Lecture 10 Topo aspect of Yang-Mills theory

1. Instanton solution to 4D Yang-Mills

2. Self-dual and anti-self-dual.

3. SU(2) monopole and the 2nd Hopf mapping.

Ref: 1. A.A. Belavin, A.M. Polyakov, A.S. Schwartz, Y.S. Tyupkin, Phys. Letts. B 59 (1) 85-87, 1975.

2. C.N. Yang. J. of Math. Phys. 19, 320, 1978.

1. Instanton Solution to Yang-Mills

$$S = \frac{1}{2g^2} \int d^4x \, tr[F_{\mu\nu} F_{\mu\nu}]$$ ← Euclidean version

Here $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]$ ← normalization

Define $\tilde{F}_{\mu\nu} = \frac{1}{2} \varepsilon_{\mu\nu\lambda\delta} F_{\lambda\delta}$, then

$\varepsilon_{\mu\nu\lambda\delta} \varepsilon_{\mu\nu\chi\delta'}$
$= 2[\delta_{\lambda\chi}\delta_{\delta\delta'} - \delta_{\lambda\sigma'}\delta_{\chi\sigma}]$

$$\tilde{F}_{\mu\nu}\tilde{F}_{\mu\nu} = \frac{1}{4} \varepsilon_{\mu\nu\lambda\delta}\varepsilon_{\mu\nu\chi\delta'} F_{\lambda\delta} F_{\chi\delta'}$$
$$= \frac{1}{2}[\delta_{\lambda\chi'}\delta_{\delta\delta'} - \delta_{\lambda\sigma'}\delta_{\chi\sigma}] F_{\lambda\delta} F_{\chi\delta'} = F_{\lambda\delta} F_{\lambda\delta}.$$

Consider identity

$$tr[(F_{\mu\nu} \pm \tilde{F}_{\mu\nu})(F_{\mu\nu} \pm \tilde{F}_{\mu\nu})] = tr[F_{\mu\nu}F_{\mu\nu}] + tr[\tilde{F}_{\mu\nu}\tilde{F}_{\mu\nu}]$$
$$\pm tr(F_{\mu\nu}\tilde{F}_{\mu\nu}] \pm tr[\tilde{F}_{\mu\nu}F_{\mu\nu}]$$

$$= 2tr[F_{\mu\nu}F_{\mu\nu}] \pm 2tr[F_{\mu\nu}\tilde{F}_{\mu\nu}] \geq 0$$

$$\Rightarrow \quad S = \frac{1}{2g^2}\int d^4x \, tr[F_{\mu\nu}F_{\mu\nu}] \geq \mp \frac{1}{2g^2}\int d^4x \, tr[F_{\mu\nu}\tilde{F}_{\mu\nu}]$$

The equality holds at $F_{\mu\nu} = \pm \tilde{F}_{\mu\nu}$.

$$tr[\tilde{F}_{\mu\nu} F_{\mu\nu}] = \frac{1}{2}\varepsilon^{\mu\nu\lambda\delta} tr[F_{\mu\nu}F_{\lambda\delta}]$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu A_\nu - A_\nu A_\mu)$$

$$\varepsilon^{\mu\nu\lambda\delta} tr[F_{\mu\nu}F_{\lambda\delta}] = 4\varepsilon^{\mu\nu\lambda\delta} tr[(\partial_\mu A_\nu + iA_\mu A_\nu)(\partial_\lambda A_\delta + iA_\lambda A_\delta)]$$

$$\mathrm{tr}[\tilde{F}_{\mu\nu} F_{\mu\nu}] = 2\varepsilon^{\mu\nu\lambda\delta}\, \mathrm{tr}[\partial_\mu A_\nu \partial_\lambda A_\delta + 2i(\partial_\mu A_\nu) A_\lambda A_\delta + A_\mu A_\nu A_\lambda A_\delta]$$

$$\varepsilon^{\mu\nu\lambda\delta}\, \mathrm{tr}[A_\mu A_\nu A_\lambda A_\delta] = \varepsilon^{\nu\lambda\delta\mu}\, \mathrm{tr}[A_\nu A_\lambda A_\delta A_\mu]$$

$$= -\varepsilon^{\mu\nu\lambda\delta}\, \mathrm{tr}[A_\mu A_\nu A_\lambda A_\delta] = 0$$

$$\partial_\mu A_\nu \partial_\lambda A_\delta = \partial_\mu (A_\nu \partial_\lambda A_\delta) - \underbrace{A_\nu \partial_\mu \partial_\lambda A_\delta}_{=0}$$

$$\varepsilon^{\mu\nu\lambda\delta}\, \mathrm{tr}[(\partial_\mu A_\nu) A_\lambda A_\delta] = \varepsilon^{\mu\nu\lambda\delta}\, \mathrm{tr}[\partial_\mu (A_\nu A_\lambda A_\delta)$$

$$- \cancel{\partial_\mu} A_\nu (\partial_\mu A_\lambda) A_\delta - A_\nu A_\lambda (\partial_\mu A_\delta))]$$

$$\Rightarrow 3\varepsilon^{\mu\nu\lambda\delta}\, \mathrm{tr}[(\partial_\mu A_\nu) A_\lambda A_\delta] = \varepsilon^{\mu\nu\lambda\delta}\, \mathrm{tr}[\partial_\mu (A_\nu A_\lambda A_\delta))]$$

$$\Rightarrow \mathrm{tr}[\tilde{F}_{\mu\nu} F_{\mu\nu}] = 2\varepsilon^{\mu\nu\lambda\delta}\, \partial_\mu\, \mathrm{tr}[A_\nu \partial_\lambda A_\delta + \tfrac{2}{3} i A_\nu A_\lambda A_\delta]$$

Define $j_\mu = 2\varepsilon^{\mu\nu\lambda\delta}\, \mathrm{tr}[A_\nu \partial_\lambda A_\delta + \frac{2}{3} i A_\nu A_\lambda A_\delta]$

$$\mathrm{tr}[\tilde{F}_{\mu\nu} F_{\mu\nu}] = \partial_\mu j_\mu$$

$$\Rightarrow \int d^4x\, \mathrm{tr}[\tilde{F}_{\mu\nu} F_{\mu\nu}] = \int d^4x\, \partial_\mu j_\mu = \oint_{S^3,\, R\to\infty} j_\mu\, d\sigma_\mu$$

~~In order to have a finite resutt j_μ must decay than R^{-3}, no slower,~~

~~$\rightarrow$ A decay faster or at the same order of R^{-1}.~~

~~In this case, Fde F must~~

In order to have a finite resut, $F_{\mu\nu}$ must decay faster than R^{-2} as $R \to +\infty$.

on the other hand, j_μ should be at the order of R^{-3}, which means $A \propto 1/R$. Then as $R \to \infty$, we can express

$$A_\mu = \underbrace{-i g^\dagger \partial_\mu g}_{\text{must be a pure gauge part}}^{\leftarrow R^{-1}} + A'_\mu \leftarrow R^{-(1+\delta)}$$

must be a pure gauge part yields $A_\mu \propto 1/R$ and $F = 0$, certainly satisfy $\tilde{F} = F = 0$

A'_μ decays faster than R^{-1}, say $R^{-(1+\delta)}$.

Then as $R \to \infty$, $F \propto R^{-(2+\delta)}$, $j_\mu \propto R^{-3} + R^{-3+\delta}$

As $R \to +\infty$, we can drop the term of A'_μ. Then.

$$\int d^4x \; tr[\tilde{F}_{\mu\nu} F_{\mu\nu}] = \oint_{S^3 \text{ with } R \to \infty} j_\mu \cdot d\sigma_\mu$$

where $j_\mu = 2 \varepsilon^{\mu\nu\lambda\sigma} tr[(-i g^\dagger \partial_\nu g)\, \partial_\lambda (-i g^\dagger \partial_\sigma g)$

$+ \frac{2}{3} i \, (-i)\, g^\dagger \partial_\nu g \, (-i)\, g^\dagger \partial_\lambda g \, (-i)\, g^\dagger \partial_\sigma g \,]$

$$(-i)^2 g^\dagger \partial_\nu g \, \partial_\lambda (g^\dagger \partial_\sigma g) = - g^\dagger \partial_\nu g \, (\partial_\lambda g^\dagger)\, \partial_\sigma g - g^\dagger \partial_\nu g \, \overset{\nearrow 0}{g^\dagger \partial_\lambda \partial_\sigma g}$$

$$= - g^\dagger \partial_\nu g \, \partial_\lambda g^\dagger \, g \, g^\dagger \partial_\sigma g = (g^\dagger \partial_\nu g)(g^\dagger \partial_\lambda g)(g^\dagger \partial_\sigma g)$$

$$-\frac{2}{3} (i)^4 (g^\dagger \partial_\nu g)(g^\dagger \partial_\lambda g)(g^\dagger \partial_\sigma g)$$

$$\Rightarrow \boxed{j_\mu = \frac{2}{3} \varepsilon^{\mu\nu\lambda\sigma} \, tr[(g^\dagger \partial_\nu g)(g^\dagger \partial_\lambda g)(g^\dagger \partial_\sigma g)]}$$

define $g = n_4 + n_i(-i\sigma_i) = n_\mu e_\mu$, $\begin{cases} e_4 = 1 \\ e_i = -i\sigma_i \end{cases}$

$$n_4^2 + n_1^2 + n_2^2 + n_3^2 = 1$$

$$\Rightarrow \quad g^\dagger \partial_\mu g = (n_4 + i n_i \sigma_i)\,\partial_\mu (n_4 - i n_j \sigma_j)$$

