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DIRAC MONOPOLE WITHOUT STRINGS: MONOPOLE HARMONICS

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Using the ideas developed in a previous paper which are borrowed from the mathematics of fibre bundles, it is shown that the wave function ψ of a particle of charge Ze around a Dirac monopole of strength g should be regarded as a *section*. The section is without discontinuities. Thus the monopole *does not* possess strings of singularities in the field around it. The eigensections of the angular momentum operators are monopole harmonics which are explicitly exhibited.

1. Introduction

In this paper, and a later on classical Lagrangian dynamics, we study the formulation of Dirac's magnetic monopoles without strings. The two papers are, however, logically and technically independent, and may be read separately.

Very soon after Dirac's original paper [1], Tamm [2] studied the wave function of an electrically charged particle around a magnetic monopole. He introduced "generalized spherical harmonics" for such wave functions. These harmonics possess discontinuities or cusps. Later Fierz [3] discussed these harmonics from a different point of view†.

Since the space around a monopole is spherically symmetrical and without singularities, the wave function of a positon or electron around the monopole *should* have no singularities. An examination of this question using the concepts developed

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† There are many papers after the publication of refs. [1–3] dealing with magnetic monopoles and generalized spherical harmonics which are more or less similar to the essence of these papers. We shall not refer to these later papers here.

in a recent paper [4] shows that this is indeed the case. By a conceptual change, we shall look at the generalized spherical harmonics from a new view point and shall call them *monopole harmonics*. The monopole harmonics are *everywhere analytic* and possess no discontinuities or cusps at all. They form a complete orthonormal set and can be used as the basis of expansion of any wave function around the monopole.

In this new view point, the wave function of an electrically charged particle of charge Ze around a monopole of strength g *should not be thought of as an ordinary function.* It should instead be considered as a "*section*" characterized by a number q defined by

$$q = \tfrac{1}{2}DZ \,, \tag{1}$$

where $D = 2eg$ = monopole strength in Dirac's unit which is $(2e)^{-1}$. We have put

$$c = \hbar = 1 \,.$$

D is an integer which may be positive, negative or zero. So is $2q$.

The concept of a section is familiar in the mathematics of fibre bundles. For the case in question, the wave function is mathematically [5] a section on a C^1 vector bundle, or a line bundle.

2. Wave function as a section

The basic new point is understandable as follows. The cusps and discontinuities arise because any choice of the vector potential $\boldsymbol{A}$ around the monopole must [4] have singularities. The situation is similar to that encountered in the choice of a coordinate system on the surface of a sphere, such as the longitude and latitude system. No choice is possible which does not have some singularities. Yet the geometry of the sphere is clearly without intrinsic singularities. To avoid introducing singularities in the coordinate system one divides the sphere into more than one overlapping region and defines a singularity-free coordinate system in each region. In the overlap one has singularity-free coordinate transformations between the different coordinate systems.

Imitating this method, we divide [4] the space outside of a magnetic monopole into two regions, R_a and R_b, and define a vector potential $(A_\mu)_\mathrm{a}$ in R_a and a vector potential $(A_\mu)_\mathrm{b}$ in R_b. Using spherical coordinates r, θ, ϕ with the monopole at the origin we choose

$$R_\mathrm{a}: \quad 0 \leqslant \theta < \tfrac{1}{2}\pi + \delta \,, \quad 0 < r \,, \quad 0 \leqslant \phi < 2\pi \,, \tag{2}$$

$$R_\mathrm{b}: \quad \tfrac{1}{2}\pi - \delta < \theta \leqslant \pi \,, \quad 0 < r \,, \quad 0 \leqslant \phi < 2\pi \,, \tag{3}$$

$$R_\mathrm{ab}: \quad \tfrac{1}{2}\pi - \delta < \theta < \tfrac{1}{2}\pi + \delta \,, \quad 0 < r \,, \quad 0 \leqslant \phi < 2\pi \quad \text{(overlap)}, \tag{4}$$

where we choose δ such that $0 < \delta \leqslant \frac{1}{2}\pi$.

The vector potentials are chosen to be

$$(A_r)_a = (A_\theta)_a = 0\,, \qquad (A_\phi)_a = \frac{g}{r\sin\theta}(1-\cos\theta)\,,$$

$$(A_r)_b = (A_\theta)_b = 0\,, \qquad (A_\phi)_b = \frac{-g}{r\sin\theta}(1+\cos\theta)\,, \tag{5}$$

where A_r, A_θ, A_ϕ are the projections of $\boldsymbol{A}$ in the three local orthogonal directions. One has

$$(A_\mu)_a = (A_\mu)_b + \frac{i}{Ze} S_{ab} \frac{\partial S_{ab}^{-1}}{\partial x^\mu}\,, \tag{6}$$

where

$$S = S_{ab} = e^{2iq\phi} = \text{transition function}\,. \tag{7}$$

S is the gauge transformation phase factor for changing from $(A_\mu)_b$ to $(A_\mu)_a$ in the overlap R_{ab},

