Lecture 3 Dimensional analysis (III)

§1 Unit systems (heat, ~~temperature~~)

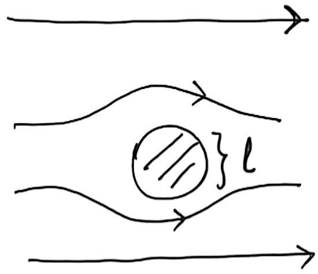
§2. E & M — SI units

— Gaussian units.

§3. natural units.

§ Unit systems

Rayleigh's problem: Heat conduction in a flow.



Incompressible viscosity free fluid.

The object's size l, the ~~temperature~~ difference relative to the infinity $\Delta\Theta$, fluid velocity v, heat capacitity per volume C, heat conductivity K. We need to analysize heat redeasing $\underbrace{\text{rate}}_{H}$ from the object to the fluid.

① Rayleigh's method.

He treats ~~temperature~~ and heat as fundamental quantities, then there're five fundemental ones, (M, L, T, Θ, Q).

Then $[l] = L$, $[v] = LT^{-1}$, $[\Theta] = \Theta$, $[C] = Q\Theta^{-1}L^{-3}$, $[K] = QL^{-1}\Theta^{-1}T^{-1}$

$$\frac{Q}{A\cdot t} = K \cdot \nabla T \qquad [H] = QT^{-1}$$

Then

	M	L	T	Θ	Q
l	0	1	0	0	0
v	0	1	-1	0	0
$\Delta\Theta$	0	0	0	1	0
C	0	-3	0	-1	1
K	0	-1	-1	-1	1
H	0	0	-1	0	1

The first $\underbrace{\text{column}}_{n}$ is all 0's. Hence, the rank is 4. There're $6-4=2$ dimensionless quantites. The first one

$$\left.\begin{aligned} \Pi_1 &= \frac{H}{Kl\Delta\Theta}, \\ \Pi_2 &= \frac{lvc}{K}. \end{aligned}\right\} \Rightarrow H = Kl\Theta\,\Phi\left(\frac{lvc}{K}\right)$$

Nevertheless, now we know heat and energy are the same, We should be able to use the same unit. Then we have M, L, T, Ⓗ 4 fundamental units.

$[l] = L$, $[v] = LT^{-1}$, $[\theta] = Ⓗ$, $[C] = [M][L]^{-1}[T]^{-2}[Ⓗ]^{-1}$

$[K] = [M][L][T]^{-3}[Ⓗ]^{-1}$, $[H] = [M][L]^2[T]^{-3}$

	M	L	T	Ⓗ
l	0	1	0	0
v	0	1	-1	0
$\Delta\theta$	0	0	0	1
C	1	-1	-2	-1
K	1	1	-3	-1
H	1	2	-3	0

Now Q is absent, but M appears.

$\Rightarrow$ still $\Pi_1 = \dfrac{H}{Kl\Delta\theta}$

$\Pi_2 = \dfrac{lvc}{K}$, which is the same as before.

Whether we should treat Q as an independent quantity, depends on our viewpoint. If it's viewed as independent, we should introduce the coefficient of mechanical equivalent of heat $J = 4.2\,J/cal$. Then we add it into table $[J] = [M][L]^2[T]^{-2}[Q]^{-1}$

	M	L	T	Ⓗ	Q
J	1	2	-2	0	-1
l	0	1	0	0	0
v	0	1	-1	0	0
$\Delta\theta$	0	0	0	1	0
C	0	-3	0	-1	1
K	0	-1	-1	-1	1
H	0	0	-1	0	1

$\rightarrow$

	M	L	T	Ⓗ	Q
l	0	1	0	0	0
v	0	1	-1	0	0
$\Delta\theta$	0	0	0	1	0
$C \cdot J$	1	-1	-2	-1	0
$K \cdot J$	1	1	-3	-1	0
$H \cdot J$	1	2	-3	0	0

if we absorb J's unit,

The next step question is whether we should treat ~~temperature~~ and energy as the same. If they're different, we should have $k_B T = E$.

