Lect 12. Quantum Inverse method

§ Quantum Inverse method

§ Nested Bethe ansatz

§ Lieb-Wu solution to the Hubbard model

§ Ogata-Shiba solution

Ref: 1. C. N. Yang. Phys. Rev. Lett. 19. 1312 (1967)

2. Lieb, Wu Phy. Rev. Lett. 20, 1415 (1968)

3. M. Ogata, H. Shiba, Phy. Rev. B 41, 2326 (1990)

Lect 6: Quantum inverse method — Fadeev's method ①

In last class, we have found

$$\left[S_{j+1,j}\, S_{j+2,j} \cdots S_{N,j}\, S_{1j}\, S_{2j} \cdots S_{j-1,j} \right] A_{\sigma_1 \cdots \sigma_N}(12\cdots N, 12\cdots N) = e^{ik_j L} A_{\sigma_1 \cdots \sigma_N}(12\cdots ; 12\cdots$$

with $S_{ij} = \dfrac{k_i - k_j + ic\, P_{\sigma_i \sigma_j}}{k_i - k_j + ic}$

In order to solve this problem, we use the following method of algebraic BA. We define an auxilary space A, and $\vec{\tau}$ as a Pauli matrix in such a space. Define $P^{j,A} = \frac{1}{2}(1 + \vec{\sigma}_j \cdot \vec{\tau})$, and the auxillary S-matrix

$$\boxed{S^{jA}[u] = \frac{k_j - cu}{k_j - cu + ic} + \frac{ic}{k_j - cu + ic} P^{j,A}}$$

The monodromy matrix

$T(u) = S^{1A}[u]\, S^{2A}[u] \cdots S^{NA}[u]$, where the matrix product only acts in the auxillary space A.

$$T_{\sigma_1 \sigma_2 \cdots \sigma_N u;\, \sigma_1' \sigma_2' \cdots \sigma_N' v} = S^{1A}_{\sigma_1 u;\, \sigma_1' \omega_1}[u]\, S^{2A}_{\sigma_2 \omega_1;\, \sigma_2' \omega_2}[u] \cdots S^{NA}_{\sigma_N \omega_{N-1};\, \sigma_N' v}[u]$$

Trace over the auxillary space $\Rightarrow$

$$tr_A [T(u_j)] = S_{j+1,j}\, S_{j+2,j} \cdots S_{N,j}\, S_{1,j}\, S_{2j} \cdots S_{j-1,j}$$

where $u_j = k_j / c$.

Proof: $tr_A[T(u_j)] = S^{1A}_{\sigma_1 u_1, \sigma_1' \underbrace{u_2 \quad S^{2A}_{\sigma_2 u_2}}, \sigma_2' u_3} \cdots S^{NA}_{\sigma_N u_N, \sigma_N' u_1}$

we can cyclic rotation over trace indices in the A-space

$$tr_A[T(u_j)] = tr_A[S^{j+1,A}[u_j] S^{j+2,A} \cdot\ S^{N,A}\ S^{1A} \cdots S^{j-1,A}\ S^{j,A}[u_j]]$$

since $S^{j,A}(u_j) = \dfrac{k_j - cu_j + icP^{j,A}}{k_j - cu_j + ic} = P^{j,A}$

and we can prove an identity that $\boxed{S^{lA}(u_j)\, P^{jA} = P^{jA}\, S^{jl}(u_j)}$

Then $tr_A[T(u_j)] = tr_A[S^{j+1,A}[u_j] \cdots S^{N,A}[u_j]\ S^{1A}[u_j] \cdots\ S^{j-1,A}[u_j]\, P^{jA}]$

$$= tr_A[S^{j+1,A}[u_j] \cdots\cdots S^{j-2,A}\ P^{jA}\ S^{j-1,j}[u_j]]$$

$$= tr_A[P^{jA}]\quad S^{j+1,j}[u_j]\ S^{j+2,j}[u_j] \cdots S^{N,j}[u_j]\ S^{1,j} \cdots S^{j-1,j}[u_j]$$

$P^{j,A} = \frac{1}{2}(1 + \vec{\sigma}_j \cdot \vec{\tau}) \Rightarrow tr_A[P^{j,A}] = 1$

$\Rightarrow \boxed{tr_A[T(u_j)] = S_{j+1,j} \cdots S_{N,j}\ S_{1,j} \cdots S_{j-1,j}}$, where $u_j = k_j/c$.

check: $S^{lA}(u_j)\, P^{jA} = P^{jA}\, S^{jl}(u_j) \longrightarrow$ only need to check $P^{lA} P^{jA} = P^{jA} P^{lj}$

$$P^{lA} P^{jA} = \tfrac{1}{4}(1 + \vec{\sigma}_l \cdot \vec{\tau})(1 + \vec{\sigma}_j \cdot \vec{\tau}) = \tfrac{1}{4}(1 + \vec{\sigma}_l \cdot \vec{\tau} + \vec{\sigma}_j \cdot \vec{\tau} + \sigma_l^{\alpha} \sigma_j^{\beta} \tau^{\alpha} \tau^{\beta})$$

$$P^{jA} P^{lj} = \tfrac{1}{4}(1 + \vec{\sigma}_j \cdot \vec{\tau})(1 + \vec{\sigma}_l \cdot \vec{\sigma}_j) = \tfrac{1}{4}(1 + \vec{\sigma}_j \cdot \vec{\tau} + \vec{\sigma}_l \cdot \vec{\sigma}_j + \sigma_j^{\alpha} \sigma_j^{\beta} \tau^{\alpha} \sigma_l^{\beta})$$

$$\tau^{\alpha} \tau^{\beta} = \delta_{\alpha\beta} + i \epsilon_{\alpha\beta\gamma} \tau^{\gamma}, \quad \sigma_j^{\alpha} \sigma_j^{\beta} = \delta_{\alpha\beta} + i \epsilon_{\alpha\beta\gamma} \sigma_j^{\gamma}$$

$\Rightarrow$ it's easy to check $\boxed{P^{lA} P^{jA} = P^{jA} P^{lj}}$

Then the Bethe ansatz equation is reduced to

$$\boxed{\left[tr_A T(k_j/c)\right]_{\sigma_1\cdots\sigma_N,\,\sigma_1'\cdots\sigma_N'} A_{\sigma_1'\cdots\sigma_N'}(12\cdots N; 12\cdots N) = e^{ik_jL} A^{(12\cdots N, 12\cdots}_{\sigma_1\cdots\sigma_N}}$$

we need to diagonalize $tr_A T(k_j/c)$ simultanously for $j=1,2,3,\cdots N$.

