Lecture 4 : Scaling analysis (I)

§1. Metabolism and animal size

2. take-off speed of bird

3. bending stability of a rod

4. airplane's flying

§ Scaling of size — animals at different size

Metabolism rate

① Conside an animal size l, then its body area $S \propto l^2$, the body weight $W \propto l^3$. How much food does it need to eat?

Suppose the animal maintains a fixed ~~temperature~~ difference from the enviroment $\Delta\theta$. Then the heat released $Q \propto l^2$. This amount of energy comes from food, hence the food needed $\propto W^{\frac{2}{3}}$. Or per unit body weight, the food needed $\propto W^{\frac{2}{3}}/W = W^{-1/3}$.

A man of body weight 60 kg, his/her basic metabolism rate is roughly 2000 kcal. A squirrel body weight ~ 0.5 kg. Hence it needs energy $\simeq \left(\frac{0.5}{60}\right)^{\frac{2}{3}} \times 2000\,\text{kcal} = 0.04 \times 2000 \approx 85\,\text{kcal}$.

The metabolism rate per unit body weight is 5 times large than human.

Carbonhydrate food energy density 4 kcal/gram, hence, a man roughly need 500g grains to live. A squirrel needs 20g (starchy) nuts to live.

How much food does a cat or a dog need? Please observe your cat and dog.

② take off speed — Why does ostrich not fly?

The lift force f; area of Wing S, air density ρ, take-off speed v

$\Rightarrow$ dimensional analysis $f = C \cdot \rho v^2 \cdot S$ — C: Constant at order of 1.

$f = mg$

$$\left.\begin{aligned} &\Rightarrow v \propto \sqrt{\frac{mg}{\rho S}} \propto \sqrt{m/S} \\ &m \propto l^3,\quad S \propto l^2 \end{aligned}\right\} \Rightarrow v \propto l^{1/2}$$

A swallow's the minimal flying speed is about 20 km/h.

An ostrich size is about 25 times larger in size, hence $v \simeq 5 \times 20$ km $= 100$ km/h.

Or ~~reduce taken~~ increasing wing area S, but not too much on body weight.

③ ~~fuel cost rate — air-plane or car?~~

stability of a rod under gravity

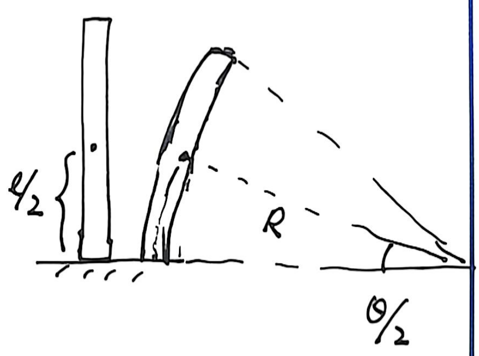
In previous lectures, we figured out the internal storge

$$M_i \propto \frac{Y r^4}{l^2}\theta \cdot l = \frac{Y r^4}{l} \cdot \theta$$

hence $\Delta E_{elastic} \propto M_i \cdot \theta \sim \frac{Y r^4}{l}\theta^2$

and the gravity potential energy gain

$$\Rightarrow \Delta E_p \simeq mg\Delta h = mg\frac{l\theta^2}{48}$$



$$R \cdot \frac{\theta}{2} = \frac{l}{2}$$

$$\Delta h = \frac{l}{2} - R\sin\frac{\theta}{2}$$

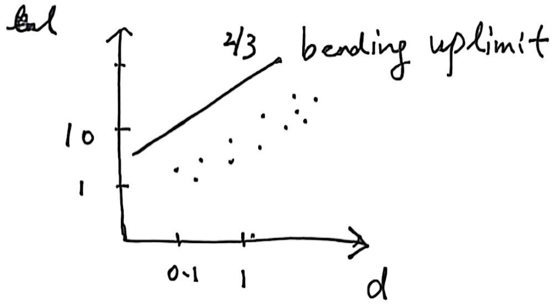
$$= \frac{l}{2} - R\left(\frac{\theta}{2} - \frac{1}{3!}\left(\frac{\theta}{2}\right)^2\right)$$

$$\simeq \frac{l\theta^2}{48}$$

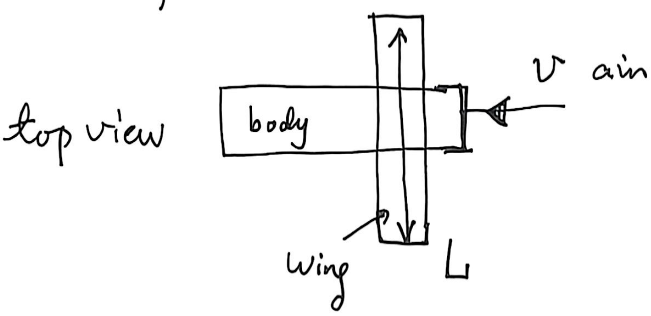
Hence, the instability occurs when

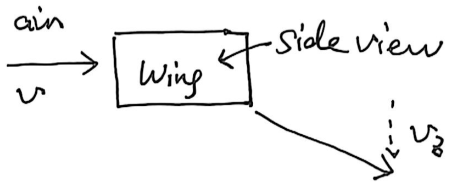
$$\left.\begin{array}{l} \dfrac{mg\,l_c}{48} > \dfrac{Y r^4}{l_c} \\ m = \rho \pi r^2 \cdot l_c \end{array}\right\} \Rightarrow l_c^3 \propto \frac{Y r^2}{\rho g}$$