$$= \underset{0}{\underset{\|}{n_i \partial_\mu n_i}} + [n_4 \partial_\mu n_i - n_i \partial_\mu n_4][-i\sigma_i] - n_i \partial_\mu n_j\, \epsilon_{ijk}(-i\sigma_k)$$

$$\mathrm{tr}[g^\dagger \partial_\mu g \; g^\dagger \partial_\nu g \; g^\dagger \partial_\delta g] = \mathrm{tr}[i\sigma_{k_1}\, i\sigma_{k_2} \cdot i\sigma_{k_3}$$

$$(n_{k_1}\partial_\mu n_4 - n_4 \partial_\mu n_{k_1} + \varepsilon_{k_1 i_1 j_1}\, n_{i_1} \partial_\mu n_{i_2})$$

$$(n_{k_2}\partial_\mu n_4 - n_4 \partial_\nu n_{k_2} + \varepsilon_{k_2 i_2 j_2}\, n_{i_2} \partial_\nu n_{j_2})$$

$$(n_{k_3}\partial_\lambda n_4 - n_4 \partial_\lambda n_{k_3} + \varepsilon_{k_3 i_3 j_3}\, n_{i_3} \partial_\lambda n_{j_3})$$

$$= 2\,\varepsilon_{k_1 k_2 k_3} \{ (n_{k_1}\partial_\mu n_4 - n_4 \partial_\mu n_{k_1} + \varepsilon_{k_1 i j_1}\, n_{i_1} \partial_\mu n_{i_2})$$
$$(\cdot \cdot \cdot \cdot \cdot)$$
$$(\cdot \cdot \cdot \cdot \cdot \cdot \cdot) \}$$

Near the north pole, $n_4 \simeq 1$, $n_{i,j,k}$ are first order infinitsmall,

Keep to the leading order, $-2\,\varepsilon_{k_1 k_2 k_3}\, \partial_\mu n_{k_1} \partial_\nu n_{k_2} \partial_\lambda n_{k_3}$

The area element near the n_0-direction

$$\cancel{j_0\, d\sigma_0 = -\frac{4}{3}\varepsilon_{0 k_1 k_2 k_3}\, \partial_\mu n_{k_1} \partial_\nu n_{k_2} \partial_\lambda n_{k_3}}$$
$$\cancel{\left(\frac{1}{3!}\varepsilon_{\mu\nu\lambda}\, d\hat{x}_\mu \wedge d\hat{x}_\nu \wedge d\hat{x}_\lambda\right) \cdot R^3}$$

$$j_4 = \frac{2}{3}\varepsilon_{\nu\lambda\delta 4}\, \mathrm{tr}[(g^\dagger \partial_\nu g)(g^\dagger \partial_\lambda g)(g^\dagger \partial_\delta g)]$$

$$d\sigma_4 = \frac{R^3}{3!}\varepsilon_{\nu'\lambda'\delta'}\, d\hat{x}_{\nu'} \wedge d\hat{x}_{\lambda'} \wedge d\hat{x}_{\delta'}$$

⑤

$$j_4 = \frac{-4R^3}{3} \varepsilon_{0\nu\lambda\delta}\, \varepsilon_{k_1k_2k_3}\, \partial_\mu \hat{n}_{k_1} \partial_\lambda \hat{n}_{k_2} \partial_\delta \hat{n}_{k_3}$$

$$= -\frac{4R^3}{3R^3} 3!\, \varepsilon_{0\nu\lambda\delta} \det\left(\frac{\partial \hat{n}_{k_1}}{\partial \hat{x}_\nu} \frac{\hat{n}_{k_2}}{\partial \hat{x}_\lambda}, \frac{n^3_k}{\hat{\partial} x_\delta} \right) = -\frac{4R^3}{3} \times 3!\, \det\left(\frac{\partial \hat{n}}{\partial \hat{x}}\right)$$

$$d\sigma_4 = \frac{1}{3!} \varepsilon_{0\nu'\lambda'\delta'}\, dx_{\nu'} \wedge dx_{\nu'} \wedge dx_{\delta'}$$

$$\Rightarrow j_4 d\sigma_4 = (-8) \det\left(\frac{\partial \hat{n}}{\partial \hat{x}}\right)\Big|_{\text{north pole}} \cdot d\sigma_0$$

$$\Rightarrow \oint j_\mu \cdot d\sigma_\mu = -8 \times C_2 \underset{\oint d\sigma}{} \quad \leftarrow S_3 \text{ sphere}$$

$$\begin{cases} n_4 = \cos\alpha & \alpha \in [0, \pi] \\ n_1 = \sin\alpha \cos\theta & \theta \in [0, \pi] \\ n_2 = \sin\alpha \sin\theta \cos\varphi & \varphi \in [0, 2\pi) \\ n_3 = \sin\alpha \sin\theta \sin\varphi & \end{cases}$$

$$\cancel{\oint d\sigma = \int d\alpha \int_0^{2\pi} \sin\theta d\theta \int_0^{2\pi} d\varphi}$$

$$ds^2 = (d\alpha)^2 + \sin^2\alpha \left[(d\theta)^2 + (\sin\theta)^2 (d\varphi)^2 \right]$$

$$\Rightarrow \int d\sigma = \int_0^\pi d\alpha \cdot \sin^2\alpha \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\varphi = \frac{\pi}{2} \cdot 2 \cdot 2\pi = 2\pi^2$$

$$\int dV = \int_0^1 r^3 dr \int d\sigma = \frac{2\pi^2}{4} = \frac{\pi^2}{2}$$

$$\Rightarrow \boxed{\int d^4x\, \mathrm{tr}[\tilde{F}_{\mu\nu} F_{\mu\nu}] = 16\pi^2 C_2}$$

Hence, we have

$$\boxed{S \geq \frac{8\pi^2}{g^2} (\mp C_2),}$$

C_2 takes integer values

Instanton configuration:

$$g(x) = \frac{x_4}{r} - i\frac{x_i}{r}\sigma_i \qquad x_4^2 + x_i x_i = r^2$$

$$A_\mu(x) = -i g^\dagger \partial_\mu g = -i\left(\frac{x_4}{r} + i\frac{x_i}{r}\sigma_i\right)\partial_\mu\left(\frac{x_4}{r} - i\frac{x_j}{r}\sigma_j\right)$$

$$= -i\left[\frac{x_4}{r}\partial_\mu\left(\frac{x_4}{r}\right) + \frac{x_i}{r}\partial_\mu\left(\frac{x_i}{r}\right)\right] + \left[-\frac{x_4}{r}\partial_\mu\left(\frac{x_i}{r}\right) + \frac{x_i}{r}\partial_\mu\left(\frac{x_4}{r}\right)\right]\sigma_i$$

$$+ \epsilon_{ijk}\left[\frac{x_i}{r}\partial_\mu\left(\frac{x_j}{r}\right)\right] \qquad \sigma_k$$

$$= \left[\frac{x_k}{r}\partial_\mu\left(\frac{x_4}{r}\right) - \frac{x_4}{r}\partial_\mu\left(\frac{x_k}{r}\right) + \varepsilon_{ijk}\left(\frac{x_i}{r}\partial_\mu\left(\frac{x_j}{r}\right) - \frac{x_j}{r}\partial_\mu\left(\frac{x_i}{r}\right)\right]\sigma_k$$

$$\frac{x_k}{r}\partial_\mu\left(\frac{x_4}{r}\right) - \frac{x_4}{r}\partial_\mu\left(\frac{x_k}{r}\right) = \frac{x_k\delta_{\mu 4} - x_4\delta_{\mu k}}{r^2} + \frac{-x_k x_4 x_\mu + x_4 x_k x_\mu}{r^4}$$

$$= \frac{1}{r^2}\left(x_k\delta_{\mu 4} - x_4\delta_{\mu k}\right)$$

$$\varepsilon_{ijk}\frac{x_i}{r}\partial_\mu\left(\frac{x_j}{r}\right) = \varepsilon_{ijk}\left[\frac{x_i\delta_{j\mu}}{r^2} - \frac{x_i x_j x_\mu}{r^4}\right] = \varepsilon_{ijk}\frac{x_i\delta_{j\mu}}{r^2}$$

$$\Rightarrow \quad A_\mu(x) = \frac{1}{r^2}\left(x_k\delta_{\mu 4} - x_4\delta_{\mu k} + \varepsilon_{kij}x_i\delta_{j\mu}\right)\sigma_k \qquad (k=1,2,3)$$

$$\begin{cases} A_4(x) = \frac{1}{r^2}\left[x_k\delta_{44}\right]\sigma_k = \frac{1}{r^2}\left(x_k\sigma_k\right) \\ A_i(x) = \frac{1}{r^2}\left[-x_4\delta_{ik} + \varepsilon_{ki'i}x_{i'}\right)\sigma_k \\ \quad = \frac{1}{r^2}\left[-x_4\sigma_i - \varepsilon_{ijk}x_j\sigma_k\right) \end{cases}$$

Such configuration is singular since $A \propto \frac{1}{r}$ as $r \to 0$.