$[k_B] = [M][L^2][T]^{-2}[\Theta]^{-1}$, ← trade off replace $[\Theta]$ by $[M][L^2][T^{-2}]$

	M	L	T	Θ	Q
k_B	1	2	-2	-1	0
J	1	2	-2	0	-1
l	0	1	0	0	0
v	0	1	-1	0	0
$\Delta\Theta$	0	0	0	1	0
C	0	-3	0	-1	1
κ	0	-1	-1	-1	1
H	0	0	-1	0	1

$\xrightarrow{\text{transformation}}$

	M	L	T	Θ	Q
l	0	1	0	0	0
v	0	1	-1	0	0
$k_B\Delta\Theta$	1	2	-2	0	0
$J\,C/k_B$	0	-3	0	0	0
$J\,\kappa/k_B$	0	-1	-1	0	0
$J H$	1	2	-3	0	0

Now we have $6-3=3$ independent unit. Hence in principle, Rayleigh was not so correct. We can have

$$\Pi_1 = \frac{JH}{\frac{J\kappa}{k_B}\, l \cdot k_B\Delta\Theta} = \frac{H}{\kappa l \Delta\Theta}, \qquad \Pi_2 = \frac{l v\,(JC/k_B)}{(J\kappa/k_B)} = \frac{lvc}{\kappa}$$

$$\Pi_3 = \frac{JC}{k_B}\, l^3$$

Hence, we arrive at $H = \kappa l \Delta\Theta\, \Phi\left(\frac{lvc}{\kappa}, \frac{JC}{k_B} l^3\right)$.

It seems more reasonable. Nevertheless, we are not considering stat-mech, and no heat and mechanical energy transfer. Hence k_B and J should not appear. We recover $H = \kappa l \Delta\Theta\, \Phi\left(\frac{lvc}{\kappa}\right)$.

§ E & M units

Whether we should introduce a new unit for E&M? In the SI units, electric current I is defined as a fundmental unit, $[I][T] = [\text{charge}]$, then it's linearly independent from M, L and T. For convenience, we use charge instead of current. If we use charge, denoted as Q (not heat in this section!)

Then $F = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{r^2}$. An additional constant ε_0 is introduced. $[F][\varepsilon_0] = [e^2]/L^{2}$ $\quad [F] = [M][L][T]^{-2}$

$$\Rightarrow \boxed{[\varepsilon_0] = Q^2 M^{-1} L^{-3} T^{2}}$$

- electric field $[E] = [F]/[e] = Q^{-1} M L T^{-2}$

potential voltage $[U] = [E]\,L = Q^{-1} M L^2 T^{-2}$

electric current $[I] = Q T^{-1}$

density $[j] = Q L^{-2} T^{-1}$

conducivity $[\sigma] = [j]/[E] = Q^2 M^{-1} L^{-3} T$

Magnetic field $\vec{F} = e\vec{v}\times\vec{B} \Rightarrow [B] = \frac{[F][v]^{-1}}{[Q]^{-1}} = Q^{-1} M T^{-1}$

Ampère's law $\oint B\cdot dl = \mu_0 I \Rightarrow [\mu] = [B]\cdot L/Q \cdot T = Q^{-2} M L$

→ energy density

$$\boxed{\begin{aligned} u &= \frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2\mu_0}B^2 \\ [u] &= M L^{-1} T^{-2} \end{aligned}}$$

$$\boxed{c^2 = \frac{1}{\mu_0\varepsilon_0}}$$

Nevertheless, ε_0, μ_0 are too cumbersome. Physicists do not like them — E and M are not symmetric either. Rather, we absorb $\frac{e^2}{4\pi\varepsilon_0} \to e^2$ and define $\vec{F} = q(\vec{E} + \frac{\vec{v}}{c} \times \vec{B})$, such that E and B carry the same unit.