$$\psi_a = S_{ab}\,\psi_b\,, \tag{8}$$

where ψ_a and ψ_b are the wave function of a particle of charge Ze in R_a and R_b, respectively. A function ξ which assumes values ξ_a and ξ_b in R_a and R_b and satisfies

$$\xi_a = S_{ab}\,\xi_b = e^{2iq\phi}\,\xi_b \tag{9}$$

in the overlap R_{ab} is called a *section. ψ is thus a section.*

Let the charged particle interact with the monopole and with a potential $V(r)$ which is spherically symmetrical. We assume $V(r)$ to be without singularities for $r > 0$. Then

$$\frac{1}{2m}(\boldsymbol{p} - Ze\boldsymbol{A})^2\,\psi + V\psi = E\psi\,, \tag{10}$$

meaning

$$\frac{1}{2m}(\boldsymbol{p} - Ze\boldsymbol{A}_a)^2\,\psi_a + V\psi_a = E\psi_a\,, \qquad \text{in } R_a\,, \tag{10a}$$

$$\frac{1}{2m}(\boldsymbol{p} - Ze\boldsymbol{A}_b)^2\,\psi_b + V\psi_b = E\psi_b\,, \qquad \text{in } R_b\,. \tag{10b}$$

It is obvious that these equations are compatible with (8) because of (6).

3. Hilbert space of sections

It is clear that if ξ is a section, $x\xi$ is also a section. Also $(p - ZeA)_x \xi$ is a section. Thus $\boldsymbol{r}$ and $\boldsymbol{p} - Ze\boldsymbol{A}$ are operators on the Hilbert space of sections. We define the scalar product of two sections ξ, η as

$$(\eta, \xi) \equiv \int \eta^* \xi \, \mathrm{d}^3 r \,. \tag{11}$$

(The question of convergence at $r = \infty$ and $r = 0$ is here ignored.) This integral is well defined because in the overlap R_{ab}

$$\eta_{\mathrm{b}}^* \xi_{\mathrm{b}} = \eta_{\mathrm{a}}^* \xi_{\mathrm{a}} \,.$$

It is clear that $\boldsymbol{r}$ and $\boldsymbol{p} - Ze\boldsymbol{A}$ are Hermitian operators. Following Fierz [3] we shall now try to construct angular momentum operators.

Define

$$\boldsymbol{L} = \boldsymbol{r} \times (\boldsymbol{p} - Ze\boldsymbol{A}) - \frac{q\boldsymbol{r}}{r} \,. \tag{12}$$

It is clear that L_x, L_y, L_z are Hermitian operators on the Hilbert space of sections. The following commutation rules can be easily verified:

$$[L_x, x] = 0 \,, \qquad [L_x, y] = iz \,, \qquad [L_x, z] = -iy \,,$$

$$[L_x, p_x - ZeA_x] = 0 \,, \qquad [L_x, p_y - ZeA_y] = i(p_z - ZeA_z) \,,$$

$$[L_x, p_z - ZeA_z] = -i(p_y - ZeA_y) \,. \tag{13}$$

It follows from these that

$$[L_x, L_y] = iL_z \,, \qquad \text{etc} \,. \tag{14}$$

Eq. (13), together with its consequence (14), show that L_x, L_y, L_z are *the angular momentum operators* [3]. We emphasize that neither the Hilbert space, nor these operators, possess any "singularities". (The singularities of A_{a} and A_{b} are not real singularities because they occur outside of R_{a} and R_{b}, respectively.)

4. Monopole harmonics $Y_{q,l,m}$

Since $[r^2, \boldsymbol{L}] = 0$, we can diagonalize r^2 and study operators $\boldsymbol{L}$ for fixed r^2. I.e. we shall study sections of the form

$$\delta(r^2 - r_0^2)\xi \,,$$

where ξ is a section dependent only on angular coordinates θ and ϕ.

$\boldsymbol{L}$ operates then on "angular sections". In the rest of this paper except sect. 11, we shall be dealing with angular sections only.

Eq. (14) shows that $[\boldsymbol{L}^2, L_z] = 0$. Simultaneous diagonalization produces the familiar multiplets with eigenvalues $l(l+1)$ and m,

$$\boldsymbol{L}^2 Y_{q,l,m} = l(l+1)\, Y_{q,l,m}\,; \; L_z Y_{q,l,m} = m Y_{q,l,m}\,, \tag{15}$$

where $l = 0, \frac{1}{2}, 1, \ldots$ and, for each value of l, m ranges from $-l$ to $+l$ in integral steps of increment. The $Y_{q,l,m}$ are the eigensections which we shall call *monopole harmonics.* We shall show later that the allowed values of l and m are

$$l = |q|, |q|+1, |q|+2, \ldots, \qquad m = -l, -l+1, \ldots, l\,, \tag{16}$$

and that each of these l, m combinations occur exactly once. We shall choose each Y normalized so that

$$\int_0^{\pi} \sin\theta \, d\theta \int_0^{2\pi} |Y_{q,l,m}|^2 \, d\phi = 1\,. \tag{17}$$