Define $b(x) = \dfrac{-x}{-x+i}$, $c(x) = \dfrac{i}{-x+i}$

and R-matrix in two auxillary spaces $A \otimes B$

$$R^{AB}(u) = c(u) + b(u) P_{AB} = (b(u) + c(u) P_{AB}) P_{AB}$$

$$S_{jA}(u) = b(u - k_j/c) + c(u - k_j/c) P_{jA}$$

$$S_{jB}(u) = b(u - k_j/c) + c(u - k_j/c) P_{jB}$$

We have the fundamental commutation relation:

$$\boxed{R^{AB}(u-v)\left[S^{jA}(u) \otimes S^{jB}(v)\right] = \left[S^{jA}(v) \otimes S^{jB}(u)\right] R^{AB}(u-v)}$$

Proof: $\text{LHS} = P_{AB}\left[\dfrac{v-u+iP_{AB}}{v-u+i}\right]\left[\dfrac{u_j-u+iP_{jA}}{u_j-u+i}\right]\left[\dfrac{u_j-v+iP_{jB}}{u_j-v+i}\right]$

$$= \left[\frac{v-u+iP_{AB}}{v-u+i}\right]\left[\frac{u_j-u+iP_{Bj}}{u_j-u+i}\right] P_{AB} \left[\frac{u_j-v+iP_{jB}}{u_j-v+i}\right]$$

$$= \left[\frac{v-u+iP_{AB}}{v-u+i}\right]\left[\frac{u_j-u+iP_{Bj}}{u_j-u+i}\right]\left[\frac{u_j-v+iP_{jA}}{u_j-v+i}\right] P_{AB}$$

compare with RHS $\left[\dfrac{u_j-v+iP_{jA}}{u_j-v+i}\right]\left[\dfrac{u_j-u+iP_{jB}}{u_j-u+i}\right]\left[\dfrac{v-u+iP_{AB}}{v-u+i}\right] P_{AB}$

Check $P_{AB} P_{Bj} P_{jA} = P_{AB} P_{jA} P_{AB} = P_{jA} P_{jB} P_{AB}$ ✓

$$(v-u) P_{Bj} P_{jA} + (u_j - u) P_{AB} P_{jA} + (u_j - v) P_{AB} P_{Bj}$$

$$= (v-u) P_{AB} P_{Bj} + (u_j - u) P_{AB} P_{jA} + (u_j - v) P_{AB} P_{Bj}$$

$$= (u_j - u) P_{AB} (P_{Bj} + P_{Aj})$$

$$(u_j - v) P_{jB} P_{AB} + (u_j - u) P_{jA} P_{AB} + (v-u) P_{jA} P_{jB} \leftarrow P_{jB} P_{AB}$$

$$= (u_j - v + v - u) P_{jB} P_{AB} + (u_j - u) P_{jA} P_{AB} .$$

$$= (u_j - u) (P_{jB} + P_{Aj}) P_{AB} = (u_j - u) P_{AB} (P_{Aj} + P_{Bj})$$ ✓

terms only involving one permutation are the same ✓.

Now we can generalize it

$$R^{AB}_{(u-v)} [S^{j-1,A}_{(u)} S^{jA}(u) \otimes S^{j-1B}(v) S^{jB}(v)]$$

$$= R^{AB}_{(u-v)} [S^{j-1A}(u) \otimes S^{j-1B}(v) . S^{jA}(u) \otimes S^{jB}(v)]$$

$$= [S^{j-1,A}(v) \otimes S^{j-1B}(u)] R^{AB}(u-v) [S^{jA}(u) \otimes S^{jB}(v)]$$

$$= [S^{j-1,A}(v) \otimes S^{j-1,B}(u)] [S^{j,A}(v) \otimes S^{jB}(u)] R^{AB}(u-v)$$

$$= [S^{j-1A}(v) S^{jA}(v) \otimes S^{j-1B}(u) S^{jB}(u)] R^{AB}(u-v)$$

$$\Rightarrow R^{AB}(u-v) [S^{1A}(u) \cdots S^{NA}(u) \otimes S^{1B}(v) \cdots S^{NB}(v)]$$

$$= [S^{1A}(v) \cdots S^{NA}(v) \otimes S^{1B}(u) \cdots S^{NB}(u)] R^{AB}(u-v)$$

RTT

i.e. $$\boxed{R^{AB}(u-v)\, T^{A}(u) \otimes T^{B}(v) = T^{A}(v) \otimes T^{B}(u) \quad R^{AB}(u-v)}$$

R^{AB} is a 4×4 matrix in the $A\otimes B$ space

$$R^{AB}(u) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c(u) & b(u) & 0 \\ 0 & b(u) & c(u) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and $T^{A}(u)$ is a 2×2 matrix in the A space

$$T(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}$$ and A, B, C, D are $2^N \times 2^N$ matrices in the physical Hilbert space

Based on the "RTT" relation, we have quite a few usefully relations among operators $A(u)$, $B(u)$, $C(u)$ and $D(u)$.

