Lecture 5 Scaling analysis (II)

§ Diffusion and drift — Einstein relation

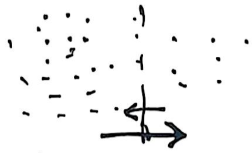
§ random walk

§ Similarity solution

§ How long does a photon take to escape from the core of the sun?

§ Diffusion and drift

Suppose a distribution of particles $n(x,t)$, the current due to diffusion $\vec{J}_{diff} = -D\nabla n$.



particles also move under external force field. The friction coefficent μ, then $\underbrace{\vec{F} = \mu\vec{v} = -f}$.

the force is balanced by friction.

Can we relate D to μ ?

Cheack dimension $[D] = [n][v]\cdot[L]/[n] = [L]^2[T^{-1}]$

$$[\mu] = [F][v]^{-1} = [F\cdot v][v]^{-2} = [E][T^{-1}]^{+2}[L]^2$$
$$= [E][D]^{-1}$$

$\Rightarrow \boxed{[D][\mu] = [E]}$ $\leftarrow$ interesting, where to find an characteristic energy? — temperature k_BT

$\boxed{D \sim T/\mu}$

Einstein relation: $\vec{J}_{drift} = n\vec{v} = -n\vec{\nabla}V/\mu$

$$\left.\begin{array}{l}\vec{J}_{diff} = -D\nabla n \\ n = n_0 e^{-V/k_BT} \Rightarrow \nabla n = n(-\nabla V)/k_BT\end{array}\right\} \Rightarrow \vec{J}_{diff} = \frac{-Dn}{k_BT}\nabla V$$

$$\Rightarrow \frac{Dn}{k_BT} = \frac{n}{\mu}, \text{ i.e. } D = \frac{k_BT}{\mu}$$ } at equlibrium $\vec{J}_{drift} + \vec{J}_{diff} = 0$

§ Random walk — binomial distribution to normal distribut.

step length l; m steps to left and $N-m$ steps to right

$x = (N-2m)l \Rightarrow m = \frac{N}{2} - \frac{x}{2l}$. then the probability

$$p(x) = 2^{-N}\binom{N}{m} = \frac{N!}{2^N\, m!\,(N-m)!} \leftarrow \text{Stirling's}$$

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

$$= \frac{(N/2)^N}{\left(\frac{N}{2}-\frac{x}{2l}\right)^{\frac{N}{2}-\frac{x}{2l}}\left(\frac{N}{2}+\frac{x}{2l}\right)^{\frac{N}{2}+\frac{x}{2l}}} \cdot \underbrace{\frac{\sqrt{N}}{\sqrt{\frac{N}{2}-\frac{x}{2l}}\sqrt{\frac{N}{2}+\frac{x}{2l}}}}_{\text{indep on } x}$$

as $N \gg x$.

$$\to \left(1-\frac{x}{Nl}\right)^{-\left(\frac{N}{2}-\frac{x}{2l}\right)}\left(1+\frac{x}{Nl}\right)^{-\left(\frac{N}{2}+\frac{x}{2l}\right)}$$

$$\ln p(x) = -\left(\frac{N}{2}-\frac{x}{2l}\right)\ln\left(1-\frac{x}{Nl}\right) - \left(\frac{N}{2}+\frac{x}{2l}\right)\ln\left(1+\frac{x}{Nl}\right)$$

$\Rightarrow p(x) =$

$$\approx -\left(\frac{N}{2}-\frac{x}{2l}\right)\left(-\frac{x}{Nl}-\frac{1}{2}\left(\frac{x}{Nl}\right)^2\right) - \left(\frac{N}{2}+\frac{x}{2l}\right)\left(\frac{x}{Nl}-\frac{1}{2}\left(\frac{x}{Nl}\right)^2\right)$$

$$= -\frac{x^2}{2Nl^2}\cdot 2 + \frac{N}{4}\left(\frac{x}{Nl}\right)^2\cdot 2 = -\frac{x^2}{2Nl^2}$$

$$\Rightarrow p(x) \propto e^{-\frac{x^2}{2\sigma}} \quad \text{with } \sigma = \langle x^2\rangle = Nl^2$$

$$\boxed{p(x) = \frac{1}{\sqrt{2\pi\sigma}}\, e^{-\frac{x^2}{2\sigma}}}$$

$$\langle x^2\rangle = \int_{-\infty}^{+\infty} dx\, x^2 e^{-\frac{x^2}{2\sigma}} \Big/ \int_{-\infty}^{+\infty} dx\, e^{-\frac{x^2}{2\sigma}} = ?$$

$$\boxed{\langle x^2\rangle = \frac{1}{2}\cdot 2\sigma = \sigma}$$

$$f(a) = \int_{-\infty}^{+\infty} dx\, e^{-ax^2} = \sqrt{\pi/a}$$

$$\frac{d}{da}f(a) = \int_{-\infty}^{+\infty} dx\,(-x^2)e^{-ax^2} = -\frac{1}{2}\sqrt{\frac{\pi}{a^3}} \Rightarrow \int_{-\infty}^{+\infty} dx\, x^2 e^{-ax^2} \Big/ \int_{-\infty}^{+\infty} dx\, e^{-ax^2} = \frac{1}{2a}$$