✱ t' Hooft symbol

Define $e_\mu = (1, -i\vec{\sigma})$, $\bar{e}_\mu = (1, i\vec{\sigma})$

then $2i\sigma_{\mu\nu} = \bar{e}_\mu e_\nu - \bar{e}_\nu e_\mu$

$\Rightarrow$ $2i\sigma_{i4} = \bar{e}_i e_4 - \bar{e}_4 e_i = 2i\sigma_i \Rightarrow \sigma_{i4} = \sigma_i$

$2i\sigma_{ij} = \bar{e}_i e_j - \bar{e}_j e_i = i(-i)\epsilon_{ijk} i\sigma_k \times 2 \Rightarrow \sigma_{ij} = \sigma_k \epsilon_{ijk}$

t' Hooft symbol $\sigma_{\mu\nu} = \eta^i_{\mu\nu}\sigma^i$ ($i = 1, 2, 3$ ← isospin space; $\mu, \nu = 1234$) ← 4D space index

$$\begin{cases} \eta^i_{\mu\nu} = \varepsilon_{i\mu\nu4} + \delta_{i\mu}\delta_{\nu4} - \delta_{i\nu}\delta_{\mu4} \\ \bar{\eta}^i_{\mu\nu} = \varepsilon_{i\mu\nu4} - \delta_{i\mu}\delta_{\nu4} + \delta_{i\nu}\delta_{\mu4} \end{cases}$$

✱ Prove that $\eta^i_{\mu\nu}$ is self-dual $\eta^i_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}\eta^i_{\alpha\beta}$

check: $\boxed{\eta^i_{jk} = \varepsilon_{ijk},\ \eta^i_{\mu4} = \delta_{i\mu},\ \eta^i_{4\nu} = -\delta_{i\nu}.}$

$$\frac{1}{2}\varepsilon_{jk\mu\nu}\eta^i_{\mu\nu} = \frac{1}{2}\left[\varepsilon_{jk4l}\eta^i_{ol} + \varepsilon_{jkl4}\eta^i_{lo}\right]$$

$$= \varepsilon_{jklo}\delta_{il} = \varepsilon_{ijk} = \eta^i_{jk}$$

$$\boxed{\begin{gathered} \frac{1}{2}\varepsilon_{4\nu jk}\eta^i_{jk} = -\frac{1}{2}\varepsilon_{\nu jk}\varepsilon_{ijk} = -\delta_{i\nu} = \eta^i_{4\nu} \\ \Rightarrow \eta^i_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}\eta^i_{\alpha\beta} \text{ i.e. } \eta^i_{\mu\nu} \text{ is self-dual} \\ \text{similarly} \Rightarrow \bar{\eta}^i_{\mu\nu} = -\frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}\bar{\eta}^i_{\alpha\beta},\ \bar{\eta}^i_{\mu\nu} \text{ is anti self-dual.} \end{gathered}}$$

(*) $\eta^i_{\alpha\lambda}\,\eta^j_{\beta\lambda} = \delta^{ij}\delta_{\alpha\beta} + \varepsilon^{ijk}\,\eta^k_{\alpha\beta}$

left $= (\varepsilon_{i\alpha\lambda4} + \delta_{i\alpha}\delta_{4\lambda} - \delta_{i\lambda}\delta_{4\alpha})(\varepsilon_{j\beta\lambda4} + \delta_{j\beta}\delta_{4\lambda} - \delta_{j\lambda}\delta_{4\beta})$

$= \varepsilon_{i\alpha\lambda}\varepsilon_{j\beta\lambda} - \varepsilon_{i\alpha j4}\delta_{4\beta} + \delta_{i\alpha}\delta_{j\beta} - \varepsilon_{j\beta i4}\delta_{4\alpha} + \delta_{ij}\delta_{4\alpha}\delta_{4\beta}$

$= (\delta_{ij}\delta_{\alpha\beta} - \delta_{i\beta}\delta_{j\alpha}) - \varepsilon_{i\alpha j4}\delta_{4\beta} + \delta_{i\alpha}\delta_{j\beta} - \varepsilon_{j\beta i4}\delta_{4\alpha}$
$+ \delta_{ij}\delta_{44}$

$= \delta_{ij}\delta_{\alpha\beta} + \underline{\delta_{i\alpha}\delta_{j\beta} - \delta_{i\beta}\delta_{j\alpha}} - \varepsilon_{i\alpha j4}\delta_{\beta4} - \varepsilon_{j\beta i4}\delta_{\alpha4}$

check $\varepsilon^{ijk}\,\eta^k_{\alpha\beta} = \underline{\varepsilon^{ijk}(\varepsilon_{k\alpha\beta4}} + \delta_{k\alpha}\delta_{4\beta} - \delta_{k\beta}\delta_{4\alpha})$

—— term $= \delta_{i\alpha}\delta_{j\beta} - \delta_{i\beta}\delta_{j\alpha}$

$\sim$ term $= \varepsilon^{ijk}\delta_{k\alpha}\delta_{4\beta} = \varepsilon^{ij\alpha}\delta_{\beta4} = -\varepsilon_{i\alpha j4}\delta_{4\beta}$

$\approx$ term $= \varepsilon^{ijk}\delta_{k\beta}\delta_{4\alpha} = -\varepsilon^{ij\beta}\delta_{4\alpha} = -\varepsilon^{j\beta i4}\delta_{4\alpha}$

(*) $\eta^i_{\mu\nu}\,\eta^j_{\mu\nu} = \delta_{ij}[1+1+1+1] = 4\delta_{ij}$

example: $\eta^1_{23}\eta^1_{23} + \eta^1_{32}\eta^1_{32} + \eta^1_{01}\eta^1_{01} + \eta^1_{10}\eta^1_{10}$

(*) $\varepsilon_{ijk}\,\eta^j_{\mu\rho}\,\eta^k_{\nu\sigma} = \delta_{\mu\nu}\eta^i_{\rho\sigma} + \delta_{\rho\sigma}\eta^i_{\mu\nu}$
$+ \delta_{\mu\sigma}\eta^i_{\rho\nu} + \delta_{\rho\nu}\eta^i_{\mu\sigma}$

to be proved.

use t' Hooft symbol

$$A_\mu(x) = \frac{-i}{r^2}\,\eta^k_{\mu\nu}\,x_\nu\,(-i\sigma_k) = -\frac{1}{r^2}\,\eta^k_{\mu\nu}\,x_\nu\,\sigma_k$$

Check $\quad A_4 = -\frac{1}{r^2}\,\eta^k_{4\nu}\,x_\nu\,\sigma_k = -\frac{1}{r^2}(-\delta_{k\nu})\,x_\nu\,\sigma_k = \frac{1}{r^2}\,\vec{x}\cdot\vec{\sigma}$

$$A_i = \frac{-1}{r^2}\,\eta^k_{i\nu}\,x_\nu\,\sigma_k = -\frac{1}{r^2}\left[\varepsilon_{kij}\,x_j\,\sigma_k + \eta^k_{i4}\,x_4\,\sigma_k\right]$$

$$= -\frac{1}{r^2}\left[\varepsilon_{ijk}\,x_j\,\sigma_k + \delta_{ik}\,x_4\,\sigma_k\right]$$

$$= -\frac{1}{r^2}\left[\varepsilon_{ijk}\,x_j\,\sigma_k + x_4\,\sigma_i\right]$$

Now try the following ansatz to remove the singularity

$$A_\mu = -\frac{f(r^2)}{r^2}\,\eta^k_{\mu\delta}\,x_\delta\,\sigma_k$$

$$\partial_\mu A_\nu = -\eta^k_{\nu\delta}\,\sigma_k\left[\frac{f(r^2)}{r^2}\,\delta_{\mu\delta} + \frac{2x_\mu x_\delta\,(f'r^2 - f)}{r^4}\right]$$

$$= -\eta^k_{\nu\mu}\,\sigma_k\,\frac{f(r^2)}{r^2} - \frac{\eta^k_{\nu\delta}\,x_\mu x_\delta}{r^4}\,(f'r^2 - f)\,]$$

$$\partial_\nu A_\mu = -\eta^k_{\mu\nu}\,\sigma_k\,\frac{f(r^2)}{r^2} - \frac{\eta^k_{\mu\sigma}\,x_\nu x_\sigma}{r^4}\,(f'r^2 - f)$$

$$\partial_\mu A_\nu - \partial_\nu A_\mu = 2\eta^k_{\mu\nu}\,\sigma_k\,\frac{f}{r^2}$$

$$+ \frac{2\left(\eta^k_{\mu\sigma}\,x_\nu - \eta^k_{\nu\sigma}\,x_\mu\right)x_\sigma}{r^4}\,(f'r^2 - f)$$

$$i[A_\mu, A_\nu] = i\frac{f^2}{r^4}\eta^{k_1}_{\mu\lambda_1}\eta^{k_2}_{\nu\lambda_2}x_{\lambda_1}x_{\lambda_2}[\sigma_{k_1},\sigma_{k_2}]$$

$$= -2\frac{f^2}{r^4}\underbrace{\varepsilon_{k_1k_2k_3}\eta^{k_1}_{\mu\lambda_1}\eta^{k_2}_{\nu\lambda_2}}x_{\lambda_1}x_{\lambda_2}\sigma_{k_3}$$

$$\swarrow$$

$$\{+\underbrace{\delta_{\mu\nu}\eta^{k_3}_{\lambda_1\lambda_2}}_{\substack{\| \\ 0 \text{ after anti-sym}}} + \delta_{\lambda_1\lambda_2}\eta^{k_3}_{\mu\nu} - \delta_{\mu\lambda_2}\eta^{k_3}_{\lambda_1\nu} - \delta_{\lambda_1\nu}\eta^{k_3}_{\mu\lambda_2})$$

$$= \frac{2f^2}{r^4}(-\eta^k_{\mu\nu}r^2 + \underbrace{\eta^k_{\lambda\nu}x_\mu x_\lambda + \eta^k_{\mu\lambda}x_\nu x_\lambda}) \sigma_k$$

$$\Downarrow$$

$$(\eta^k_{\mu\sigma}x_\nu - \eta^k_{\nu\sigma}x_\mu)x_\sigma$$

$$\Rightarrow F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]$$

$$= 2\eta^k_{\mu\nu}\sigma_k\frac{1}{r^2}(f-f^2) + \frac{2}{r^4}(\eta^k_{\mu\sigma}x_\nu - \eta^k_{\nu\sigma}x_\mu)x_\sigma$$

$$\times(f'r^2 - f + f^2)$$

$$\tilde{F}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F_{\rho\sigma}$$

$$F_{\rho\sigma} = 2\eta^k_{\rho\sigma}\sigma_k\frac{f-f^2}{r^2} + \frac{2}{r^4}(f'r^2 - f + f^2)(\eta^k_{\rho\tau}x_\sigma - \eta^k_{\sigma\tau}x_\rho)x_\tau$$