(Notice that in R_{ab}, $|(Y_{q,l,m})_a|^2 = |(Y_{q,l,m})_b|^2$.) Different $Y_{q,l,m}$ (for a fixed q) are orthogonal, a fact one easily proves in the usual way from (15). We shall choose the phases of $Y_{q,l,m}$ such that the matrix elements of L_z, L_y, L_z between the Y's conform to the convention adopted in ch. 2 of Edmonds' book [6]. In particular

$$(L_x + iL_y) Y_{q,l,m} = (l-m)^{1/2}(l+m+1)^{1/2}\, Y_{q,l,m+1}\,. \tag{18}$$

These monopole harmonics will be explicitly exhibited. Each is analytic. That is, $(Y_{q,l,m})_a$ is analytic in R_a and $(Y_{q,l,m})_b$ is analytic in R_b. The set of all monopole harmonics for a fixed q forms a complete set of sections, as we shall see.

5. Explicit expressions for $Y_{q,l,m}$

Stating from (12) one easily verifies

$$\boldsymbol{L}^2 = [\boldsymbol{r} \times (\boldsymbol{p} - Ze\boldsymbol{A})]^2 + q^2\,, \tag{19}$$

$$m Y_{q,l,m} = L_z Y_{q,l,m} = (-i\partial_\phi - q) Y_{q,l,m}\,, \qquad \text{in } R_a\,,$$

$$m Y_{q,l,m} = L_z Y_{q,l,m} = (-i\partial_\phi + q) Y_{q,l,m}\,, \qquad \text{in } R_b\,. \tag{20}$$

Eq. (20) shows that

$$Y_{q,l,m} = \Theta_{q,l,m}(\theta)\, e^{i(m+q)\phi} \text{ in } R_a\,,$$

$$Y_{q,l,m} = \Theta_{q,l,m}(\theta)\, e^{i(m-q)\phi} \text{ in } R_b\,. \tag{21}$$

The condition for a section, (9), shows that $[\Theta_{q,l,m}(\theta)]_a = [\Theta_{q,l,m}(\theta)]_b$ in the overlap. They are, in fact, the same function. Apply (19) to $Y_{q,l,m}$. An explicit evaluation of the operator $[\boldsymbol{r} \times (\boldsymbol{p} - Ze\boldsymbol{A})]^2$ acting on $Y_{q,l,m}$ gives

$$[l(l+1) - q^2]\, \Theta_{q,l,m} = \left[-\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}(m + q\cos\theta)^2\right]\Theta_{q,l,m}\,. \tag{22}$$

Writing $\cos\theta = x$, this gives

$$[l(l+1) - q^2]\,\Theta = -(1-x^2)\,\Theta'' + 2x\Theta' + \frac{1}{1-x^2}(m+qx)^2\,\Theta\;,$$

$$-1 \leqslant x \leqslant 1\;, \tag{23}$$

where prime means differentiation with respect to x. This equation can be treated in the usual way, through analyzing the indical equations at $x = \pm 1$. We shall, however, pursue a different method which yields the normalization constant and phase factor automatically.

Before proceeding we note that since Y is single valued in each region, (21) shows that

$$m - q = \text{integer}\,.$$

Thus

$$l - q = \text{integer}\,. \tag{24}$$

Now (19) shows that

$$l(l+1) \geqslant q^2\,. \tag{25}$$

Eqs. (24) and (25) show that the allowed values of l are among those given in (16).

We shall now show that each value of l in (16) is allowed, by constructing, for each of them, the explicit function $\Theta_{q,l,m}$:

$$\Theta_{q,l,-l} = N_{q,l}\sqrt{1-x}^{\,l-q}\sqrt{1+x}^{\,l+q}\,, \qquad l - |q| = \text{integer} \geqslant 0\,, \tag{26}$$

where

$$N_{q,l} = \left[\frac{(2l+1)!}{4\pi(2^{2l})(l-q)!\,(l+q)!}\right]^{1/2} > 0\,. \tag{27}$$

To show this one substitutes (26) into (23) and verifies that the latter is satisfied. The factor $N_{q,l}$ is inserted so that $Y_{q,l,-l}$ is normalized in the sense of (17).

Repeated application of (18) onto the monopole harmonics $Y_{q,l,-l}$ (given by (21) and (26)) leads to, (for l, m satisfying (26)), the explicit expression for $Y_{q,l,m}$

given below. (As stated above, this method leads to automatically normalized $Y_{q,l,m}$ starting from normalized $Y_{q,l,-l}$.)