① $A(u)\,A(v) = A(v)\,A(u)$

② $A(u)\,B(v) = c(u-v)\,A(v)\,B(u) + b(u-v)\,B(v)\,A(u)$

③ $B(u)\,A(v) = c(u-v)\,B(v)\,A(u) + b(u-v)\,A(v)\,B(u)$

④ $B(u)\,B(v) = B(v)\,B(u)$

⑤ $A(v)\,B(u) = c(u-v)\,A(u)\,B(v) + b(u-v)\,B(u)\,A(v)$
the same as ②

⑥ $c(u-v)\left[A(u)\,D(v) - A(v)\,D(u)\right] = b(u-v)\left[B(v)\,C(u) - C(u)\,B(v)\right]$

⑦ $c(u-v)\left[B(u)\,C(v) - B(v)\,C(u)\right] = b(u-v)\left[A(v)\,D(u) - D(u)\,A(v)\right]$

⑧ $B(v)\,D(u) = c(u-v)\,B(u)\,D(v) + b(u-v)\,D(u)\,B(v)$

(9) $C(v)A(u) = c(u-v)\,C(u)\,A(v) + b(u-v)\,A(u)\,C(v)$

(10) $c(u-v)\left[C(u)\,B(v) - C(v)\,B(u)\right] = b(u-v)\left[D(v)A(u) - A(u)D(v)\right]$

(11) $c(u-v)\left[D(u)A(v) - D(v)A(u)\right] = b(u-v)\left[C(v)\,B(u) - B(u)\,C(v)\right]$

(12) $D(v)\,B(u) = c(u-v)\,D(u)\,B(v) + b(u-v)\,B(u)\,D(v)$

(13) $C(u)\,C(v) = C(v)\,C(u)$

(14) $C(u)\,D(v) = c(u-v)\,C(v)\,D(u) + b(u-v)\,D(v)\,C(u)$

(15) $D(u)\,C(v) = c(u-v)\,D(v)\,C(u) + b(u-v)\,C(v)\,D(u)$

(16) $D(u)\,D(v) = D(v)\,D(u)$

The proof is straight forward from the RTT relation.

$$R^{AB}(u-v)\,T^A(u)\otimes T^B(v) = T^A(v)\otimes T^B(u)\,R^{AB}(u-v)$$

which is a 4x4 matrix identity in the $A\otimes B$ space. Expand it and check term by ~~te~~rm. See Appendix for proof. Now we will use it to prove

$$\boxed{\left[\mathrm{tr}_A T(u),\ \mathrm{tr}_A T(v)\right] = 0\ .}$$

Proof: $tr_A T(u) = A(u) + D(u)$, $tr\, T(v) = A(v) + D(v)$

$$[tr_A T(u), tr_A T(v)] = [A(u)+D(u), A(v)+D(v)]$$

$$= [A(u), A(v)] + [D(u), D(v)] + (A(u)D(v) - A(v)D(u) + D(u)A(v) - D(v)A(u)]$$

From relation ① $\Rightarrow [A(u), A(v)] = 0$

⑯ $\Rightarrow [D(u), D(v)] = 0$

⑥ $\Rightarrow$ $\underline{A(u)D(v) - A(v)D(u) = \frac{b(u-v)}{c(u-v)}[B(v)C(u) - C(u)B(v)]}$

⑪ exchang u and v

$$\underline{D(v)A(u) - D(u)A(v) = \frac{b(v-u)}{c(v-u)}[C(u)B(v) - B(v)C(u)]}$$

since $b(x)/c(x) = -x/i = ix \Rightarrow \frac{b(u-v)}{c(u-v)} = -\frac{b(v-u)}{c(v-u)}$

$$\Rightarrow A(u)D(v) - A(v)D(u) = D(v)A(u) - D(u)A(v)$$

$$\Rightarrow [tr_A T(u), tr_A T(v)] = 0.$$

★ In the A-space, $S^{jA}(u) = b(u-u_j) + c(u-u_j)\frac{1}{2}(1 + \vec{\sigma}_j \cdot \vec{\tau}^A)$

$$= b(u-u_j) + \frac{c(u-u_j)}{2}\left[(1+\sigma_j^z)\frac{\tau_A^z+1}{2} + (1-\sigma_j^z)\frac{1-\tau_A^z}{2} + 2\sigma_+\frac{\tau_x - i\tau_y}{2} + 2\sigma_-\frac{\tau_x + i\tau_y}{2}\right]$$

$$\boxed{S^{jA}(u) = b(u-u_j) + c(u-u_j)\begin{bmatrix} \frac{1+\sigma_j^z}{2} & \sigma_- \\ \sigma_+ & \frac{1-\sigma_j^z}{2} \end{bmatrix}}$$

In the Physical Hilbert space, $H_1 \otimes H_2 \cdots \otimes H_n$, we define the vacuum state $|0\rangle = |\uparrow\uparrow\cdots\uparrow\rangle$.

Apply $S^{jA}(u)$ on $|0\rangle$, we have

$$S^{jA}(u)|0\rangle = \begin{bmatrix} b(u-u_j)+c(u-u_j) & c(u-u_j)\sigma_- \\ c(u-u_j)\sigma_+ & b(u-u_j) \end{bmatrix} |0\rangle$$

$$= \begin{bmatrix} 1 & c(u-u_j)\sigma_-^j \\ 0 & b(u-u_j) \end{bmatrix} |0\rangle ,$$

we have used $b(u-u_j)+c(u-u_j)=1$

$$\Rightarrow T(u)|0\rangle = S^{1A}(u)\, S^{2A}(u) \cdots S^{NA}(u)\, |0\rangle$$

$$= \begin{bmatrix} 1 & c(u-u_1)\sigma_-^1 \\ 0 & b(u-u_1) \end{bmatrix} \begin{bmatrix} 1 & c(u-u_2)\sigma_-^2 \\ 0 & b(u-u_2) \end{bmatrix} \cdots \begin{bmatrix} 1 & c(u-u_N)\sigma_-^N \\ 0 & b(u-u_N) \end{bmatrix} |0\rangle$$

$$= \begin{pmatrix} 1 & B(u) \\ 0 & \prod_{j=1}^{N} b(u-u_j) \end{pmatrix} |0\rangle$$

← complicated

can be proved by induction.

$$\boxed{T(u)|0\rangle = \begin{pmatrix} 1 & B(u) \\ 0 & \prod_{j=1}^{N} b(u-u_j) \end{pmatrix} |0\rangle}$$

Hence

$$A(u)|0\rangle = |0\rangle$$

$$D(u)|0\rangle = \prod_{j=1}^{N} b(u-u_j)\, |0\rangle$$

$$C(u)|0\rangle = 0$$

$$B(u)|0\rangle = \sum_j \# |\uparrow\cdots\uparrow \underset{j\text{th}}{\downarrow} \uparrow\cdots\rangle$$

← one magnon state.

