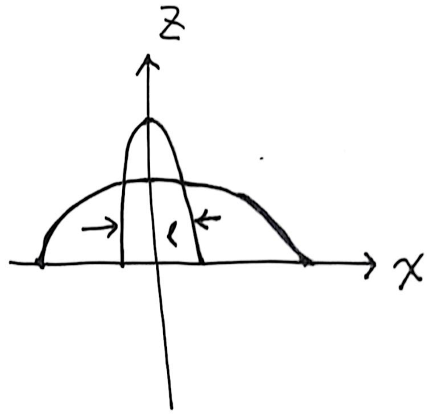
Lecture 6. Scaling analysis (Ⅲ)

§. Groundwater flow $\partial_t H = \kappa \partial_x^2 H^2$

§ self-similar solution

§. The intermediate asymptotics



$$\Pi_2 = \frac{l}{(I\kappa t)^{1/3}}$$

$$\Pi_3 = \frac{H_i}{(I^2/\kappa t)^{1/3}}$$

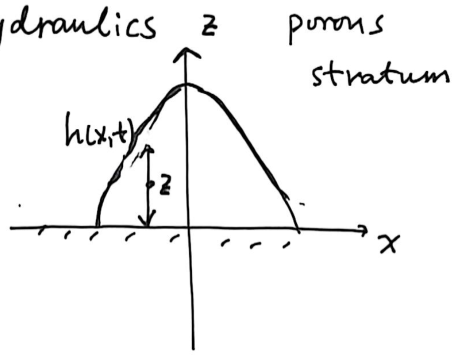
$$H = H_0(t)\, f\left(\frac{x}{x_f(t)}\right)$$

$$H_0(t) \propto \left(\frac{I^2}{\kappa t}\right)^{1/3}$$

$$x_f \propto (I\kappa t)^{1/3}$$

$$H = \rho g h + p \,.= \rho g z$$

§ Gently sloping groundwater flow — hydraulics



$p = \rho g(h-z) \rightarrow$ water head

$H = p + \rho g z = \rho g h$

Darcy's law $\quad q = -\frac{k}{\mu} h \partial_x H$

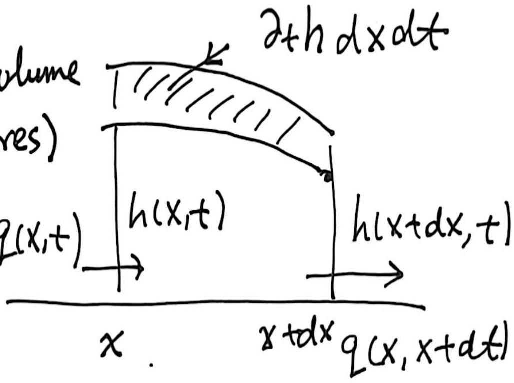
total flux through an area of unit width $[q] = \frac{[V]}{T \cdot L} = \frac{L^2}{T}$

μ: viscocity $\quad [\mu] = [E] \cdot T / L^3$

water head $\quad [H] = [E] \cdot L^{-3}$

$$\Rightarrow \frac{L^2}{T} = \frac{[k]}{[E] \cdot T\, L^{-3}} \cdot \frac{[E]}{L^3} \Rightarrow [k] = L^2$$

porosity m (fraction of volume occupied by pores)



$m \partial_t h\, dt\, dX$ — change of water

$= -\partial_x q\, dt\, dX \qquad$ similarly to $\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{j} = 0$

$$\Rightarrow m \partial_t h = \frac{k}{\mu} \partial_x (h \partial_x H) = \frac{k}{\mu \rho g} \partial_x (H \partial_x H)$$

$\boxed{\partial_t H = \chi \partial_x^2 H^2}$ where $\chi = \frac{k}{2m\mu} \qquad [\chi] = \frac{L^3 T}{M}$

check $\quad [T]^{-1} = [\chi][L]^{-2} [M][L]^{-1}[T]^{-2}$

(Boussinesq Eq).