~~we will use ① $\eta^i_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}\eta^k_{\rho\sigma}$~~

② $\frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}\cancel{\eta^k_{\rho}}\ x_\sigma$

$$\tilde{F}_{\mu\nu} = \varepsilon_{\mu\nu\rho\sigma}\eta^k_{\rho\sigma}\sigma_k\frac{1}{r^2}(f-f^2)$$

$$+ \varepsilon_{\mu\nu\rho\sigma}(\eta^k_{\rho\tau}x_\tau x_\sigma - \eta^k_{\sigma\tau}x_\rho x_\tau)\frac{1}{r^4}(f'r^2 - f + f^2)$$

$$\frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}\eta^k_{\rho\sigma} = \eta^k_{\mu\nu}$$

$$\varepsilon_{\mu\nu\rho\delta}\,\eta^k_{\rho\tau}x_\tau x_\sigma = (\delta_{\sigma\tau}\eta^{ak}_{\mu\nu} + \delta_{\mu\tau}\eta^k_{\nu\sigma} - \delta_{\nu\tau}\eta^k_{\mu\sigma})x_\tau x_\sigma$$

$$= \left[\eta^k_{\mu\nu}r^2 + \eta^k_{\nu\sigma}x_\mu x_\sigma - \eta^k_{\mu\sigma}x_\nu x_\sigma\right]$$

$$\varepsilon_{\mu\nu\rho\sigma}\,\eta^{ak}_{\sigma\tau}x_\tau x_\rho = -(\delta_{\rho\tau}\eta^k_{\mu\nu} + \delta_{\mu\tau}\eta^k_{\nu\rho} - \delta_{\nu\tau}\delta^k_{\mu\rho})x_\tau x_\rho$$

$$= -\eta^k_{\mu\nu}r^2 - \eta^k_{\nu\sigma}x_\mu x_\sigma + \eta^k_{\mu\sigma}x_\nu x_\sigma$$

$$\Rightarrow \varepsilon_{\mu\nu\rho\sigma}\left[\eta^k_{\rho\tau}x_\tau x_\sigma - \eta^k_{\sigma\tau}x_\tau x_\rho\right]$$

$$= 2\eta^k_{\mu\nu}r^2 + 2\eta^k_{\nu\sigma}x_\mu x_\sigma - 2\eta^k_{\mu\sigma}x_\nu x_\sigma$$

$$\Rightarrow \hat{F}_{\mu\nu} = \eta^k_{\mu\nu}\,\sigma_k\left[\frac{2}{r^2}(f-f^2) + \frac{2r^2}{r^4}(f'r^2 - f + f^2)\right]$$

$$+ \frac{2}{r^4}(\eta^k_{\nu\sigma}x_\mu - \eta^k_{\mu\sigma}x_\nu)x_\sigma\;(f'r^2 - f + f^2)\Big]$$

~~We require~~ $\cancel{F_{\mu\nu} = \widetilde{F}_{\mu\nu}}$

$$= \eta^k_{\mu\nu}\,\sigma_k\,2f' - \frac{2}{r^4}(\eta^k_{\mu\sigma}x_\nu - \eta^k_{\nu\sigma}x_\mu)x_\sigma\,(f'r^2 - f + f^2)$$

we require $\tilde{F} = F \Rightarrow$

$$f' = f(1-f)/r^2$$

please note $f' = \dfrac{df}{dr^2}$

$$\Rightarrow \quad \frac{df}{f(1-f)} = \frac{d(r^2)}{r^2} = \ln r^2 + C$$

$$\ln \frac{f}{1-f} = \ln r^2 + C \Rightarrow \frac{f}{1-f} = \frac{r^2}{\rho^2} \Rightarrow f = \frac{r^2}{r^2+\rho^2}$$

ρ is an arbitrary constant

$$\begin{cases} f(r^2)\big|_{r^2\to\infty} = 1 \\ f(r^2)\big|_{r^2\to 0} = \text{const.}\, r^2 \end{cases}$$

then there's no singularity.

$$\Rightarrow A_\mu = -i \frac{1}{r^2+\rho^2} \eta^k_{\mu\sigma} x_\sigma \sigma_k .$$

This a self-dual ~~ga~~ gauge potential $\rightarrow$ a finite action

$$S = \frac{8\pi^2}{g} .$$

Such a solution a soliton.

§1. 2nd Hopf map — SU(2) monopole

$$\Gamma^i = \begin{pmatrix} 0 & i\sigma_i \\ -i\sigma_i & 0 \end{pmatrix}, \quad \Gamma^4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$\Gamma^i = -\sigma_i \otimes \tau_2$, $\Gamma^4 = 1 \otimes \tau_1$, $\Gamma^5 = 1 \otimes \tau_3$, Clearly $\{\Gamma^a, \Gamma^b\} = 2\delta^{ab}$

- define 4d spinor

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{1+x_5}{2}} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \\ \sqrt{\frac{1}{2(1+x_5)}} (x_4 - i\vec{x}\cdot\vec{\sigma}) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \end{pmatrix}$$

with $\sum_{i=1}^{5} x_i^2 = 1$, $|u_1|^2 + |u_2|^2 = 1$.

- Check normalization

$$\psi^\dagger \psi = \frac{1+x_5}{2} + \frac{x_1^2 + \cdots x_4^2}{2(1+x_5)} = \frac{1+x_5}{2} + \frac{1-x_5^2}{2(1+x_5)} = 1$$

- 验证 $x_i = \psi^\dagger \Gamma_i \psi$

for $i=1,2,3$.

$$\psi^\dagger \Gamma_i \psi = \sqrt{\frac{1+x_5}{2}}\, u^\dagger (i\sigma_i) \sqrt{\frac{1}{2(1+x_5)}} (x_4 - i\vec{x}\cdot\vec{\sigma})\, u + h.c$$

$$= \frac{1}{2} u^\dagger (i\sigma_i x_4 + x_j \sigma_i \sigma_j) u + h.c$$

$$= \frac{1}{2} u^\dagger (i\sigma_i x_4 + x_i + i\epsilon_{ijk} x_j \sigma_k) u + h.c$$

$$= \frac{1}{2} u^\dagger u\, x_i \times 2 = x_i$$

$$\psi^+ \Gamma_4 \psi = \frac{1}{2} u^+ (x_4 - i\vec{x}\cdot\vec{\sigma}) u + h.c = x_4$$

$$\psi^+ \Gamma_5 \psi = \frac{1+x_5}{2} u^+ u - u^+ \frac{x_1^2 + \cdots x_4^2}{2(1+x_5)} u = u^+ u \left(\frac{1+x_5}{2} - \frac{1-x_5}{2}\right) = x_5$$

⊛ Any SU(2) rotation $u' = R u$, maps into the same point on the S^4 sphere.

⊛ prof $(x_i \cdot \Gamma_i) \psi = \psi$. eigenvalue $= 1$.

$$x_i \cdot \Gamma_i = \begin{pmatrix} x_5 & x_4 + i\vec{x}\cdot\vec{\sigma} \\ x_4 - i\vec{x}\cdot\vec{\sigma} & -x_5 \end{pmatrix}$$

define projection operator $P_+ = \frac{1}{2}(1 + x_i \Gamma_i)$

$$H P_+ = \frac{1}{2} (x_i \Gamma_i)(1 + x_j \Gamma_j) = P_+$$

take $\psi_0 = \begin{pmatrix} u \\ 0 \end{pmatrix}$ $\Rightarrow$ $H(P_+ \psi_0) = (P_+ \psi_0)$

then
$$P_+ \psi_0 = \begin{pmatrix} \frac{1+x_5}{2} & \frac{x_4 + i\vec{x}\cdot\vec{\sigma}}{2} \\ \frac{x_4 - i\vec{x}\cdot\vec{\sigma}}{2} & \frac{1-x_5}{2} \end{pmatrix} \begin{pmatrix} u \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1+x_5}{2} u \\ \frac{x_4 - i\vec{x}\cdot\vec{\sigma}}{2} u \end{pmatrix}$$

normalization
$$|P_+ \psi_0|^2 = \left(\frac{1+x_5}{2}\right)^2 + \frac{x_4^2 + x_1^2 + x_2^2 + x_3^2}{4}$$

$$= \frac{1}{4}[1 + 1 + 2x_5] = \frac{1+x_5}{2}$$

$$\Rightarrow \psi = \left(\frac{1+x_5}{2}\right)^{-1/2} P_+ \psi_0 = \begin{pmatrix} \sqrt{\frac{1+x_5}{2}}\, u \\ \sqrt{\frac{1}{2(1+x_5)}} (x_4 - i\vec{x}\cdot\vec{\sigma}) u \end{pmatrix}$$

singular point is located at $x_5 = -1$

if we take $\psi_0 = \begin{pmatrix} 0 \\ u \end{pmatrix}$

$$P_+ \psi_0 = \begin{pmatrix} \frac{x_4 + i\vec{x}\cdot\vec{\sigma}}{2} & u \\ \frac{1-x_5}{2} & u \end{pmatrix} \qquad |P_+\psi_0|^2 = \frac{1}{4}(x_4^2 + x_1^2 + x_2^2 + x_3^2) + \frac{1}{4}(1 + x_5^2 - 2x_5)$$

$$= \frac{1}{2}(1 - x_5)$$

$$\rightarrow \psi' = \sqrt{\frac{1}{2(1-x_5)}} \begin{pmatrix} (x_4 + i\vec{x}\cdot\vec{\sigma})u \\ (1-x_5)\,u \end{pmatrix}$$

then x' has ~~a~~ the singular point at north point, $x_5 = 1$.