✯ $|0\rangle$ is the eigenstate of $tr_A(T(u)) = A(u) + D(u)$

with the eigenvalue $1 + \prod_{j=1}^{N} b(u-u_j)$.

✯ $B(u)$ operator behaves like S^- to creat "magnon" excitation.

To get all the other eigenstates of $tr_A(T(u))$, we apply the flipping operator B on the "ferro" state:

$B(v_1)\, B(v_2) \cdots B(v_M)\, |0\rangle$ is $N-M$ spin-up, M spin-down.

we prove that it is an eigenstate of $tr\, T(u)$.

From relation ④ $\Rightarrow$ $B(u)\, B(v) = B(v)\, B(u)$ ← commutate

⑧ $\Rightarrow$ $$D(u)\, B(v) = \frac{1}{b(u-v)}\, B(v)\, D(u) - \frac{c(u-v)}{b(u-v)}\, B(u)\, D(v)$$

③ exchange $u \leftrightarrow v$

$$A(u)\, B(v) = \frac{1}{b(v-u)}\, B(v)\, A(u) - \frac{c(v-u)}{b(v-u)}\, B(u)\, A(v)$$

Check: the case that $M=2$

$$A(u)\, B(v_1)\, B(v_2) = \left[\frac{B(v_1)\, A(u)}{b(v_1-u)} - \frac{c(v_1-u)}{b(v_1-u)}\, B(u)\, A(v_1)\right] B(v_2)$$

$$= \Big\{ \frac{1}{b(v_1-u)}\, B(v_1) \Big[\frac{1}{b(v_2-u)}\, B(v_2)\, A(u) - \frac{c(v_2-u)}{b(v_2-u)}\, B(u)\, A(v_2) \Big]$$

$$- \frac{c(v_1-u)}{b(v_1-u)}\, B(u) \Big[\frac{1}{b(v_2-v_1)}\, B(v_2)\, A(v_1) - \frac{c(v_2-v_1)}{b(v_2-v_1)}\, B(v_1)\, A(v_2) \Big] \Big\}$$

$$D(u)\, B(v_1) \cdots B(v_M) = \left[\frac{1}{b(u-v_1)}\, B(v_1)\, D(u) - \frac{c(u-v_1)}{b(u-v_1)}\, B(u)\, D(v_1) \right] B(v_2)$$

$$= \Big\{ \frac{1}{b(u-v_1)}\, B(v_1) \Big[\frac{1}{b(u-v_2)}\, B(v_2)\, D(u) - \frac{c(u-v_2)}{b(u-v_2)}\, B(u)\, D(v_2) \Big]$$

$$- \frac{c(u-v_1)}{b(u-v_1)}\, B(u) \Big[\frac{1}{b(v_1-v_2)}\, B(v_2)\, D(v_1) - \frac{c(v_1-v_2)}{b(v_1-v_2)}\, B(v_1)\, D(v_2) \Big].$$

$A(u)|0\rangle = |0\rangle$

$D(u)|0\rangle = \prod_{j=1}^{N} b(u-u_j)|0\rangle$

$$[A(u)+D(u)]\, B(v_1)\, B(v_2)\, |0\rangle$$

$$= \left\{ \frac{1}{b(v_1-u)\, b(v_2-u)} + \frac{\prod_{j=1}^{N} b(u-u_j)}{b(u-v_1)\, b(u-v_2)} \right\} B(v_1) B(v_2) |0\rangle$$

+ unwanted terms.

If unwanted terms were zero, then $B(v_1) \cdots B(v_M)|0\rangle$ is $tr_A T(u)$'s eigenstate with the eigenvalue $\frac{1}{b(v_1-u)\, b(v_2-u)} + \frac{\prod_{j=1}^{N} b(u-u_j)}{b(u-v_1)\, b(u-v_2)}$

Set $u = u_j \Rightarrow tr_A T(u_j)$'s eigenvalue $\frac{1}{b(v_1-u_j)\, b(v_2-u_j)}$ since $b(0)=0$.

Then we have
$$\boxed{e^{ik_jL} = \frac{1}{b(v_1-u_j)\, b(v_2-u_j)}}$$

the 1st order unwanted terms

$$-\frac{c(v_1-u)}{b(v_1-u)} \left[\frac{1}{b(v_2-v_1)} - \frac{\prod_{j=1}^{N} b(v_1-u_j)}{b(v_1-v_2)} \right] B(u) B(v_2) |0\rangle$$

Set $\prod_{j=1}^{N} b(v_1-u_j) = \frac{b(v_1-v_2)}{b(v_2-v_1)} = \frac{\prod_{\alpha\neq 1} b(v_1-v_\alpha)}{\prod_{\alpha \neq 1} b(v_\alpha - v_1)} \Rightarrow$ this term vanish