$$(Y_{q,l,m})_a = M_{q,l,m}(1-x)^{\alpha/2}(1+x)^{\beta/2} P_n^{\alpha,\beta}(x)\, e^{i(m+q)\phi} ,$$

$$(Y_{q,l,m})_b = (Y_{q,l,m})_a\, e^{-2iq\phi} , \tag{28}$$

where

$$\alpha = -q - m, \qquad \beta = q - m, \qquad n = l + m, \qquad x = \cos\theta , \tag{29}$$

$$M_{q,l,m} = 2^m \left[\frac{2l+1}{4\pi}\, \frac{(l-m)!\,(l+m)!}{(l-q)!\,(l+q)!}\right]^{1/2} , \tag{30}$$

and $P_n^{\alpha,\beta}(x)$ are [7] the Jacobi polynomials,

$$P_n^{\alpha,\beta}(x) = \frac{(-1)^n}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^n}{dx^n}[(1-x)^{\alpha+n}(1+x)^{\beta+n}] , \tag{31}$$

which are defined if

$$n,\ n+\alpha,\ n+\beta \text{ and } n+\alpha+\beta \text{ are all integers} \geqslant 0 . \tag{32}$$

Eq. (28) will be proved in appendix A, and some properties of the Jacobi polynomials will be discussed in appendix B.

6. Completeness of monopole harmonics

For a given q (q may be negative) the set of $Y_{q,l,m}$ with l, m satisfying (16) form a complete set of orthonormal sections. I.e. every continuous section (i.e. a section satisfying (9), with ξ_a and ξ_b being continuous in R_a and R_b) can be expanded as a series

$$\sum_{l,m} a_{l,m} Y_{q,l,m} .$$

Proof: According to appendix C, $Y_{q,l,m}$ can be expressed in terms of $P_\nu^{|\alpha|,|\beta|}(x)$. Now for fixed q = integer or half-integer, and $q+m$ = integer, there are four possible cases:

$$\alpha \geqslant 0,\ \beta \geqslant 0,\ \text{so that } -m \geqslant |q| \quad \text{and} \quad \nu = l+m , \tag{33}$$

$$\alpha \geqslant 0,\ \beta \leqslant 0,\ \text{so that } |m| \leqslant -q,\quad q \leqslant 0 \quad \text{and } \nu = l+q , \tag{34}$$

$$\alpha \leqslant 0,\ \beta \geqslant 0,\ \text{so that } |m| \leqslant q,\ q \geqslant 0 \text{ and } \nu = l-q , \tag{35}$$

$$\alpha \leqslant 0,\ \beta \leqslant 0,\ \text{so that } m \geqslant |q| \quad \text{and } \nu = l-m . \tag{36}$$

In case (33), the allowed values of l, according to (16), are $l = |m|, |m| + 1, \ldots$ which are precisely

$$\nu = 0, 1, 2, \ldots . \tag{37}$$

In case (34), the allowed values of l according to (16) are $l = -q, -q + 1, \ldots$ which are also precisely (37). Continuing this way we conclude that given q = integer or half-integer, $q + m$ = integer, the allowed values of l according to (16) are always precisely those given by (37).

Now for fixed $|\alpha|$, $|\beta|$, the Jacobi polynomials $P_\nu^{|\alpha|, |\beta|}$, ($\nu = 0, 1, 2, \ldots$) form [7] a complete set. The exponential functions $e^{i\phi(m+q)}$, ($m + q$ = all integers) also form a complete set. It can be proved from these results that $Y_{q,l,m}$ forms a complete set of sections for fixed q.

7. Examples and analyticity of $Y_{q,l,m}$

For the case $q = 0$, $\alpha = \beta$, and (31) shows that

$$P_{l+m}^{-m, -m} = \frac{(-1)^m}{2^m} \frac{l!}{(l+m)!} (1 - x^2)^{m/2} P_l^m , \tag{38}$$

Table 1
Examples of $\sqrt{4\pi} Y_{q,l,m}$ in region a

q	l	m	$(\sqrt{4\pi} Y_{q,l,m})_a$
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-e^{i\phi}\sqrt{1-x}$
	$\frac{1}{2}$	$-\frac{1}{2}$	$e^0\sqrt{1+x}$
	$\frac{3}{2}$	$\frac{3}{2}$	$\sqrt{3/2}\, e^{2i\phi}\sqrt{1+x}(1-x)$
	$\frac{3}{2}$	$\frac{1}{2}$	$-\sqrt{1/2}\, e^{i\phi}\sqrt{1-x}(1+3x)$
	$\frac{3}{2}$	$-\frac{1}{2}$	$-\sqrt{1/2}\, e^0\sqrt{1+x}(1-3x)$
	$\frac{3}{2}$	$-\frac{3}{2}$	$\sqrt{3/2}\, e^{-i\phi}\sqrt{1-x}(1+x)$
1	1	1	$\sqrt{3/4}\, e^{2i\phi}(1-x)$
	1	0	$-\sqrt{3/2}\, e^{i\phi}\sqrt{1-x^2}$
	1	-1	$\sqrt{3/4}\, e^0 (1+x)$

$x = \cos\theta$. To obtain $Y_{q,l,m}$ in R_b apply (9).

where P^l_m is the associated Legendre function. Substitution of (38) into (28) shows that

$$Y_{0,l,m} = \text{usual spherical harmonics } Y_{l,m} \ .$$

We tabulate in table 1 a few of the monopole harmonics for $q = \frac{1}{2}$, 1, These examples illustrate the fact that $Y_{q,l,m}$ *is analytic everywhere.* I.e., $(Y_{q,l,m})_a$ is analytic in R_a and $(Y_{q,l,m})_b$ is analytic in R_b. For example, $(Y_{\frac{1}{2}\frac{1}{2}\frac{1}{2}})_a$ is clearly analytic in R_a, which includes the point $\theta = 0$, and

$$(Y_{\frac{1}{2}\frac{1}{2}\frac{1}{2}})_{b\,1/2})_b = \sqrt{1-\cos\theta}/\sqrt{4\pi} \qquad (39)$$

is clearly analytic in R_b which includes the point $\theta = \pi$.

