglect its compressibility at $v \ll v_{sound}$).

⊛ ~~lx~~ laminar flow v.s. turbulance

We expect two different behavior of $\Phi(R)$, or $P(R)$. For laminar flow, there's no acceleration. The flow velocity is only determined by the friction, not by its density ρ. Hence, $P(R)$ should be a constant, i.e. $Q = \frac{\Delta P}{\Delta l} \frac{d^4}{\eta} \cdot C$ ← $C = \frac{\pi}{128}$ from theory calculation.

$\boxed{\text{(Poiseuille formula)}}$ $\Rightarrow \Phi(R) \propto R^{-1}$

But for turbulance $\Phi(R)$ behaves very differently.



⊛ Drag force revisit

physical quantities: ρ, v, d, f, η $\rightarrow$ 2 dimensionless quantities.

$$[f] = [\text{Pressure}][A] = \frac{[\text{energy}]}{[V]}[A]$$

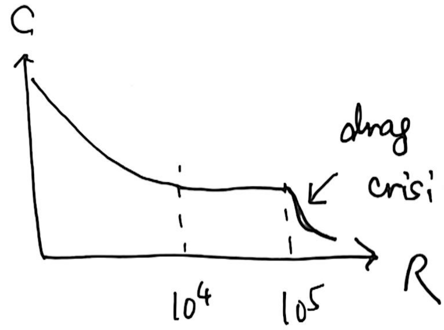
$$= \frac{[M][v]^2}{[V]}[A] = [\rho][v]^2[d]^2$$

Additionally $C(R)$ with $R = \rho v d/\eta$

$$\Rightarrow f = \rho v^2 d^2 \, C(R), \quad C(R) = \frac{C_1}{R} + C_2 + \cdots$$

At R small, drag coefficient $\propto R^{-1}$,

as $R \sim 10^3 \sim 10^5$ $C(R)$ is independent on R.



Wind tunnel: $R = \rho v d/\eta$

If we use model d is smaller than the real object, nevertheless, we can use larger v, or larger ρ to maintain the same Renyolds number.