Check terms for $B(v_1)B(u)|0\rangle$ — the calculation is complicated

− … but since $B(v_1)B(v_2) = B(v_2)B(v_1)$

we should arrive at

$$-\frac{c(v_2-u)}{b(v_2-u)} \left[\frac{1}{b(v_1-v_2)} - \frac{\prod_{j=1}^{N} b(v_2-u_j)}{b(v_2-v_1)} \right] B(u) B(v_1) |0\rangle$$

set $\prod_{j=1}^{N} b(v_2-u_j) = \frac{b(v_2-v_1)}{b(v_1-v_2)} = \frac{\prod_{\alpha\neq 2} b(v_2-v_\alpha)}{\prod_{\alpha\neq 2} b(v_\alpha-v_2)}$

Then we can generalize to M - "magon" case

$$
\begin{aligned}
&\left[A(u)+D(u)\right] B(v_1)B(v_2)\cdots B(v_M)\,|0\rangle \\
&= \left\{ \prod_{\alpha=1}^{M} \frac{1}{b(v_\alpha-u)} + \frac{\prod_{j=1}^{N} b(u-u_j)}{\prod_{\alpha=1}^{M} b(u-v_\alpha)} \right\} B(v_1)\cdots B(v_M)\,|0\rangle \\
&\quad - \frac{c(v_1-u)}{b(v_1-u)} \left\{ \frac{\overbrace{1}^{A(v_1)|0\rangle}}{\prod_{\alpha=2}^{M} b(v_\alpha-v_1)} - \frac{\overbrace{\prod_{j=1}^{N} b(v_1-u_j)}^{D(v_1)|0\rangle}}{\prod_{\alpha=2}^{M} b(v_1-v_\alpha)} \right\} B(u)B(v_2)\cdots B(v_M)\,|0\rangle \\
&\quad - \frac{c(v_k-u)}{b(v_k-u)} \left\{ \frac{\overbrace{1}^{A(v_k)|0\rangle}}{\prod_{\alpha\neq k}^{M} b(v_\alpha-v_k)} - \frac{\overbrace{\prod_{j=1}^{N} b(v_k-u_j)}^{D(v_k)|0\rangle}}{\prod_{\alpha\neq k}^{M} b(v_k-v_\alpha)} \right\} B(v_1)\cdots B(u)\,B(v_{k+1})\cdots B(v_M)\,|0\rangle
\end{aligned}
$$

* Due to the permutation symmetry

Hence, we set

$$
\boxed{
\begin{aligned}
&\prod_{\alpha=1}^{M} \frac{1}{b(v_\alpha-u_j)} = e^{ik_jL} \qquad \text{for } j=1,2,\cdots N \\
&\prod_{j=1}^{N} b(v_k-u_j) = \frac{\prod_{\alpha\neq k}^{M} b(v_k-v_\alpha)}{\prod_{\alpha\neq k}^{M} b(v_\alpha-v_k)} \qquad \text{for } k=1,\cdots M
\end{aligned}
}
$$

All unwanted terms vanish, and

$$
tr_A\left[T(u_j)\right]\left\{B(v_1)B(v_2)\cdots B(v_M)\,|0\rangle\right\} = e^{ik_jL}\left[B(v_1)\cdots B(v_M)\right]|0\rangle .
$$

Plug in $u_j = \frac{k_j}{c}$, $v_\alpha = \frac{\Lambda_\alpha}{c} + \frac{i}{2}$, $b(u) = \frac{-u}{-u+i}$

$$\Rightarrow \quad b(v_\alpha - u_j) = \frac{-\frac{\Lambda_\alpha}{c} + \frac{k_j}{c} - \frac{i}{2}}{-\frac{\Lambda_\alpha}{c} + \frac{k_j}{c} + \frac{i}{2}} = \frac{k_j - \Lambda_\alpha - \frac{ic}{2}}{k_j - \Lambda_\alpha + \frac{ic}{2}}$$

$$b(v_\alpha - v_\kappa) = \frac{\Lambda_\kappa - \Lambda_\alpha}{\Lambda_\kappa - \Lambda_\alpha + ic},$$

$$b(v_\kappa - v_\alpha) = \frac{\Lambda_\alpha - \Lambda_\kappa}{\Lambda_\alpha - \Lambda_\kappa + ic}$$

$$\Rightarrow \quad \boxed{e^{ik_jL} = \prod_{\alpha=1}^{M} \frac{k_j - \Lambda_\alpha + \frac{ic}{2}}{k_j - \Lambda_\alpha - \frac{ic}{2}}} \leftarrow \text{for each "} k_j \text{" } j=1,\cdots N$$

$$\prod_{j=1}^{N} \frac{(k_j - \Lambda_\kappa) - \frac{ic}{2}}{k_j - \Lambda_\kappa + \frac{ic}{2}} = \prod_{\alpha=1,\alpha\neq\kappa}^{M} \frac{\Lambda_\alpha - \Lambda_\kappa - ic}{\Lambda_\alpha - \Lambda_\kappa + ic}$$

or rewrite

$$\boxed{\prod_{j=1}^{N} \frac{k_j - \Lambda_\alpha + \frac{ic}{2}}{k_j - \Lambda_\alpha - \frac{ic}{2}} = -\prod_{\beta=1}^{M} \frac{\Lambda_\beta - \Lambda_\alpha + ic}{\Lambda_\beta - \Lambda_\alpha - ic} \quad \text{for } \alpha = 1,\cdots M}$$

Appendix:

$$\left[R^{AB}(u-v)\,T^{A}(u)\,T^{B}(v)\right]_{u\omega,u'\omega'} = \left[c(u-v)\,\delta_{u\omega,u''\omega''} + b(u-v)\,P_{u\omega,u''\omega''}\right] \overbrace{T^{A}_{u''u'}(u)\,T^{B}_{\omega''\omega'}(v)}$$

$$= c(u-v)\,T^{A}_{uu'}(u)\,T^{B}_{\omega\omega'}(v) + b(u-v)\,T^{A}_{\omega u'}(u)\,T^{B}_{u\omega'}(v)$$

$$\left[T^{A}(v)\,T^{B}(u)\,R^{AB}(u-v)\right]_{u\omega,u'\omega'} = T^{A}_{uu''}(v)\,T^{B}_{\omega\omega''}(u)\left[c(u-v)\,\delta_{u''\omega'',u'\omega'} + b(u-v)\,P_{u''\omega'',u'\omega'}\right]$$

$$= c(u-v)\,T^{A}_{uu'}(v)\,T^{B}_{\omega\omega'}(u) + b(u-v)\,T^{A}_{u\omega'}(v)\,T^{B}_{\omega u'}(u)$$