§2. SU(2) monopole field on S^4

Based on $\psi = \sqrt{\frac{1}{2(1+x_5)}} \begin{pmatrix} (1+x_5)u \\ (x_4 - i\vec{x}\cdot\vec{\sigma})u \end{pmatrix}$

$$d\psi = \frac{-\frac{1}{2}dx_5}{(1+x_5)}\psi + \sqrt{\frac{1}{2(1+x_5)}} \begin{pmatrix} dx_5\, u \\ (dx_4 - i\,d\vec{x}\cdot\vec{\sigma})u \end{pmatrix}$$

$$+ \sqrt{\frac{1}{2(1+x_5)}} \begin{pmatrix} (1+x_5)\,du \\ (x_4 - i\vec{x}\cdot\vec{\sigma})\,du \end{pmatrix}$$

$$\psi^\dagger d\psi = \frac{-\frac{1}{2}dx_5}{1+x_5} + \frac{(1+x_5)dx_5 + u^\dagger(x_4 + i\vec{x}\cdot\vec{\sigma})(dx_4 - i\,d\vec{x}\cdot\vec{\sigma})u}{2(1+x_5)}$$

$$+ \frac{u^\dagger du\,(1+x_5)^2 + u^\dagger(x_4 + i\vec{x}\cdot\vec{\sigma})(x_4 - i\vec{x}\cdot\vec{\sigma})du}{2(1+x_5)}$$

其中 $(x_4 + i\vec{x}\cdot\vec{\sigma})(dx_4 - i\,d\vec{x}\cdot\vec{\sigma}) = \underline{x_4 dx_4} + i(x_i dx_4 - x_4 dx_i)\sigma_i$

2nd term $\quad + \underline{x_i dx_i} + i(x_i dx_j - x_j dx_i)\epsilon_{ijk}\sigma_k/2$

$x_1 dx_1 + \cdots x_5 dx_5 = 0$,

$$u_1^* du_1 + u_2^* du_2$$

3rd term: $u^\dagger du \dfrac{(1+x_5)^2 + (1-x_5^2)}{2(1+x_5)} = u^\dagger du$ ← a pure gauge

$$\Rightarrow \psi^\dagger d\psi = i u^\dagger \left[\frac{(x_i dx_4 - x_4 dx_i)\sigma_i + \frac{1}{2}\varepsilon_{ijk}(x_i dx_j - x_j dx_i)\sigma^k}{2(1+x_5)} \right] u$$

$$\langle \psi^\dagger | d | \psi \rangle = i u_\sigma^\dagger (A_\mu \cdot dx_\mu)_{\sigma\sigma'} u_{\sigma'}$$

$$\Rightarrow A_\mu dx_\mu = \frac{1}{2(1+x_5)} (dx_i (-x_4 \sigma_i - \epsilon_{ijk} x_j \sigma_k)$$
$$+ dx_4 \, x_i \cdot \sigma_i]$$

$$\Rightarrow \boxed{\begin{cases} A_i = -\dfrac{1}{2(1+x_5)} (\varepsilon_{ijk} x_j \sigma_k + x_4 \sigma_i) \\[2ex] A_4 = \dfrac{1}{2(1+x_5)} x_i \sigma_i \\[2ex] A_5 = 0 \end{cases}}$$

Discuss why we drop $u^\dagger du$ term;

promote $u \to \begin{pmatrix} u_1 & -u_2^* \\ u_2 & u_1^* \end{pmatrix} = \mathcal{U}$ we can do the same operation to 2nd coloumn

SU(2) phase

$$\Rightarrow \mathcal{U}^\dagger d\mathcal{U} = i \mathcal{U}^\dagger dA \, \mathcal{U} \Rightarrow d\mathcal{U} = i\, dA\, \mathcal{U}$$

$$\Rightarrow dA = d\mathcal{U} \cdot \mathcal{U}^{-1} = -\mathcal{U} d\mathcal{U}^{-1}, \Rightarrow$$ this contribution is a pure gauge.

t' Hooft symbol $\eta^j_{\mu\nu} = \varepsilon_{j\mu\nu 4} + \delta_{j\mu}\delta_{4\nu} - \delta_{j\nu}\delta_{4\mu}$ $\quad j = 1,2,3$; $\mu\nu = 1234$

then $$A_\mu = -\frac{1}{1+x_5}\,\eta^j_{\mu\nu}\,x_\nu\,\frac{\sigma_j}{2}$$

check for $\mu = 1,2,3 \Rightarrow \eta^j_{i\nu} = \varepsilon_{ji\nu} + \delta_{ij}\delta_{4\nu}$

$$A_i = -\frac{1}{1+x_5}\left[\varepsilon_{i\nu j}\,x_\nu\frac{\sigma_j}{2} + x_4\frac{\sigma_i}{2}\right]$$

for $\mu = 4 \Rightarrow \eta^j_{4\nu} = -\delta_{j\nu}$

$$A_4 = \frac{1}{1+x_5}\,(\delta_{j\nu}\,x_\nu\frac{\sigma_j}{2} = \frac{1}{1+x_5}\,\frac{x_j\sigma_j}{2}$$

Define $A_\mu = A^i_\mu\,\tau_i$ where $\tau_i = \frac{\sigma_i}{2}$ are $SU(2)$ generators

$$\Rightarrow \boxed{A^i_\mu(x_5, x_\mu) = \frac{-1}{R(R+x_5)}\,\eta^i_{\mu\nu}\,x_\nu}$$

restore the radius from 1 to R.

Next Calculate the field strength F_{ab} ⑥

we need to use a few properties

① $A^i_a x_a = 0$

Proof: $A^i_a x_a = \frac{-1}{R(R+x_5)} \eta^i_{a\nu} x_\nu x_a = 0$

$\eta^i_{a\nu}$ is anti-symmetric for exchanging $a\nu$.

② $\eta^i_{\alpha\lambda} \eta^j_{\beta\lambda} = \delta^{ij} \delta_{\alpha\beta} + \varepsilon^{ijk} \eta^k_{\alpha\beta}$.

Proof: left $= (\varepsilon_{i\alpha\lambda 4} + \underline{\delta_{i\alpha}} \delta_{4\lambda} - \delta_{i\lambda} \delta_{4\alpha})(\varepsilon_{j\beta\lambda 4} + \delta_{j\beta} \delta_{4\lambda} - \underline{\delta_{j\lambda}} \delta_{4\beta})$

$= \underset{\sim\sim}{\varepsilon_{i\alpha\lambda 4} \varepsilon_{j\beta\lambda 4}} - \varepsilon_{i\alpha j 4} \delta_{4\beta} + \delta_{i\alpha} \delta_{j\beta}$

$- \varepsilon_{j\beta i 4} \delta_{4\alpha} + \underset{\sim\sim}{\delta_{ij} \delta_{4\alpha} \delta_{4\beta}}$

$= (\delta_{ij} \delta_{\alpha\beta} - \delta_{i\beta} \delta_{j\alpha}) - \varepsilon_{i\alpha j 4} \delta_{4\beta} + \delta_{i\alpha} \delta_{j\beta} - \varepsilon_{j\beta i 4} \delta_{4\alpha}$

two $\sim$ term add together

i, j take 1,2,3

$\alpha\beta$ take 1234

$= \delta_{ij} \delta_{\alpha\beta} + (\underline{\delta_{i\alpha} \delta_{j\beta} - \delta_{i\beta} \delta_{j\alpha}}$

$- \underset{\sim\sim}{\varepsilon_{i\alpha j 4} \delta_{\beta 4}} - \underset{\approx}{\varepsilon_{j\beta i 4} \delta_{\alpha 4}})$

Check $\varepsilon^{ijk} \eta^k_{\alpha\beta} = \varepsilon^{ijk} (\underline{\varepsilon_{k\alpha\beta 4}} + \underset{\sim}{\delta_{k\alpha} \delta_{4\beta}} - \underset{\approx}{\delta_{k\beta} \delta_{4\alpha}})$