8. Zeros of $Y_{q,l,m}$

Table 1 shows that each of the $Y_{q,l,m}$ exhibited has at least one zero. This is in fact a special case of a general topological theorem that for $q \neq 0$, any continuous section must have at least one zero. This theorem can be proved as follows. If ξ is a continuous section and has no zeros, trace the value of $\xi_a|\xi_a|^{-1}$ in *the complex plane* as one goes along the parallel $r = 1, \theta = \theta_0$, from $\phi = 0 \to 2\pi$. $\xi_a|\xi_a|^{-1}$ (= the phase of ψ_a) describes a loop which is confined to the unit circle. As θ_0 changes, the loop is *continuously* distorted, remaining always on the unit circle. As $\theta_0 \to 0$, the loop shrinks to a point. Thus for any θ_0 satisfying $0 < \theta_0 < \frac{1}{2}\pi + \delta$ (see (2)), the loop is always shrinkable, along the circle, to a point. The same is true for the loop described by $\xi_b|\xi_b|^{-1}$ for $\frac{1}{2}\pi - \delta < \theta_0 < \pi$. Now take $\theta_0 = \frac{1}{2}\pi$. These two last statements together contradict (9) if $q \neq 0$.

9. Global gauge transformation on $Y_{q,l,m}$

The monopole harmonics exhibited above are for a special gauge [4] in which the regions R_a, R_b and the vector potential A_μ were chosen to be that given in (2) → (5). One can make global gauge transformations [4] which change the regions, the vector potential A_μ, and the value of $Y_{q,l,m}$ in a coordinated manner.

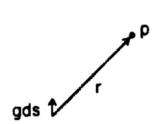

Fig. 1. Pseudomagnetic field produced by current segment ds. It is equal to $g\boldsymbol{ds} \times \boldsymbol{r}r^{-3}$ where g = current.

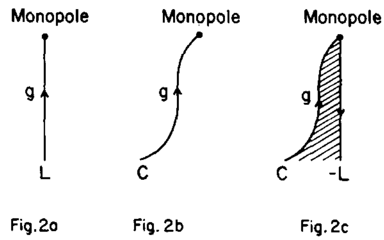

Fig. 2. Half line L and half curve C carrying current g, and complete circuit $C + (-L)$.

For example, one could make a global gauge transformation by merely contracting and expanding the regions R_a and R_b without changing either A_μ or $Y_{q,l,m}$, provided R_a and R_b together always fill the whole space outside of the origin, and provided R_a does not include the line $\theta = \pi$ while R_b does not include the line $\theta = 0$.

To discuss a more interesting global gauge transformation we shall first try to find some other possible vector potential A'_a in R_a. To this end, define as in fig. 1 the pseudomagnetic field produced, according to Biot–Savart's law, from a segment of electric current g. (The field is called a pseudomagnetic field and not a magnetic field, because of two related facts: (i) The current segment itself does not give conserved current. (ii) The pseudomagnetic field is not curlless outside of the current segment. The total pseudomagnetic field produced by a *complete* electric circuit is the magnetic field.) It is easy by straightforward integration to find that (5) satisfies

$$(A)_a = \text{total pseudomagnetic field generated by half line } L\,, \quad \text{carrying current } g \text{ (fig. 2a)}\,. \tag{40}$$

Now define

$$(A')_a = \text{total pseudomagnetic field generated by any half curve } C\,, \quad \text{carrying current } g \text{ (fig. 2b)}\,. \tag{41}$$

Then

$$\begin{aligned}(A')_a - (A)_a &= \text{total pseudomagnetic field generated by } C + (-L)\,, \\ &\qquad \text{carrying current } g \text{ (fig. 2c)}\,. \\ &= \text{magnetic field generated by same}\,.\end{aligned} \tag{42}$$

Thus

$$\nabla \times [(A')_a - (A)_a] = 0 \text{ outside of } L \text{ and } C\,. \tag{43}$$

If C is chosen completely outside of R_a, then (43) asserts that $\nabla \times (A')_a$ is the magnetic field of the monopole in R_a and thus we may use $(A')_a$ as the vector potential in R_a.