$u\omega\ u'\omega'$

$(11,11)$: $\quad T_{11}(u)T_{11}(v) = T_{11}(v)T_{11}(u)$, $\Rightarrow$ $\underline{A(u)A(v) = A(v)A(u)}$,

$(11,12)$ $\quad c(u-v)\,T_{11}(u)\,T_{12}(v) + b(u-v)\,T_{11}(u)\,T_{12}(v)$

$$= c(u-v)\,T_{11}(v)\,T_{12}(u) + b(u-v)\,T_{12}(v)\,T_{11}(u)$$

$$\Rightarrow \quad \underline{A(u)B(v) = c(u-v)\,A(v)\,B(u) + b(u-v)\,B(v)\,A(u)}$$

$(11, \cancel{13}\ 21)$ $\quad \cancel{c(u-v)\,T_{11}(u)\,T_{13}(v) + b(u-v)\,T_{11}(u)\,T_{13}(v)}$

$$\cancel{= c(u-v)\,T_{11}(v)\,T_{13}(u) + b(u-v)\,T_{13}(v)\,T_{11}(u)}$$

$$\Rightarrow \quad c(u-v)\,T_{12}(u)\,T_{11}(v) + b(u-v)\,T_{12}(u)\,T_{11}(v)$$

$$= c(u-v)\,T_{12}(v)\,T_{11}(u) + b(u-v)\,T_{11}(v)\,T_{12}(u)$$

$$\underline{B(u)A(v) = c(u-v)\,B(v)\,A(u) + b(u-v)\,A(v)\,B(u)}$$

$$\Rightarrow \quad A(v)B(u) = \frac{1}{b(u-v)}\,B(u)A(v) - \frac{c(u-v)}{b(u-v)}\,B(v)A(u)$$

$uw\ u'w'$
$(11,22)$

$$\left.\begin{aligned} & c(u-v)\, T_{12}(u)\, T_{12}(v) + b(u-v)\, T_{12}(u)\, T_{12}(v) \\ &= c(u-v)\, T_{12}(v)\, T_{12}(u) + b(u-v)\, T_{12}(v)\, T_{12}(u) \end{aligned}\right\} \Rightarrow \boxed{\begin{aligned} & B(u)\, B(v) \\ &= B(v)\, B(u) \end{aligned}}$$

$uw\ u'w'$
$(12,11)$

$$c(u-v)\, T_{11}(u)\, T_{21}(v) + b(u-v)\, T_{21}(u)\, T_{11}(v)$$

$$= c(u-v)\, T_{11}(v)\, T_{21}(u) + b(u-v)\, T_{11}(v)\, T_{21}(u)$$

$$\underline{c(u-v)\, A(u)\, B(v) + b(u-v)\, B(u)\, A(v) = A(v)\, B(u)}$$

$uw\ u'w'$
$(12,12)$

$$c(u-v)\, T_{11}(u)\, T_{22}(v) + b(u-v)\, T_{21}(u)\, T_{12}(v)$$

$$= c(u-v)\, T_{11}(v)\, T_{22}(u) + b(u-v)\, T_{12}(v)\, T_{21}(u)$$

$\star$ $$\underline{c(u-v)\, A(u)\, D(v) + b(u-v)\, C(u)\, B(v) = c(u-v)\, A(v)\, D(u) + b(u-v)\, B(v)\, C(u)}$$

$uw\ u'w'$
$12\ 21$

$$c(u-v)\, T_{12}(u)\, T_{21}(v) + b(u-v)\, T_{22}(u)\, T_{11}(v)$$

$$= c(u-v)\, T_{12}(v)\, T_{21}(u) + b(u-v)\, T_{11}(v)\, T_{22}(u)$$

$\star$ $$\underline{\begin{aligned} & c(u-v)\, B(u)\, C(v) + b(u-v)\, D(u)\, A(v) \\ &= c(u-v)\, B(v)\, C(u) + b(u-v)\, A(v)\, D(u) \end{aligned}}$$

$uw,\ u'w'$
$12\ \ 22$

$$c(u-v)\, T_{12}(u)\, T_{22}(v) + b(u-v)\, T_{22}(u)\, T_{12}(v)$$

$$= c(u-v)\, T_{12}(v)\, T_{22}(u) + b(u-v)\, T_{12}(v)\, T_{22}(u)$$

$$\underline{\begin{aligned} & c(u-v)\, B(u)\, D(v) + b(u-v)\, D(u)\, B(v) \\ &= \cancel{c(u-v)}\ B(v)\, D(u) \end{aligned}} \Rightarrow \boxed{D(u)\, B(v) = \frac{B(v)\, D(u)}{b(u-v)} - \frac{c(u-v)}{b(u-v)}\, B(u)\, D(v)}$$

$uw \; u'w'$

$21 \; 11$ $\quad c(u-v)\, T_{21}(u)\, T_{11}(v) + b(u-v)\, T_{11}(u)\, T_{21}(v)$

$$= c(u-v)\, T_{21}(v)\, T_{11}(u) + b(u-v)\, T_{21}(v)\, T_{11}(u)$$

$$c(u-v)\, C(u)\, A(v) + b(u-v)\, A(u)\, C(v) = C(v)\, A(u)$$

$21 \; 12$ $\quad c(u-v)\, T_{21}(u)\, T_{12}(v) + b(u-v)\, T_{11}(u)\, T_{22}(v)$

$$= c(u-v)\, T_{21}(v)\, T_{12}(u) + b(u-v)\, T_{22}(v)\, T_{11}(u)$$

$$c(u-v)\, C(u)\, B(v) + b(u-v)\, A(u)\, D(v) = c(u-v)\, C(v)\, B(u) + b(u-v)\, D(v)\, A(u)$$