The correspondence is as follows: $F = \frac{e^2}{4\pi\varepsilon_0 r^2} \Leftrightarrow F' = \frac{e'^2}{r^2} = F$

$$\Rightarrow \quad e' = \frac{e}{\sqrt{4\pi\varepsilon_0}} \qquad\qquad E = \frac{e}{4\pi\varepsilon_0 r^2} \qquad E' = \frac{e'}{r^2} = \sqrt{4\pi\varepsilon_0}\, E$$

(those with a $'$, means it's in Gaussian unit).

$$e\vec{v} \times \vec{B} = e' \frac{\vec{v}}{c} \times \vec{B}'$$

$$\Rightarrow B' = c\, e/e'\, B = c\sqrt{4\pi\varepsilon_0}\, B.$$

$$= \sqrt{\frac{4\pi}{\mu_0}}\, B$$

Hence, the formula in the SI unit

$$u = \frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2\mu_0} B^2 \leftarrow \boxed{\begin{cases} E = \frac{1}{\sqrt{4\pi\varepsilon_0}} E' \\ B = \sqrt{\mu_0/4\pi}\, B' \end{cases}}$$

$$= \frac{1}{8\pi}\left(E'^2 + B'^2\right)$$

This set of unit is called the Gaussian unit.

$$j = \frac{e}{\Delta t A} \qquad j' = \frac{e'}{\Delta t A} = \frac{1}{\sqrt{4\pi\varepsilon_0}}\, j$$

Now check Maxwell equation

$$\nabla \cdot \vec{E} = \rho/\varepsilon_0 \quad \longrightarrow \quad \nabla \cdot \frac{1}{\sqrt{4\pi\varepsilon_0}} \vec{E}' = \sqrt{4\pi\varepsilon_0}\, \rho'/\varepsilon_0 \Rightarrow \nabla \cdot \vec{E}' = 4\pi\rho'$$

$$\nabla \cdot \vec{B} = 0 \quad \longrightarrow \quad \nabla \cdot \vec{B}' = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \longrightarrow \quad \nabla \times \frac{1}{\sqrt{4\pi\varepsilon_0}} \vec{E}' = -\frac{\partial}{\partial t}\sqrt{\frac{\mu_0}{4\pi}}\, \vec{B}' \Rightarrow \nabla \times \vec{E}' = -\frac{1}{c}\frac{\partial}{\partial t}\vec{B}'$$

$$\nabla \times \vec{B} = \mu_0 \vec{j} + \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} \quad \rightarrow \quad \nabla \times \sqrt{\frac{\mu_0}{4\pi}}\, B' = \mu_0\sqrt{4\pi\varepsilon_0}\, j' + \frac{1}{c^2}\frac{\partial}{\partial t}\frac{1}{\sqrt{4\pi\varepsilon_0}} E'$$

$$\nabla \times \vec{B}' = 4\pi\sqrt{\mu_0\varepsilon_0}\, j' + \frac{1}{c^2}\frac{1}{\sqrt{\mu_0\varepsilon_0}}\frac{\partial}{\partial t} E' \Rightarrow \nabla \times \vec{B}' = \frac{4\pi}{c} j + \frac{1}{c}\frac{\partial}{\partial t}\vec{E}'$$

§ Natural unit (SI)

atomic length $a = \frac{\hbar^2}{me^2} \cdot 4\pi\varepsilon_0 \simeq 0.5\,\mathring{A}$, $\frac{\hbar^2}{ma^2} = \frac{e^2}{4\pi\varepsilon_0 a}$

Ryderberg energy $E = \frac{e^2}{4\pi\varepsilon_0 a} = \frac{me^4}{(4\pi\varepsilon_0)^2\hbar^2} = 27.2\,V$

	Q	M	L	T
$4\pi\varepsilon_0$	2	-1	-3	2
e^2	2	0	0	0
$\hbar$	0	1	2	-1
c	0	0	1	-1
m	0	1	0	0
I	1	0	0	-1
U	-1	1	2	-2
R	-2	1	2	-1
C	2	-1	-2	2
ϕ	-1	1	2	-1
L	-2	1	2	0
E	-1	1	1	-2
B	-1	1	0	-1
$\frac{\mu_0}{4\pi}$	-2	1	1	0
P	1	0	1	0
磁矩 $m = IA$	1	0	2	-1
$\frac{\mu_0}{4\pi}$	-2	1	1	0