Check terms ——, $\varepsilon^{ijk} \varepsilon_{\alpha\beta k 4} = \delta_{i\alpha} \delta_{j\beta} - \delta_{i\beta} \delta_{j\alpha}$

$$\left\{\begin{array}{l} \varepsilon^{ijk}\,\delta_{k\alpha}\,\delta_{4\beta} = \varepsilon^{ij\alpha}\,\delta_{\beta 4} = -\,\varepsilon^{i\alpha j}\,\delta_{\beta 4} = -\,\varepsilon^{i\alpha j 4}\,\delta_{4\beta} \\ -\,\varepsilon^{ijk}\,\delta_{k\beta}\,\delta_{4\alpha} = -\,\varepsilon^{ij\beta}\,\delta_{4\alpha} = -\,\varepsilon^{j\beta i 4}\,\delta_{4\alpha}\,. \end{array}\right.$$

(*) Now we calculate

$$F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu - \epsilon^{ijk} A^j_\mu A^k_\nu$$

① $A_5 = 0$. but $F^i_{5\mu} = \partial_5 A^i_\mu = -\,\partial_5 \left[\frac{1}{R(R+X_5)}\, \eta^i_{\mu\nu} X_\nu \right]$

$$= \left[\frac{1}{R^2(R+X_5)} \frac{X_5}{R} + \frac{1}{R(R+X_5)^2} \left(\frac{X_5}{R} + 1 \right) \right] \eta^i_{\mu\nu} X_\nu \qquad \underline{\nu \neq 5}$$

$$\frac{(R+X_5)X_5 + R(X_5+R)}{R^3(R+X_5)^2} = \frac{1}{R^3}$$

$$\Rightarrow \boxed{F^i_{5\mu} = \frac{1}{R^3}\, \eta^i_{\mu\nu} X_\nu = -\,\frac{R+X_5}{R^2}\, A^i_\mu .}$$

② for $\mu, \nu = 1, 2, 3, 4$

$$\partial_\mu A^i_\nu = -\,\partial_\mu \left(\frac{1}{R(R+X_5)}\, \eta^i_{\nu\lambda}\, x_\lambda \right)$$

$$= \left[\frac{1}{R^2(R+X_5)} \frac{X_\mu}{R} + \frac{1}{R(R+X_5)^2} \frac{X_\mu}{R} \right] \eta^i_{\nu\lambda}\, x_\lambda - \frac{1}{R(R+X_5)}\, \eta^i_{\nu\mu}$$

$$= \frac{2R + X_5}{R^3(R+X_5)}\, x_\mu\, \eta^i_{\nu\lambda}\, X_\lambda - \frac{1}{R(R+X_5)}\, \eta^i_{\nu\mu}$$

$$= -\frac{2R+X_5}{R^2(R+X_5)} X_\mu A^i_\nu - \frac{1}{R(R+X_5)} \eta^i_{\nu\mu}$$

$$\underbrace{\partial_\mu A^i_\nu - \partial_\nu A^i_\mu} = \frac{2R+X_5}{R^2(R+X_5)} (X_\nu A^i_\mu - X_\mu A^i_\nu) + \frac{2}{R(R+X_5)} \eta^i_{\mu\nu}$$

$$\varepsilon_{ijk} A^j_\mu A^k_\nu = \frac{1}{R^2(R+X_5)^2} \underline{\varepsilon_{ijk} \eta^j_{\mu\rho} \eta^k_{\nu\sigma}} X_\rho X_\sigma$$

property

$$\boxed{\varepsilon_{ijk} \eta^j_{\mu\rho} \eta^k_{\nu\sigma} = \delta_{\mu\nu} \eta^i_{\rho\sigma} - \delta_{\rho\nu} \eta^i_{\mu\sigma} - \delta_{\mu\sigma} \eta^i_{\rho\nu} + \delta_{\rho\sigma} \eta^i_{\mu\nu}}$$

$$\varepsilon_{ijk} A^j_\mu A^k_\nu = \frac{1}{R^2(R+X_5)^2} [\underbrace{\delta_{\mu\nu} \eta^i_{\rho\sigma}}_{\text{antisym with } \sigma \leftrightarrow \rho = 0} - \underline{\delta_{\rho\nu} \eta^i_{\mu\sigma}} - \delta_{\mu\sigma} \eta^i_{\rho\nu} + \underline{\delta_{\rho\sigma} \eta^i_{\mu\nu}}] X_\rho X_\sigma$$

$$—— = \frac{1}{R^2(R+X_5)^2} [-\eta^i_{\mu\sigma} X_\nu X_\sigma - \eta^i_{\rho\nu} X_\rho X_\mu]$$

$$= \frac{1}{R^2(R+X_5)^2} [-X_\nu \eta^i_{\mu\sigma} X_\sigma + X_\mu \eta^i_{\nu\rho} X_\rho]$$

$$= \frac{1}{R(R+X_5)} [X_\nu A^i_\mu - X_\mu A^i_\nu]$$

$\underline{\underline{\text{term}}}$ $\quad \frac{1}{R^2(R+X_5)^2} \eta^i_{\mu\nu} [x_1^2 + \cdots x_4^2] = \frac{R-X_5}{R^2(R+X_5)} \eta^i_{\mu\nu}$

$$\Rightarrow \varepsilon_{ijk} A^j_\mu A^k_\nu = \frac{-1}{R(R+X_5)} (X_\nu A^i_\mu - X_\mu A^i_\nu) + \frac{R-X_5}{R^2(R+X_5)} \eta^i_{\mu\nu}$$

$$F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu - \epsilon_{ijk} A^j_\mu A^k_\nu$$

$$= \frac{2R + X_5 - R}{R^2 (R + X_5)} \left(X_\nu A^i_\mu - X_\mu A^i_\nu \right) + \frac{2R - (R - X_5)}{R^2 (R + X_5)} \eta^i_{\mu\nu}$$

$$= \frac{1}{R^2} \left(X_\nu A^i_\mu - X_\mu A^i_\nu \right) + \frac{1}{R^2} \eta^i_{\mu\nu}$$

$\rightarrow$ unit sphere

$$\boxed{\begin{aligned} F_{\mu\nu} &= X_\nu A_\mu - X_\mu A_\nu + \eta^i_{\mu\nu} \frac{\sigma_i}{2} \\ F_{\mu 5} &= (1 + X_5) A_\mu \end{aligned}}$$

2nd chern number

The above definition is valid in the northern hemisphere. We could choose another gauge $\psi^0 = \frac{1+P_+}{2}\begin{pmatrix}0\\u\end{pmatrix}$. After normalization, we arrive at

$$\psi' = \frac{1}{\sqrt{2(1-x_5)}}\begin{pmatrix}(x_4+ix_i\sigma_i)u\\(1-x_5)\,u\end{pmatrix}.$$

Under this convention, we arrive at

$$\begin{cases} A'_\mu = -\frac{1}{1-x_5}\,\bar\eta^i_{\mu\nu}\,x_\nu\,\frac{\sigma_i}{2}\\ A'_5 = 0\end{cases}$$

where
$$\begin{cases}\bar\eta^i_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\,\eta^i_{\rho\sigma} = \epsilon_{i\mu\nu4} - \delta_{i\mu}\delta_{\nu4} + \delta_{i\nu}\delta_{\mu4}\\ \eta^i_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\,\bar\eta^i_{\rho\sigma}\end{cases}$$

Then
$$\boxed{\begin{aligned}&F'_{\mu\nu} = x_\mu A'_\nu - x_\nu A'_\mu + \bar\eta^i_{\mu\nu}\frac{\sigma_i}{2}\\ \text{or } &F^{i'}_{\mu\nu} = \frac{1}{R^2}(x_\mu A^{i'}_\nu - x_\nu A^{i'}_\mu) + \frac{1}{R^2}\bar\eta^i_{\mu\nu}\end{aligned}}$$
← on S^4 sphere

Let us compare $\psi^\alpha = \frac{1}{\sqrt{2(1+x_5)}}\begin{pmatrix}(1+x_5)u^\alpha\\(x_4-i\vec{x}\cdot\vec{\sigma})u^\alpha\end{pmatrix}$. $\psi'^\alpha = \frac{1}{\sqrt{2(1-x_5)}}\begin{pmatrix}(x_4+i\vec{x}\cdot\vec{\sigma})u^\alpha\\(1-x_5)u^\alpha\end{pmatrix}$

where $u^1 = \begin{pmatrix}1\\0\end{pmatrix}$, $u^2 = \begin{pmatrix}0\\1\end{pmatrix}$, $\alpha,\beta = 1,2$