Using $(A'_\mu)_a$ and $(A_\mu)_b$ as the vector potentials requires a global gauge transformation from $(A_\mu)_a$ and $(A_\mu)_b$. In other words, we can find a transformation phase factor $T_{a'a}$ such that

$$(A'_\mu)_a = (A_\mu)_a + \frac{i}{Ze} T_{a'a} \frac{\partial T_{a'a}^{-1}}{\partial x^\mu} \text{ in } R_a . \tag{44}$$

For any section ξ,

$$\xi_{a'} = T_{a'a} \xi_a . \tag{45}$$

Comparison of (44) and (42) shows that

$$T_{a'a} = \exp(-i\, Zeg\, \Omega) = \exp(-i\, q\Omega) , \tag{46}$$

where Ω is the solid angle subtended by circuit $C + (-L)$ in fig. 2c at the point where $T_{a'a}$ is evaluated. Actually Ω is defined and is continuous not just in R_a but in all space outside of a surface bordered by the closed circuit $C + (-L)$. Take the surface to be the shaded area in fig. 2c. On the surface, but outside of the border $C + (-L)$, Ω increases discontinuously by 4π in going from above the diagram to underneath the diagram. Since D is an integer, $T_{a'a}$ is single valued.

The transition function in the overlap region R_{ab} is now

$$S_{a'b} = T_{a'a} S_{ab} = \exp(2iq\phi - iq\Omega) . \tag{47}$$

We have thus defined completely the new gauge: regions R_a, R_b and transition function $S_{a'b}$. We have also defined the gauge field in this new gauge: $(A'_\mu)_a$, $(A_\mu)_b$. We have, in addition, exhibited the global gauge transformation $T_{a'a}$ between the old gauge and the new.

Notice that the continuation of $(A'_\mu)_a$ into R_b yields singularities not on L, but on C.

If we had taken another half curve C' outside of R_a, we would have gotten another new vector potential $(A''_\mu)_a$. Any linear combination

$$(A'''_\mu)_a = \alpha(A_\mu)_a + \alpha'(A'_\mu)_a + \alpha''(A''_\mu)_a , \tag{48}$$

with

$$\alpha + \alpha' + \alpha'' = 1 ,$$

where α, α' and α'' are positive or negative real numbers, is also a possible vector potential in R_a. Notice that the continuation of $(A'''_\mu)_a$ into R_b has, in general, singularities on L, C and C'. Thus the position of the singularities of $(A_\mu)_a$, when continued into space not covered by R_a, is in general quite arbitrary.

10. Rotation of coordinate axes

A rotation of coordinate axes generates [6] a linear combination of the usual spherical harmonics,

$$Y_{0,l,m}(\theta', \phi') = \sum_{m'=-l}^{l} Y_{0,l,m'}(\theta,\phi)\mathcal{D}^{(l)}_{m'm} , \tag{49}$$

where $\mathcal{D}$ depends on the rotation. Does this hold also for the case $q \neq$ o? Define

$$Z_{q,l,m}(\theta', \phi') = \sum_{m'=-l}^{l} Y_{q,l,m'}(\theta, \phi)\mathcal{D}^{(l)}_{m'm} , \tag{50}$$

for $q \neq 0$. $Z_{q,l,m}$ is $Y_{q,l,m}(\theta'\phi')$ but in *a different gauge*, because the vector potential A_μ has not yet been changed to that which conforms with convention (5) for the new coordinate system. If one performs a global gauge transformation on $Z_{q,l,m}$ by first changing $(A_\mu)_a$ so that its singularities after continuation become the new negative z-axis, and then changing $(A_\mu)_b$ so that its singularities after continuation become the new positive z-axis then $Z_{q,l,m} \rightarrow Y_{q,l,m}(\theta',\phi') \times$ (phase factor which is independent of m).

11. Schrödinger equation

It is simple to show by explicit evaluation, and with the aid of (19) that

$$(\boldsymbol{p} - Ze\boldsymbol{A})^2 = -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\,[\boldsymbol{r} \times (\boldsymbol{p} - Ze\boldsymbol{A})]^2$$

$$= -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\,[\boldsymbol{L}^2 - q^2] . \tag{51}$$

The Hamiltonian in (10) thus commutes with $\boldsymbol{L}^2$ and L_z. Hence in solving (10) we can choose specific eigenvalues for $\boldsymbol{L}^2$ and L_z. I.e. we take

$$\psi = R(r)Y_{q,l,m} , \tag{52}$$

obtaining

$$\left[-\frac{1}{2mr^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{l(l+1) - q^2}{2mr^2} + V - E\right]R = 0 . \tag{53}$$

For the case that $V = 0$ this equation was solved by Tamm [2] who found that R is a Bessel function, if $E > 0$,

$$R = \frac{1}{\sqrt{kr}}\,J_\mu(kr) , \tag{54}$$

where

$$\mu = \sqrt{l(l+1) - q^2 + \tfrac{1}{4}} = \sqrt{(l+\tfrac{1}{2})^2 - q^2} > 0\,, \tag{55}$$

$$k = \sqrt{2mE}\,.$$

If $E \leqslant 0$, (53) has no meaningful solution.

It is a pleasure to thank Professor Shiing-shen Chern for enlightening us on the mathematical concepts of fibre bundles and sections.