Self-similar solution:

Consider an intense flood: $\int_{-l}^{l} H_0(x, t=0)\, dx = I$.

$H_0(x) = I/l\, f_0(x/l)$. Then $H = f(t, I, \kappa, l, x)$

$[H] = M/LT^2$, $[t] = T$, $[I] = M/T^2$, $[\kappa] = \frac{L^3 T}{M}$, $[l] = [x] = L$

treat t, I, κ as fundamental quantities,

$$\begin{cases} [H] = [t]^{-1/3}[I]^{2/3}[\kappa]^{-1/3} \\ [x] = [l] = [(I\kappa t)]^{1/3} \end{cases} \Rightarrow \frac{H}{I^{2/3}(\kappa t)^{-1/3}} = \Phi\left(\frac{x}{(I\kappa t)^{1/3}}, \frac{l}{(I\kappa t)^{1/3}}\right)$$

Consider the limit $l \to 0$, $\Rightarrow$

$$H = \left(\frac{I^2}{\kappa t}\right)^{1/3} \Phi(\xi, 0) = \left(\frac{I^2}{\kappa t}\right)^{1/3} f(\xi). \text{ with } \xi = \frac{x}{(I\kappa t)^{1/3}}$$

$$\rightarrow \partial_t H = -\frac{1}{3t} H + \left(\frac{I^2}{\kappa t}\right)^{1/3} \frac{df}{d\xi}\left(-\frac{1}{3t}\right)\xi$$

$$\partial_x H^2 = \left(\frac{I^2}{\kappa t}\right)^{2/3} \frac{df^2}{d\xi} \frac{1}{(I\kappa t)^{1/3}}.$$

$$\partial_x^2 H^2 = \left(\frac{I^2}{\kappa t}\right)^{2/3} \frac{1}{(I\kappa t)^{2/3}} \frac{d^2 f^2}{d\xi^2}$$

$$\Rightarrow \kappa \left(\frac{I^2}{\kappa t}\right)^{2/3} \frac{1}{(I\kappa t)^{2/3}} \frac{d^2 f^2}{d\xi^2} + \frac{1}{3t}\left(\frac{I^2}{\kappa t}\right)^{1/3}\left(f(\xi) + \xi \frac{df}{d\xi}\right) = 0$$

$$\frac{I^{\frac{2}{3}}}{\kappa^{1/3} t^{4/3}}$$

$$\Rightarrow \frac{d^2 f^2}{d\xi^2} + \frac{\xi}{3}\frac{df}{d\xi} + \frac{f}{3} = 0.$$

$$\frac{d}{d\xi}\left(\frac{df^2}{d\xi} + \frac{\xi F}{3}\right) = 0.$$

At $x = 0, \rightarrow \xi = 0$

$$df^2/d\xi = f\frac{df}{d\xi} = 0$$

$$\xi f = 0$$

$$\Rightarrow \frac{df^2}{d\xi} = -\frac{\xi}{3} f \quad \text{or} \quad \frac{1}{f}\frac{df^2}{} = -\frac{\xi}{3} d\xi$$

$$df = -\frac{\xi d\xi}{6}$$

$f = -\frac{1}{12}\xi^2 + \text{Const} \leftarrow$ denote as $\frac{1}{12}\xi_f^2$

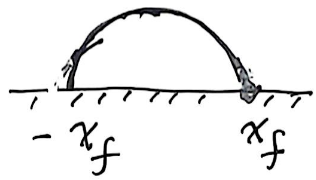
then $f = \begin{cases} \frac{1}{12}(\xi_f^2 - \xi^2), & 0 \le |\xi| \le \xi_f \\ 0 & |\xi| > \xi_f \end{cases}$ with $\xi = \frac{x}{(I\kappa t)^{1/3}}$

and $H = \begin{cases} \frac{1}{12}\left(\frac{I^2}{\kappa t}\right)^{1/3}\left(\xi_f^2 - \frac{x^2}{(I\kappa t)^{2/3}}\right) & 0 \le |x| \le x_f = \xi_f (I\kappa t)^{1/3} \\ 0 & |x| > x_f \end{cases}$

At $H \neq 0$, $\partial_x H$ should be continuous.