Define
$$g_{\alpha\beta} = \langle\psi^\alpha|\psi'^\beta\rangle = \frac{1}{2\sqrt{1-x_5^2}}\,u_\alpha^\dagger\left((1+x_5)+(1-x_5)\right)(x_4+i\vec{x}\cdot\vec{\sigma})\,u_\beta$$
$$= \frac{1}{\sqrt{1-x_5^2}}\left(x_4+i\vec{x}\cdot\vec{\sigma}\right)_{\alpha\beta}$$

$$\left(A'_\mu dx_\mu\right)_{\alpha\beta} = -i\langle\psi'^{\alpha}|d|\psi'^{\beta}\rangle$$

$$\begin{cases} |\psi'^{\alpha}\rangle = \sum_\lambda |\psi^\lambda\rangle.\langle\psi^\lambda|\psi'^{\alpha}\rangle = \sum_\lambda |\psi^\lambda\rangle g_{\lambda\alpha} \\ \langle\psi'^{\alpha}| = \sum_\lambda g^\dagger_{\alpha\lambda}\langle\psi^\lambda| \end{cases}$$

$$\Rightarrow (A'_\mu dx_\mu)_{\alpha\beta} = -i\sum_\lambda g^\dagger_{\alpha\lambda}\langle\psi^\lambda|d\left(|\psi^\delta\rangle g_{\delta\beta}\right)$$

$$= -i\sum_\lambda g^\dagger_{\alpha\lambda}\{\langle\psi^\lambda|d|\psi^\delta\rangle\} g_{\delta\beta} - i\sum_\lambda g^\dagger_{\alpha\lambda}\delta_{\lambda\delta}\, d g_{\delta\beta}$$

$$= \left[g^\dagger(A_\mu dx_\mu) g\right]_{\alpha\beta} - i(g^\dagger dg)_{\alpha\beta}.$$

under SU(2) transformation

$$\boxed{A'_\mu = g^\dagger A_\mu g - i g^\dagger \partial_\mu g}$$

Check field strength : $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]$

$$F'_{\mu\nu} = \partial_\mu A'_\nu - \partial_\nu A'_\mu + i[A'_\mu, A'_\nu]$$

$$\partial_\mu A'_\nu = (\partial_\mu g^\dagger) A_\nu g + g^\dagger(\partial_\mu A_\nu) g + g^\dagger A_\nu(\partial_\mu g) - i\,\partial_\mu(g^\dagger\partial_\nu g)$$

$$\partial_\nu A'_\mu = (\partial_\nu g^\dagger) A_\mu g + g^\dagger(\partial_\nu A_\mu) g + g^\dagger A_\mu(\partial_\nu g) - i\,\partial_\nu(g^\dagger\partial_\mu g)$$

$$i[A'_\mu, A'_\nu] = i[g^\dagger A_\mu g, g^\dagger A_\nu g] + [g^\dagger A_\mu g, g^\dagger\partial_\nu g] + [g^\dagger\partial_\mu g, g^\dagger A_\nu g] - i[g^\dagger\partial_\mu g, g^\dagger\partial_\nu g]$$

only underline —— terms survive $\Rightarrow g^\dagger F_{\mu\nu} g$

check ~ terms: $(\partial_\mu g^\dagger) A_\nu g + g^\dagger A_\nu (\partial_\mu g) + [g^\dagger \partial_\mu g, g^\dagger A_\nu g]$

$$= (\partial_\mu g^\dagger) A_\nu g - g^\dagger \partial_\mu g^\dagger A_\nu g + g^\dagger A_\nu \partial_\mu g - g^\dagger A_\nu \partial_\mu g = 0$$

Similarly - - - terms

$$-\partial_\mu g^\dagger A_\mu g - g^\dagger A_\mu \partial_\mu g + [g^\dagger A_\mu g \; g^\dagger \partial_\nu g] = 0.$$

check xxxx terms

$$\partial_\mu (g^\dagger \partial_\nu g) - \partial_\nu (g^\dagger \partial_\mu g) + [g^\dagger \partial_\mu g, g^\dagger \partial_\nu g]$$

$$= \partial_\mu g^\dagger \partial_\nu g - \partial_\nu g^\dagger \partial_\mu g - \partial_\mu g^\dagger \partial_\nu g + \partial_\nu g^\dagger \partial_\mu g = 0$$

we used $g^\dagger \partial_\mu g = -\partial_\mu g^\dagger g$ and $g \partial_\mu g^\dagger = -\partial_\mu g g^\dagger$

$$\Rightarrow \boxed{F'_{\mu\nu} = g^\dagger F_{\mu\nu} g}$$

Calculate the topological charge.

$$\oint_{S^4} d^4x \, tr[\tilde{F}^{ab} F_{ab}] \quad \text{where } \tilde{F}^{ab} = \frac{1}{2}\varepsilon^{abcd} F_{cd}$$

$$tr[\tilde{F}^{ab} F_{ab}] = \frac{1}{2}\varepsilon^{abcd} tr[F_{ab} F_{cd}]$$

$$F_{ab} = \partial_a A_b - \partial_b A_a + i(A_a A_b - A_b A_a)$$

$$\varepsilon^{abcd} tr[F_{ab} F_{cd}] = \varepsilon^{abcd}\{tr[(\partial_a A_b + iA_a A_b)(\partial_c A_d + iA_c A_d)]$$

$$+ tr[(\partial_b A_a + iA_b A_a)(\partial_d A_c + A_d A_c)]\}$$

$$- \varepsilon^{abcd} tr[(\partial_b A_a + iA_b A_a)(\partial_c A_d + iA_c A_d]$$

$$+ [\partial_a A_b + iA_a A_b)(\partial_d A_c + iA_d A_c)]$$

$$= 4\varepsilon^{abcd} tr[(\partial_a A_b + iA_a A_b)(\partial_c A_d + iA_c A_d)]$$

$$\frac{1}{2}\varepsilon^{abcd} tr[F_{ab} F_{cd}] = 2\varepsilon^{abcd} tr[\partial_a A_b \partial_c A_d$$

$$+ 2i(\partial_a A_b) A_c A_d + \underbrace{i^2 A_a A_b A_c A_d}_{\text{zero from cyclo of trace.}}]$$

$$\partial_a A_b \partial_c A_d = \partial_a (A_b \partial_c A_d) - \underbrace{A_b \partial_a \partial_c A_d}_{=0 \text{ after } \varepsilon^{abcd}}$$

$$\varepsilon^{abcd} tr[(\partial_a A_b) A_c A_d] = \varepsilon^{abcd} tr[\partial_a (A_b A_c A_d) - (\partial_a A_c) A_d A_b$$

$$- (\partial_a A_d) A_b A_c]$$

$$\Rightarrow 3\varepsilon^{abcd} tr[(\partial_a A_b) A_c A_d] = \varepsilon^{abcd} tr[\partial_a (A_b A_c A_d)]$$

$$\Rightarrow \frac{1}{2}\varepsilon^{abcd} tr[F_{ab} F_{cd}] = 4\varepsilon^{abcd} \partial_a tr\left[\frac{1}{2} A_b \partial_c A_d + \frac{i}{3} A_b A_c A_d\right]$$

$$\Rightarrow \oint d^4x\, \mathrm{tr}[\tilde{F}^{ab} F_{ab}] = 2\oint d^4x\, \partial_a\, \varepsilon^{abcd}\, \mathrm{tr}\left[A_b \partial_c A_d + \frac{2}{3} i A_b A_c A_d\right]$$

Define $X^a = \varepsilon^{abcd}\, \mathrm{tr}\left[\frac{1}{2} A_b \partial_c A_d + \frac{1}{3} i A_b A_c A_d\right]$

$$\Rightarrow \oint d^4x\, \mathrm{tr}[\tilde{F}^{ab} F_{ab}] = 4 \oint d^4x\, \partial_a X^a$$

$$= 2 \oint_{S^3} d\sigma_{abc}\, \mathrm{tr}\left(A_a \partial_b A_c + \frac{2}{3} i A_a A_b A_c\right]$$

↖ northern hemisphere

$$- 2 \oint_{S^3} d\sigma_{abc}\, \mathrm{tr}\left[\tilde{A}_a \partial_b \tilde{A}_c + \frac{2}{3} i \tilde{A}_a \tilde{A}_b \tilde{A}_c\right]$$

$$\tilde{A}_a = g^\dagger A_a g - i g^\dagger \partial_a g$$

$$\tilde{A}_a \partial_b \tilde{A}_c = \left[(g^\dagger A_a g) - i g^\dagger \partial_a g\right] \partial_b \left[g^\dagger A_c g - i g^\dagger \partial_c g\right]$$

$$= (g^\dagger A_a g)\, \partial_b (g^\dagger A_c g) - i (g^\dagger \partial_a g)\, \partial_b (g^\dagger A_c g)$$

$$- i (g^\dagger A_a g)\, \partial_b (g^\dagger \partial_c g) - (g^\dagger \partial_a g)\, \partial_b (g^\dagger \partial_c g)$$

first term: $g^\dagger A_a g (\partial_b g^\dagger) A_c g + g^\dagger A_a \partial_b A_c g + g^\dagger A_a A_c \partial_b g$

$$\xrightarrow{\mathrm{tr}} (g \partial_b g^\dagger) A_c A_a + A_a A_c (\partial_b g\, g^\dagger) + A_a \partial_b A_c$$

$$= (g \partial_b g^\dagger)(A_c A_a - A_a A_c) + A_a \partial_b A_c$$

$d\sigma_{abc}$

anti-sym w.s exchange $a \leftrightarrow c$.

$$d\sigma_{abc} \longrightarrow 2 g \partial_b g^\dagger A_c A_a + A_a \partial_b A_c$$

$$-i(g^\dagger\partial_a g)\,\partial_b(g^\dagger A_c g)\ -i(g^\dagger A_a g)\,\partial_b(g^\dagger\partial_c g)$$

2nd term $\rightarrow$ $i\,\partial_b(g^\dagger A_a g)\,g^\dagger\partial_c g \xrightarrow{tr} i g^\dagger\partial_c g\ \partial_b(g^\dagger A_a g)$

$$\xrightarrow{d\sigma_{abc}} -i(g^\dagger\partial_a g)\,\partial_b(g^\dagger A_c g)$$