Random walk:

continuity condition $\frac{\partial n}{\partial t} + \nabla \cdot \vec{j} = 0$

$\Rightarrow$ $\boxed{\frac{\partial n}{\partial t} = D \nabla^2 n}$ $\quad t=0 \quad n(x,t) = I_0 \, \delta(x)$

t , $n(x,t) = ?$

characteristic quantities: $[t] = T$, $[x] = L$, $[D] = L^2 T^{-1}$

$[n] = [I_0] L^{-1}$,

we treat $[N], [T], [L]$ as fundament, we could build two dimensionless quantities, $\frac{I_0}{n\sqrt{Dt}}$, $\frac{x}{\sqrt{Dt}}$, i.e. $(Dt)^{1/2}$ carries the length a unit

$$\Rightarrow \quad \frac{n\sqrt{Dt}}{I_0} = \phi\left(\frac{x}{\sqrt{Dt}}\right). \quad \text{set } \xi = \frac{x}{\sqrt{Dt}}$$

$\Rightarrow n(x,t) = \frac{I_0}{\sqrt{Dt}} \phi(\xi)$, how to determine $\phi(\xi)$.

$$\frac{\partial (n/I_0)}{\partial t} = -\frac{1}{2t} \frac{1}{\sqrt{Dt}} \phi(\xi) + \frac{1}{\sqrt{Dt}} \frac{d}{d\xi}\phi(\xi) \frac{x}{\sqrt{Dt}} \left(-\frac{1}{2t}\right)$$

$$\frac{\partial^2}{\partial x^2}\left(\frac{n}{I_0}\right) = \frac{1}{\sqrt{Dt}} \frac{d^2}{d\xi^2}\phi = \left(\frac{d^2}{d\xi^2}\phi\right)(Dt)^{-3/2}$$

$$\Rightarrow \quad \frac{d^2}{d\xi^2}\phi \cdot \cancel{D^{-1/2} t^{-3/2}} = -\frac{1}{2} \cancel{t^{-3/2} D^{-1/2}} \left(\phi(\xi) + \xi \frac{d}{d\xi}\phi \right)$$

$$\frac{d^2\phi}{d\xi^2} + \frac{\xi}{2}\frac{d\phi}{d\xi} + \frac{\phi}{2} = 0 \Rightarrow \frac{d}{d\xi}\left(\frac{d\phi}{d\xi} + \frac{\xi\phi}{2} \right) = 0$$

$\frac{d\phi}{d\xi} + \frac{\xi\phi}{2} = C_0$. As $\xi \to 0$, $\phi(\xi) \sim \xi^{-1-\alpha}$ $\quad \alpha > 0$

hence $\frac{d\phi}{d\xi} \to 0$, $\xi\phi \to 0$ as $\xi \to +\infty$

$\Rightarrow C_0 = 0 \Rightarrow \phi = C e^{-\frac{\xi^2}{4}}$

$$n = \frac{I_0}{\sqrt{4\pi D t}} e^{-\frac{x^2}{4Dt}} \quad \text{with} \int_{-\infty}^{+\infty} dx\, n(x) = I_0$$

compare $\begin{cases} \sigma = 2Dt = N\ell^2 \\ t = N\tau \end{cases} \Rightarrow \underline{2D\tau \sim \ell^2}$

mean free path and mean free time

§ How long does a photon take to escape from the center of the sun to the surface

photon - electron scattering cross section

length $r_0 = \frac{e^2}{mc^2}$, i.e. $\sigma \propto \left(\frac{e^2}{mc^2}\right)^2 \cdot \pi$

more precisely $\sigma = \frac{8}{3}\pi\left(\frac{e^2}{mc^2}\right)^2$

$$\begin{cases} \frac{e^2}{a} = 27.2\,\text{eV} \\ \frac{e^2}{r_0} = 0.5\times10^6\,\text{eV} \end{cases} \Rightarrow \sigma \approx \left(\frac{27.2}{0.5\times10^6}\times0.5\times10^{-8}\right)^2 \cdot \pi\ \text{cm}^2 \approx 2.3\times10^{-25}\ \text{cm}^2$$

The radius of sun $\sim 7\times10^5$ km, overall density $1.4\ \text{g/cm}^3$

but the core of the sun where the density is $150\ \text{g/cm}^3$

$r_c \simeq 1.4\times10^5$ km. Most time is spent to pass the

& the core.

$1\ \text{cm}^3 \simeq 150$ mole hydrogen $\Leftrightarrow$ 150 mole electrons

$6\times10^{23}\times150 \sim 1\times10^{26}$ electrons average volume per electron in the core of the sun

$10^{-26}\ cm^3$ $\Rightarrow$ mean free path $l \simeq \frac{10^{-26}}{2.3 \times 10^{-25}}\ cm$

$\simeq 0.4\ mm$

$$\left.\begin{array}{l} D \cdot T \simeq 2 r_c^2 \\ D \cdot \tau \simeq 2 l^2 \Rightarrow D \cdot \frac{l}{c} \simeq 2 l^2 \end{array}\right\} \Rightarrow 2 l c T \simeq 2 r_c^2$$

$$T = \left(\frac{r_c}{l}\right)\left(\frac{r_c}{c}\right) = \frac{1.4 \times 10^8}{0.4 \times 10^{-3}} \cdot \frac{1.4 \times 10^5}{3 \times 10^5}\ s = 3 \times 10^{11} \times \frac{1.4}{3}$$

$$\left.\begin{array}{l} \simeq 1.4 \times 10^{11}\ s \\ 1\,yr \simeq 3 \times 10^7\ s \end{array}\right\} \Rightarrow T \simeq 0.5 \times 10^4\ year \simeq 5000\ year.$$