$\partial_x(H\partial_x H) \rightarrow \partial_x H$ may be discontinuous at $H = 0$.

at the water front



Now let's determine ξ_f

$$\frac{d}{dt}\int_{-x_f}^{x_f} H(x,t)\,dx = \kappa\,\partial_x H^2 \Big|_{-x_f}^{x_f} = 0 - 0 = 0$$

Hence $I(t) = \int_{-x_f}^{x_f} H(x,t)\,dx$ is a const of time.

$$\int_{-x_f}^{x_f} \frac{1}{12}\left(\frac{I^2}{\kappa t}\right)^{1/3}\left(\xi_f^2 - \frac{x^2}{(I\kappa t)^{2/3}}\right)dx = I$$

$$\frac{1}{12}\int \frac{dx}{(I\kappa t)^{1/3}}\left(\xi_f^2 - \xi^2\right) = \frac{\xi_f^2}{12}\int_{-\xi_f}^{\xi_f} d\xi\left(1 - \frac{\xi^2}{\xi_f^2}\right) = 1$$

$\Rightarrow \xi_f = 3^{\frac{2}{3}}$

$x_f = (9I\kappa t)^{2/3}$

$$H(x,t) = \begin{cases} \frac{3\sqrt[3]{3}}{4}\left(\frac{I^2}{\kappa t}\right)^{1/3}\left(1 - \frac{x^2}{(9I\kappa t)^{2/3}}\right), & 0 < |x| < x_f \\ 0. & |x| > x_f. \end{cases}$$

Self-similar solution

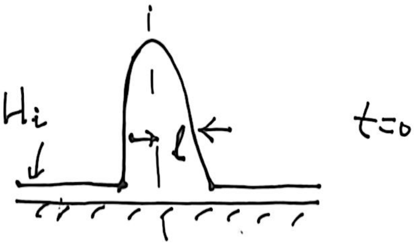
$$H = H_0(t)\, f\left(\frac{x}{x_f(t)}\right) \rightarrow \text{则 } H/H_0(t) = f\left(\frac{x}{x_f(t)}\right)$$

$$H_0(t) = \frac{\sqrt[3]{3}}{4}\left(\frac{I^2}{\kappa t}\right)^{1/3}, \qquad x_f(t) = (9 I \kappa t)^{1/3}$$

↖ single curve.

☆ Intermediate asymptotics

In principle, the above ~~solutions~~ solution is an ideal case. The inital ~~wave~~ water head distribution may have a length scale of l_0. The inital water head may have a finite value H_i as $x \rightarrow \infty$.



$$\Pi_2 = \frac{l}{(I\kappa t)^{1/3}}. \qquad \Pi_3 = \frac{H_i}{(I^2/\kappa t)^{1/3}}$$

For, Π_2, we expect that at sufficiently large time.

$$x_f >> l, \quad \text{i.e} \quad (9 I \kappa t)^{1/3} >> l$$

or $t >> \dfrac{l^3}{(I\kappa)} = T_1$ in this case $\Pi_2 \rightarrow 0$, the initial shape of the water head is no longer important

For Π_3. we need $\quad \Pi_3 << 1 \iff x_f << I/H_i$

This means the the ~~height of~~ the ~~initial~~ wave head remain $>> H_i$

$$9(I\kappa t)^{1/3} << I/H_i$$

$$\Rightarrow t << I^2/(\kappa H_i^3) = T_2$$

Within the period, $T_2 >> t >> T_1$, the self-similar solution is valid.

Correspondingly, the length scales $L_1 = l$, $L_2 = I/H_i$.