$\Rightarrow$ $(g^\dagger\partial_a g)\,\partial_b(g^\dagger A_c g) = g^\dagger\partial_a g\,(\partial_b g^\dagger)A_c g + g^\dagger\partial_a g\, g^\dagger(\partial_b A_c)g$

$$+\ g^\dagger\partial_a g\ g^\dagger A_c(\partial_b g)$$

$$\xrightarrow{tr} (\partial_a g)(\partial_b g^\dagger)A_c \ \underline{-g\,\partial_a g^\dagger\,\partial_b A_c} - \partial_a g^\dagger A_c\,\partial_b g$$

$$\swarrow$$

$$\partial_b(g\,\partial_a g^\dagger)A_c = \partial_b g\,\partial_a g^\dagger A_c + \underbrace{g\,\partial_b\partial_a g^\dagger A_c}_{\to 0}$$

$$\xrightarrow{tr} (\partial_a g)(\partial_b g^\dagger)A_c + (\partial_b g)(\partial_a g^\dagger)A_c - (\partial_b g)(\partial_a g^\dagger)A_c$$

$$= (\partial_a g\,\partial_b g^\dagger)A_c$$

$\Rightarrow$ $\cancel{-ig(g}$ $-i(g^\dagger\partial_a g)\partial_b(g^\dagger A_c g) - i(g^\dagger A_a g)\partial_b(g^\dagger\partial_c g)$

$$\xrightarrow{tr,\ d\sigma_{abc}} -2i(\partial_a g\,\partial_b g^\dagger)A_c$$

$$-(g^\dagger\partial_a g)\,\partial_b(g^\dagger\partial_c g) \longrightarrow -g^\dagger\partial_a g\ \partial_b g^\dagger\ \partial_c g\ \underbrace{-\,g^\dagger\partial_a g\ g^\dagger\partial_b\partial_c g}_{\to 0}$$

$$= -g^\dagger\partial_a g\ \partial_b g^\dagger\ g\,g^\dagger\,\partial_c g$$

$$= (g^\dagger\partial_a g)(g^\dagger\partial_b g)(g^\dagger\partial_c g)$$

$\cdot\ \tilde{A}_a \partial_b \tilde{A}_c \to A_a \partial_b A_c + \underline{2g\partial_b g^\dagger A_c A_a}$ ② $'$

$- \underline{2i(\partial_a g\, \partial_b g^\dagger) A_c}_{③'} + (g^\dagger \partial_a g)(g^\dagger \partial_b g)(g^\dagger \partial_c g)$

$\frac{2}{3} i \tilde{A}_a \tilde{A}_b \tilde{A}_c \to \frac{2}{3} i \Big[g^\dagger A_a g\, g^\dagger A_b g\, g^\dagger A_c g$ ①

$+ (-i g^\dagger \partial_a g)\, g^\dagger A_b g\, g^\dagger A_c g$ + permutations ②

$+ (-i g^\dagger \partial_a g)(-i g^\dagger \partial_b g)\, g^\dagger A_c g$ + permutations ③

$+ (-i g^\dagger \partial_a g)(-i g^\dagger \partial_b g)(-i g^\dagger \partial_c g) \Big]$ ④

① $\xrightarrow{\text{after tr}}$ $\frac{2}{3} i A_a A_b A_c$

② $\xrightarrow[\text{permutation}]{\text{after tr}}$ $\frac{2}{3} \times 3.\ \partial_a g\, g^\dagger A_b A_c = -2\, g \partial_a g^\dagger A_b A_c$

② + ②$'$ cancel $= 0$

③ $\frac{2}{3}(-i) \times 3\ \ \partial_a g\, g^\dagger \partial_b g\, g^\dagger A_c$

$= \ 2i\ \partial_a g\, \partial_b g^\dagger\, A_c$ $\Rightarrow$ ③ + ③$'$ cancel $= 0$

④ $-\frac{2}{3} \,\mathrm{tr}[g^\dagger \partial_a g\ \ g^\dagger \partial_b g\ \ g^\dagger \partial_c g]$

$$\oint_{S^3} d\sigma_{abc}\, tr[\tilde{A}_a \partial_b \tilde{A}_c + \frac{2}{3} i \tilde{A}_a \tilde{A}_b \tilde{A}_c]$$

$$= \oint_{S^3} d\sigma_{abc}\, tr[A_a \partial_b A_c + \frac{2}{3} i A_a A_b A_c]$$

$$+ \frac{1}{3} \oint_{S^3} d\sigma_{abc}\, tr[g^\dagger \partial_a g\, g^\dagger \partial_b g\, g^\dagger \partial_c g]$$

$$\Rightarrow \oint d^4x\, tr[\tilde{F}^{ab} F_{ab}] = -\frac{2}{3} \oint_{S^3} d\sigma_{abc}\, tr[g^\dagger \partial_a g\, g^\dagger \partial_b g\, g^\dagger \partial_c g]$$

This defines an mapping fun $S^3 \to S^3$. define $g = n_0 - i n_i \sigma_i$
$= n_\mu e_\mu$

where $n_0^2 + n_1^2 + n_2^2 + n_3^2 = 1$

$$g^\dagger \partial_a g = n_b \bar{e}_b \partial_a (n_c e_c) = n_b \bar{e}_b (\partial_a n_c) e_c$$

$$= n_b \partial_a n_c\, \bar{e}_b e_c = \underbrace{n_b \partial_a n_b}_{0} + n_b \partial_a n_c\, \eta^d_{bc}\, e_d$$

Define

$$\bar{e}_a e_b = \delta_{ab} + \eta^c_{ab} e_c$$

$\eta^c_{ab} = \varepsilon_{abc}$ for $a, b, c \in 1, 2, 3$

$-\eta^c_{0b} = \eta^c_{b0} = \delta_{bc}$

$$\Rightarrow \oint d^4x\, tr[\tilde{F}^{ab} F_{ab}] = -\frac{2}{3} \oint_{S^3} d\sigma_d\, \varepsilon_{dabc}\, tr\big(n_\mu \partial_a n_\nu\, \eta^{a'}_{\mu\nu}\, e_{a'}\; n_{\mu'} \partial_b n_{\nu'}\, \eta^{b'}_{\mu'\nu'}\, e_{b'}\; n_{\mu''} \partial_c n_{\nu''}\, \eta^{c'}_{\mu''\nu''}\, e_{c'} \big)$$

Define $g = n_0 - i n_i \sigma_i$

$$g^\dagger \partial_a g = (n_0 + i n_i \sigma_i)\, \partial_a (n_0 - i n_j \sigma_j)$$

$$= n_\mu \partial_a n_\mu + [n_0 \partial_a n_i - n_i \partial_a n_0](-i\sigma_i]$$

$$n_i \partial_a n_j \quad \epsilon_{ijk}\, i \sigma_k .$$

$$= i\sigma_k [(n_k \partial_a n_0 - n_0 \partial_a n_k) + \epsilon_{kij}\, n_i \partial_a n_j]$$

$$tr[g^\dagger \partial_a g \; g^\dagger \partial_b g \; g^\dagger \partial_c g]$$

$$= tr[i\sigma_{k_1}\, i\sigma_{k_2}\, i\sigma_{k_3} \; (n_{k_1} \partial_a n_0 - n_0 \partial_a n_{k_1} + \varepsilon_{k_1 i_1 j_1} n_{i_1} \partial_a n_{j_1})$$
$$(n_{k_2} \partial_b n_0 - n_0 \partial_a n_{k_2} + \varepsilon_{k_2 i_2 j_2} n_{i_2} \partial_b n_{j_2})$$
$$(n_{k_3} \partial_a n_0 - n_0 \partial_c n_{k_3} + \varepsilon_{k_3 i_3 j_3} n_{i_3} \partial_c n_{j_3})]$$

$$= 2\epsilon_{k_1 k_2 k_3} \{ (n_{k_1} \partial_a n_0 - n_0 \partial_a n_{k_1}) + \varepsilon_{k_1 i_1 j_1} n_{i_1} \partial_a n_{j_1})$$
$$(n_{k_2} \partial_b n_0 - n_0 \partial_b n_{k_2}) + \varepsilon_{k_2 i_2 j_2} n_{i_2} \partial_b n_{j_2})$$
$$(n_{k_3} \partial_c n_0 - n_0 \partial_c n_{k_3}) + \varepsilon_{k_3 i_3 j_3} n_{i_3} \partial_c n_{j_3}) \}$$

near $n_0 \simeq 1$, $n_{i,j,k}$ small ~~n~~ number

$$-2\, \varepsilon_{k_1 k_2 k_3}\, \partial_a n_{k_1} \partial_b n_{k_2} \partial_c n_{k_3} \quad \varepsilon^{abc}$$

~~Co~~ Consider the area perpendicular to n_0-direction

$$d\sigma_0\, \varepsilon_{abc}\, \varepsilon_{k_1 k_2 k_3}\, \partial_a n_{k_1} \partial_b n_{k_2} \partial_c n_{k_3}$$

$\epsilon_{k_1 k_2 k_3}\, \partial_a n_{k_1}\, \partial_b n_{k_2}\, \partial n_{k_3}$ is the determined from local $R^3 \to$ local R^3 on a S^3. Due to rotation invarince, we can ~~the~~ extend the mapping to the entire $S^3 \to S^3$ sphere.

$$\Rightarrow \oint d^4x \, tr\left[\tilde{F}^{ab} F_{ab}\right] = +\frac{4}{3} \cdot \overbrace{3! \oint_{S^3}}^{C_2} \underbrace{n_\alpha \, \partial_{\theta_1} n_\beta \, \partial_{\theta_2} n_\gamma \, \partial_{\theta_3} n_\delta}_{2\pi^2}$$

$$= 8\pi^2 C_2$$

Here C_2 means the covering number from $S^3 \to S^3$.

Then the $SU(2)$ monopole charge is also quantized in terms of a topological #, denoted as the 2nd Chern number.