$\Rightarrow \Pi_1 = \frac{e^2}{4\pi\varepsilon_0 \hbar c} = \frac{1}{137} = \alpha$

$\alpha a = \frac{\hbar}{mc}$ λ: Compton length

$\alpha^2 a = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{mc}$: electron classic radius

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Quantum unit of R. C. L

R: $\boxed{R_Q = h/e^2} = \frac{6.6\times10^{-34}}{1.6^2\times10^{-38}} = 25.8\,k\Omega$

C: $\left[\frac{C}{4\pi\varepsilon_0}\right] = L \Rightarrow \frac{C}{4\pi\varepsilon_0} = \frac{\hbar^2}{me^2}\cdot 4\pi\varepsilon_0$

$\Rightarrow \boxed{C_Q = a\cdot 4\pi\varepsilon_0} = \frac{0.5\times10^{-10}}{9\times10^{9}}\ F$

$= 5.5\times10^{-21}\ F$

L: $[L\cdot 4\pi\varepsilon_0] = L^{-1}T^2 \Rightarrow = L\left(T/L\right)^2$

$L_Q \cdot 4\pi\varepsilon_0 = \frac{\hbar^2}{me^2}\cdot 4\pi\varepsilon_0 \frac{1}{c^2}$

$L_Q = \frac{\hbar^2}{me^2}\frac{1}{c^2} = \boxed{\frac{a}{4\pi\varepsilon_0}\frac{1}{c^2} = L}$

$\omega_Q = \frac{1}{\sqrt{L_Q C_Q}} = \frac{c}{a} \sim 6\times10^{18}\,Hz$ $\quad = \frac{0.5\times10^{-10}\times9\times10^{9}}{9\times10^{16}}\,H = 5\times10^{-18}\,H$

Gaussian unit   Q is not an indepedent unit

	Q	M	L	T
$4\pi\varepsilon_0$	2	-1	-3	2
$\mu_0/4\pi$	-2	1	1	0
$e'^2 = e^2/4\pi\varepsilon_0$	0	1	3	-2
$\hbar$	0	1	2	-1
c	0	0	1	-1
m	0	1	0	0
$I' = \frac{I}{\sqrt{4\pi\varepsilon_0}}$	0	1/2	3/2	-2
$U' = U\sqrt{4\pi\varepsilon_0}$	0	1/2	1/2	-1
$\phi' = \frac{\phi}{\sqrt{\frac{\mu_0}{4\pi}}}$	0	1/2	3/2	-1
$E' = E\sqrt{4\pi\varepsilon_0}$	0	1/2	-1/2	-1
$B' = \frac{B}{\sqrt{\frac{\mu_0}{4\pi}}}$	0	1/2	-1/2	-1
$R' = R\,4\pi\varepsilon_0$	0	0	-1	1
$C' = \frac{C}{4\pi\varepsilon_0}$	0	0	1	0
$L' = L\,4\pi\varepsilon_0$	0	0	-1	2

	Q	M	L	T
$p' = \frac{p}{\sqrt{4\pi\varepsilon_0}}$	0	1/2	5/2	-1
$m' = m\sqrt{\frac{\mu_0}{4\pi}}$	0	1/2	5/2	-1

$$L'I' = LI\sqrt{4\pi\varepsilon_0}$$
$$= \Phi\sqrt{4\pi\varepsilon_0}$$
$$= \Phi'\sqrt{\frac{\mu_0}{4\pi}\cdot 4\pi\varepsilon_0} = \frac{\Phi'}{c}$$

$$\Rightarrow \boxed{L'I' = \frac{\Phi'}{c}}$$

Gaussian unit.

$$\Rightarrow \boxed{\begin{array}{l} a = \frac{\hbar^2}{m e'^2}, \quad E = \frac{m e'^4}{\hbar^2}, \quad \alpha = \frac{e'^2}{\hbar c}, \quad \alpha a = \frac{\hbar}{mc}, \quad \alpha^2 a = \frac{e'^2}{mc} \\ R'_Q = h/e'^2, \quad C'_Q = a, \quad L'_Q = L\cdot 4\pi\varepsilon_0 = \frac{a}{c^2} \end{array}}$$

It's much simpler