$21 \; 21$ $\quad \cancel{c(u)}\; c(u-v)\, T_{22}(u)\, T_{11}(v) + b(u-v)\, T_{12}(u)\, T_{21}(v)$

$$= c(u-v)\, T_{22}(v)\, T_{11}(u) + b(u-v)\, T_{21}(v)\, T_{12}(u)$$

$$c(u-v)\, D(u)\, A(v) + b(u-v)\, B(u)\, C(v) = c(u-v)\, D(v)\, A(u) + b(u-v)\, C(v)\, B(u)$$

$21 \; 22$ $\quad c(u-v)\, T_{22}(u)\, T_{12}(v) + b(u-v)\, T_{12}(u)\, T_{22}(v)$

$$= c(u-v)\, T_{22}(v)\, T_{12}(u) + b(u-v)\, T_{22}(v)\, T_{12}(u)$$

$$c(u-v)\, D(u)\, B(v) + b(u-v)\, B(u)\, D(v) = c(u-v)\, D(v)\, B(u) + b(u-v)\, D(v)\, B(u)$$

$$\boxed{= D(v)\, B(u)}$$

$22 \; 11,$ $\quad c(u-v)\, T_{21}(u)\, T_{21}(v) + b(u-v)\, T_{21}(u)\, T_{21}(v)$

$$= c(u-v)\, T_{21}(v)\, T_{21}(u) + b(u-v)\, T_{21}(v)\, T_{21}(u)$$

$$\cancel{c(u-v)}\; \boxed{C(u)\, C(v) = C(v)\, C(u)}$$

$uw\ u'w'$
22, 12 $\quad c(u-v)\, T_{21}(u) T_{22}(v) + b(u-v)\, T_{21}(u)\, T_{22}(v)$ (16)

$$= c(u-v)\, T_{21}(v) T_{22}(u) + b(u-v)\, T_{22}(v)\, T_{21}(u)$$

$$\Rightarrow \underline{C(u)\, D(v) = c(u-v)\, C(v)\, D(u) + b(u-v)\, D(v)\, C(u)}$$

$uw\ u'w'$
22 21 $\quad c(u-v)\, T_{22}(u) T_{21}(v) + b(u-v)\, T_{22}(u)\, T_{21}(v)$

$$= c(u-v)\, T_{22}(v) T_{21}(u) + b(u-v)\, T_{21}(v)\, T_{22}(u)$$

$$\underline{D(u)\, C(v) = c(u-v)\, D(v)\, C(u) + b(u-v)\, C(v) D(u)}$$

22 22 $\quad \Rightarrow \quad \underline{D(u)\, D(v) = D(v)\, D(u)}$

Next we prove:

$$S^+ \; B(v_1) \cdots B(v_m) \,|0\rangle,$$

$$S^3 \; B(v_1) \cdots B(v_m) \,|0\rangle = \frac{N-2m}{2} \, B(v_1) \cdots B(v_m) \,|0\rangle,$$

where $S^\pm = S^1 \pm i S^2$, i.e. $B(v_1) \cdots B(v_m)$ is $S_{tot} = \frac{N-2m}{2}$ state.

Proof: $$[\, S^{j,A}(u), S_j^n \,] = [\, b(u-u_j) + c(u-u_j) \frac{1+\vec{\tau}\cdot\vec{\sigma}_j}{2} \,,\; \frac{\sigma_j^n}{2} \,]$$ ($n = 1,2,3$ spin orientation)

$$= \frac{1}{4} c(u-u_j) \quad \tau^m \, [\, \sigma_j^m \, \sigma_j^n \,] \leftarrow 2i\varepsilon^{mnl} \, \tau^m \sigma^l$$

$$\parallel$$

$$= -\frac{1}{4} c(u-u_j) \quad \sigma_j^l \, [\, \tau_j^l \, \tau^n \,] \quad 2i\epsilon^{lnm} \, \sigma^l \tau^m$$

$$= \boxed{- [\, S^{j,A}(u) \,,\; \tfrac{1}{2}\tau^n \,] = [\, S^{j,A}(u) \,,\; S_j^n \,]}$$

$$T(u) = S^{1A}(u) \, S^{2A}(u) \cdots S^{NA}(u)$$

$$[\, T(u), S^n_{tot} \,] = \sum_{j=1}^{N} S^{1A} \cdots [\, S^{jA}, S^n_{tot} \,] \cdots S^{NA}$$

$$= -\sum_j L_1 \cdots [\, L_j, \tfrac{\tau^n}{2} \,] \cdots L_N = -[\, T(u), \tfrac{1}{2}\tau^n \,]$$

in the A space: $T_{\alpha\beta}(u) \leftarrow$ explicitly write the matrix element.

$$\Rightarrow [\, T_{\alpha\beta}(u) \,,\; S^n_{tot} \,] = -[\, T(u) \, \tfrac{1}{2}\tau^n \,] = \frac{1}{2} [\, \tau^n_{\alpha\beta'} \, T_{\beta'\beta} - T_{\alpha\beta'} \, \tau^n_{\beta'\beta} \,]$$

$$T_N = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

$\Rightarrow$

$$\Rightarrow \begin{pmatrix} [A,S^n], & [B,S^n] \\ [C,S^n], & [D,S^n]\end{pmatrix} = -\left[\begin{pmatrix} A & B\\ C & D\end{pmatrix}, \frac{1}{2}\tau^n\right]$$