Appendix A

Proof of (28)

A straight forward computation shows that in R_a

$$L_x + iL_y = e^{i\phi}\left[-\sqrt{1-x^2}\,\frac{\partial}{\partial x} + i\frac{x}{\sqrt{1-x^2}}\frac{\partial}{\partial\phi} - q\sqrt{\frac{1-x}{1+x}}\right], \tag{A.1}$$

where

$$x = \cos\theta\ .$$

Substitute this into (18) and use (21). One obtains, in R_a,

$$\Theta_{q,l,m+1} = \frac{1}{\sqrt{(l-m)(l+m+1)}}\left[-\sqrt{1-x^2}\,\frac{d}{dx} - \frac{mx}{\sqrt{1-x^2}} - q\frac{1}{\sqrt{1-x^2}}\right]\Theta_{q,l,m}$$

$$= \frac{-\sqrt{1-x^2}^{\,m+1}}{\sqrt{(l-m)(l+m+1)}}\left[\frac{d}{dx} + q\,\frac{1}{1-x^2}\right]\sqrt{1-x^2}^{\,-m}\,\Theta_{q,l,m}\,. \tag{A.2}$$

Repeated application of (A.2) gives

$$\Theta_{q,l,m} = (\text{const})\,(1-x^2)^{m/2}\left[\frac{d}{dx} + q\,\frac{1}{1-x^2}\right]^{l+m}(1-x^2)^{l/2}\,\Theta_{q,l,-l}\,. \tag{A.3}$$

Now use

$$\frac{d}{dx} + q\,\frac{1}{1-x^2} = \sqrt{\frac{1-x}{1+x}}^{\,q}\,\frac{d}{dx}\sqrt{\frac{1+x}{1-x}}^{\,q}\,. \tag{A.4}$$

One obtains eq. (28) from (A.3). (The constant in (A.3) can be evaluated explicitly in the repeated application of (A.2).)

Appendix B

Some properties of $P_n^{\alpha,\beta}$

$P_n^{\alpha,\beta}(x)$ as defined by (31) and (32) is a polynomial of degree n. It satisfies

$$P_n^{\alpha,\beta}(-x) = (-1)^n P_n^{\beta,\alpha}(x)\,, \tag{B.1}$$

$$P_n^{\alpha,\beta}(x) = 2^{-n} \sum_{\lambda=0}^{n} \frac{(n+\alpha)!}{\lambda!\,(n+\alpha-\lambda)!}\,\frac{(n+\beta)!}{(n-\lambda)!\,(\beta+\lambda)!}\,(x-1)^{n-\lambda}(x+1)^{\lambda}, \tag{B.2}$$

in which $m!$ is defined to be ∞ when $m < 0$ and $0! = 1$. To prove (B.2) we arrange the square bracket in (31) into a product of $2n+\alpha+\beta$ factors, each being $(1-x)$ or $(1+x)$, and choose λ factors $(1-x)$ and $n-\lambda$ factors $(1+x)$ for differentiation. (B.2) then follows

We shall now show

$$P_{n+\alpha}^{-\alpha,\beta} = 2^{-\alpha}(x-1)^{\alpha}\,\frac{n!\,(n+\alpha+\beta)!}{(n+\beta)!\,(n+\alpha)!}\,P_n^{\alpha,\beta}\,, \tag{B.3}$$

$$P_{n+\beta}^{\alpha,-\beta} = 2^{-\beta}(x+1)^{\beta}\,\frac{n!\,(n+\alpha+\beta)!}{(n+\beta)!\,(n+\alpha)!}\,P_n^{\alpha,\beta}\,, \tag{B.4}$$

$$P_{n+\alpha+\beta}^{-\alpha,-\beta} = 2^{-\alpha-\beta}(x-1)^{\alpha}(x+1)^{\beta}\,P_n^{\alpha,\beta}\,. \tag{B.5}$$

To show (B.3) we use (B.2),

$$2^{n+\alpha} P_{n+\alpha}^{-\alpha,\beta} = \sum_{\lambda=0}^{n+\alpha} \frac{n!}{\lambda!\,(n-\lambda)!}\,\frac{(n+\alpha+\beta)!}{(n+\alpha-\lambda)!\,(\beta+\lambda)!}\,(x-1)^{n+\alpha-\lambda}(x+1)^{\lambda}$$

$$= \sum_{\lambda=0}^{n} \text{same} = (x-1)^{\alpha}\,\frac{n!(n+\alpha+\beta)!}{(n+\alpha)!\,(n+\beta)!}\,2^n P_n^{\alpha,\beta}\,,$$

which leads to (B.3). Eq. (B.4) can be proved similarly. Eq. (B.5) can be proved by using (B.3) and (B.4) in succession.