$$\Rightarrow \quad l_c \propto r^{2/3}$$

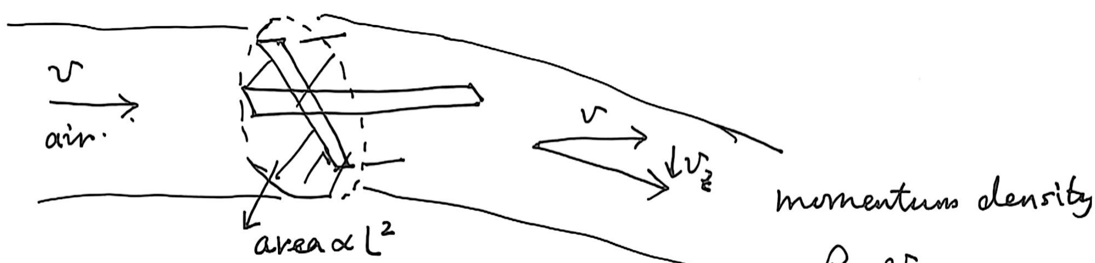


- Airplane





airplaine needs to push air to gain the lift.



The cross section $\propto L^2$: the reason is that this is largest length scale, The air flow in the ~~...~~ area $\propto L^2$ is affected. Hence

$$mg \cdot \Delta t \sim \rho_{air} v_z \cdot v \Delta t L^2 \Rightarrow v_z \sim \frac{mg}{\rho_{air} L^2 v}$$

① Then the power to produce the lift

$$P_{lift} \sim mg v_z \sim (mg)^2 / \rho_{air} L^2 v$$

Boering 747-400 jet $L \sim 60\, m$, $m \sim 4\times10^5\, Kg$ (400 ton)

v takeoff $80\, m/s \sim 290\, km/h$.

(check with swallow $l \sim 10\sim20\, cm$, then $\sqrt{L/l} \sim \sqrt{400} \sim 20$

$\Rightarrow$ $v_{swallow} \sim 15\, km/h$ reasonable)

Then $v_z \sim mg / \rho_{air} L^2 v \approx \dfrac{4\times10^6}{1.2\times60^2\times80} \approx 12\, m/s$.

$$P_{lift} \sim mg v_z \simeq 4\times10^6 \times 12 \simeq 5\times10^7\, W \approx 5\times10^4\, KW.$$

② The airplane also need to fly. The resistance

$$f_{res} \propto \rho_{air} \cdot v^2 \cdot A_{cross} = c\, \rho_{air} v^2 L^2$$ C is a coefficient

$$P_{fly} = f_{res} v = c\, \rho_{air} v^3 L^2$$

to fly over a distance of d, the time taken is $\frac{d}{v} = t$

Then $E = (P_{lift} + P_{fly})\dfrac{d}{v} = \left[\dfrac{(mg)^2}{\rho_{air} L^2 v^2} + c\,\rho_{air} v^2 L^2\right] \cdot d$

$$v_m L \sim m^{1/2} c^{-1/2}$$

$\Rightarrow$ E_{min} is reached at

$$c(\rho_{air} L^2 v^2)^2 = (mg)^2, \text{ i.e. } mg \sim \sqrt{c}\, \rho_{air} L^2 v^2$$

or $\boxed{mg \cdot \sqrt{c} \sim f_{res}}$ $\Rightarrow$ $\boxed{E_{min} = 2\sqrt{c}\, mg d}$

Hence, the maximal distance that can be traveled $d_m \sim \dfrac{E_{fuel}}{\sqrt{C}\, mg}$

$E_{fule} = \beta m \mathcal{E}_{fuel} \leftarrow$ energy density

$\Rightarrow (*)\ d_m \sim \dfrac{\beta \mathcal{E}_{fuel}}{\sqrt{c}\, g}$ $\qquad \beta$: the fuel/weight ratio

① $\beta \sim 40\%$

② $\left.\begin{array}{l} \mathcal{E}_{fuel} \simeq 10\,\text{kcal/gram} \simeq 4\times10^{7}\,\text{J/kg} \\ \text{engine efficiency} \sim 20\sim30\% \end{array}\right\} \Rightarrow \mathcal{E}_{fuel,eff} \simeq 1\times10^{7}\,\text{J/kg}.$

③ Boering 747: $f = \frac{1}{2} c' A_{wing} \rho v^2 \qquad c' \approx 0.22$

$= C L^2 \rho v^2 \Rightarrow C = \frac{1}{2} c' \dfrac{A_{wing}}{L^2} = \dfrac{c'}{2}\dfrac{l}{L} = 0.002$

assume $l/L \sim 1/6$

$\Rightarrow d_m \sim \dfrac{0.4\times10^{7}}{\sqrt{2\times10^{-3}}\times 10}\, m \simeq \dfrac{4\times10^{5}}{4\times10^{-2}}\, m \sim 10^{7} m = 10^{4}\, km$ $\Bigg\}$ Boering 747 $d_{max} \simeq 13450\, km$. !

From (*), the maximal fly distance is independent of the mass. ~~If airplan and bird, and~~ If, β, η, c are comparable in birds and airplane, then d_m are nearly the same! Indeed bird can fly across the Pacific from Alaska to New Zealand.

~~speed at the optimal~~, $\cancel{v \sim m^{1/2}/l \sim l^{\frac{3}{2}-1} = l^{1/2} \text{ or } m^{1/6}}$.