Define

$$R_n^{\alpha,\beta} \equiv (1-x)^{\alpha/2}(x+1)^{\beta/2}\,2^{-(\alpha+\beta)/2}\,P_n^{\alpha,\beta}\,. \tag{B.6}$$

Then (B.3), (B.4) and (B.5) together show that

$$R_n^{\alpha,\beta} = (-1)^{(\alpha-|\alpha|)/2} R_{\nu}^{|\alpha|,|\beta|}\,, \tag{B.7}$$

where

$$\nu = n + \tfrac{1}{2}(\alpha+\beta-|\alpha|-|\beta|)\,. \tag{B.8}$$

Using (28) as the definition of $Y_{q,l,m}$, we obtain by utilizing (B.5)

$$Y^*_{q,l,m} = (-1)^{q+m}\, Y_{-q,l,-m}\,, \tag{B.9}$$

which is a useful formula. It is correct in both regions R_a and R_b. If we take $q = 0$, (B.6) reduces to the usual formula for the complex conjugate of $Y_{l,m}$.

Appendix C

Alternative expression for $Y_{q,l,m}$

Using (28) and (B.6) we obtain

$$(Y_{q,l,m})_a = M_{q,l,m}\, 2^{(\alpha+\beta)/2} R_n^{\alpha,\beta} e^{i(m+q)\phi}\,. \tag{C.1}$$

Now use (B.7) to obtain

$$(Y_{q,l,m})_a = (\text{const})\,(1-x)^{|\alpha|/2}(1+x)^{|\beta|/2} P_\nu^{|\alpha|,|\beta|}\, e^{i(m+q)\phi}\,, \tag{C.2}$$

where ν is given by (B.8), and n, α and β are given by (29).

Appendix D

Clebsch-Gordan coefficients

We shall define the usual Clebsch-Gordan coefficients

$$\langle lml'm' \,|\, ll'jm_j\rangle \tag{D.1}$$

as in ref. [6]. Some usage of these coefficients for combining sections will be discussed below:

(a) Consider the product of two sections $Y_{q,l,m}(\theta,\phi)\, Y_{q',l',m'}(\theta,\phi)$ *of the same argument* θ, ϕ. The result is clearly a section with $q'' = q + q'$. The usual vector addition theorem applies and we have

$$\sum_{mm'} Y_{q,l,m}\, Y_{q',l',m'}\, \langle lml'm' \,|\, ll'jm_j\rangle = K\, Y_{q+q',j,m_j}\,, \tag{D.2}$$

where K depends on q, l, q', l', and j but not on m. Notice that sometimes K is zero. For example, for $Y_{q,l,m}\, Y_{q,l,m'}$, it is well known that the CG coefficients are symmetrical (with respect to $m \leftrightarrow m'$) for $j = 2l -$ even integer and antisymmetrical for $j = 2l -$ odd integer. For the latter case clearly $K = 0$. Notice also that if $j < |q + q'|$, then the right-hand side of (D.2) must vanish, since Y_{q+q',j,m_j} does not then exist.

For example, for the case $Y_{1,1,m}\,Y_{0,1,m'}$ the final j value is, *a priori*, 2, 1 or 0. But the case $j = 0$ vanishes since $Y_{1,0,m_j}$ does not exist. This can indeed be checked with the aid of tables 1, and the appropriate values of $Y_{0,1,m'}$ and the Clebsch-Gordan coefficients

$$Y_{1,1,1}(\sqrt{1-x^2}\,\mathrm{e}^{-i\phi}) - Y_{1,1,0}(\sqrt{2}x) + Y_{1,1,-1}(-\sqrt{1-x^2}\,\mathrm{e}^{i\phi}) = 0\,. \quad \text{(D.3)}$$

Similarly, $Y_{1,1,m}\,Y_{1,1,m'}$ can be linearly combined, *a priori*, to give $Y_{2,2,m_j}$ $Y_{2,1,m}$ and $Y_{2,0,m}$. But the latter two do not exist, giving rise to the following identity which can be checked with table 1

$$Y_{1,1,1}Y_{1,1,-1} - Y_{1,1,0}Y_{1,1,0} + Y_{1,1,-1}Y_{1,1,1} = 0\,. \quad \text{(D.4)}$$

(b) For a problem with two particles of different charges Ze, $Z'e$ moving in the field of a magnetic monopole, the wave function is a "double" section with respect to both $\boldsymbol{r}$ and $\boldsymbol{r}'$. Then

$$\sum_{mm'} Y_{q,l,m}(\theta,\phi)\,Y_{q',l',m'}(\theta',\phi')\langle ll'jm_j|lml'm'\rangle = F_{q,q',j,m_j}\,, \quad \text{(D.5)}$$

is a double section that transforms under a simultaneous rotation of θ and ϕ like Y_{j,m_j} does. One has to, however, remember that after the rotation one is using a different gauge, as discussed before in sect. 10.

(c) For a particle with spin $\boldsymbol{S}$, the total angular momentum is

$$\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S} = \boldsymbol{r} \times (\boldsymbol{p} - Ze\boldsymbol{A}) - \frac{q\boldsymbol{r}}{r} + \boldsymbol{S}\,. \quad \text{(D.6)}$$

The addition of $\boldsymbol{L}$ and $\boldsymbol{S}$ is achieved with the Clebsch-Gordan coefficients in the usual way with no difficulty.

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