$$-\frac{1}{2}\left[\begin{pmatrix}A&B\\C&D\end{pmatrix},\begin{pmatrix}1&\\&-1\end{pmatrix}\right] = \begin{pmatrix}0&B\\C&0\end{pmatrix}, \quad -\frac{1}{2}\left[\begin{pmatrix}A&B\\C&D\end{pmatrix}\begin{pmatrix}&1\\1&\end{pmatrix}\right] = -\frac{1}{2}\begin{pmatrix}B-C, & A-D\\ D-A, & C-B\end{pmatrix},$$

$$-\frac{1}{2}\left[\begin{pmatrix}A&B\\C&D\end{pmatrix}\begin{pmatrix}&-i\\i&\end{pmatrix}\right] = -\frac{1}{2}\begin{bmatrix}(B+C)i, & -(A-D)i\\ (D-A)i, & -(B+C)i\end{bmatrix}$$

$$\Rightarrow [A+D, S^n] = 0, \quad \text{for } n=1,2,3$$

$$[B(u)\ S^3] = B(u), \quad [B(u), S^+] = [B(u), S^1] + i[B(u), S^2]$$

$$= -\frac{1}{2}(A-D) - \frac{1}{2}(A-D) = -(A-D)$$

or rewrite $\boxed{[S^n, A+D] = 0, \quad [S^3, B(u)] = -B(u), \quad [S^+, B(u)] = A(u) - D(u)}$

Obviously $S^+|0\rangle = 0, \quad S^3|0\rangle = \frac{N}{2}|0\rangle$

$$S^3 B(u) = B(u)(S^3-1) \Rightarrow$$

$$S^3 B(v_1)\cdots B(v_m)\,|0\rangle = B(v_1)(S^3-1)B(v_2)\cdots\ |0\rangle$$

$$= B(v_1)B(v_2)(S^3-2)\cdots\ |0\rangle = B(v_1)\cdots B(v_m)(S^3-M)|0\rangle$$

$$= \frac{N-2M}{2}\ B(v_1)\cdots B(v_m)\,|0\rangle$$

$$[S^+, B(v_1)\cdots B(v_m)] = \sum_\alpha B(v_1)\cdots[S^+, B(v_\alpha)]\cdots B(v_m)$$

$$= \sum_\alpha B(v_1)\cdots (A(v_\alpha) - D(v_\alpha))\cdots B(v_m)$$

$$\Rightarrow \quad S^+ B(v_1)\cdots B(v_m)|0\rangle = [S^+, B(v_1)\cdots B(v_m)]|0\rangle = \sum_\alpha B(v_1)\cdots(A(v_\alpha)-D(v_\alpha))\cdots B(v_m)|0\rangle$$

by using
$$A(u)B(v) = \frac{1}{b(v-u)}B(v)A(u) - \frac{c(v-u)}{b(v-u)}B(u)A(v)$$

$$D(u)B(v) = \frac{1}{b(u-v)}B(v)D(u) - \frac{c(u-v)}{b(u-v)}B(u)D(v)$$

$$D(u)|0\rangle = \prod_{j=1}^{N} b(u-u_j)|0\rangle, \quad A|0\rangle = |0\rangle.$$

We can expect that, the finial expression can be expressed as

$$S^+ B(v_1)\cdots B(v_m)|0\rangle = \sum_\alpha M_\alpha B(v_1)\cdots B(v_{\alpha-1})B(v_{\alpha+1})\cdots B(v_m)|0\rangle$$

The coefficient of M_1 can be obtained as the term missing $\boxed{B(v_1)}$

$$(A(v_1) - D(v_1))B(v_2)\cdots B(v_m)|0\rangle$$

$$M_1 \overset{\rightarrow}{=} \prod_{\alpha=2}^{M}\frac{1}{b(v_\alpha - v_1)} - \frac{\prod_{j=1}^{N} b(v_1-u_j)}{\prod_{\alpha=2}^{M} b(v_1 - v_2)} = 0. \leftarrow \text{Bethe antsatz Eq.}$$

Because all B commutate, we can do arbitary permutation of B. Thus from $M_1 = 0$, we obtain $M_\alpha = 0$.

$$\Rightarrow \boxed{S^+ B(v_1)\cdots B(v_m)|0\rangle = 0}$$

Lect 7: - Nested Bethe Ansatz

Spin-1/2 fermion model has two conserved quantities: the total particle number N and the total spin S_z. Correspondinly, we have two sets BA quantum numbers and two sets of BA equations

$$\textcircled{1} \quad e^{ik_jL} = \prod_{\alpha=1}^{M} \frac{k_j - \Lambda_\alpha + ic/2}{k_j - \Lambda_\alpha - ic/2} \qquad \text{for each } k_j \quad j=1,2,\cdots N$$

(particle number)

$$\textcircled{2} \quad \prod_{j=1}^{N} \frac{k_j - \Lambda_\alpha - ic/2}{k_j - \Lambda_\alpha + ic/2} = -\prod_{\beta=1}^{M} \frac{\Lambda_\beta - \Lambda_\alpha - ic}{\Lambda_\beta - \Lambda_\alpha + ic} \qquad \alpha = 1, \cdots M$$

(# of spin up)

k_j ($j=1,2\cdots N$) and Λ_α ($\alpha=1,\cdots M$) are two sets of physical quantities that we are going to solve.

Eq ① $\Rightarrow$

$$k_jL = 2\pi \cdot \text{Integer} + \sum_{\alpha=1}^{M} 2\tan^{-1} \frac{c}{2(k_j - \Lambda_\alpha)}$$

$$= 2\pi \cdot \text{integer} + \sum_{\alpha=1}^{M} \pi - 2\tan^{-1} \frac{2(k_j - \Lambda_\alpha)}{c}$$

$$\boxed{k_jL = 2\pi I_j - 2\sum_{\alpha=1}^{M} \tan^{-1} \frac{2(k_j-\Lambda_\alpha)}{c} \qquad (j=1,2,\cdots N)}$$

I_j: integer if M is even;
half integer if M is odd.

Eq ② $\Rightarrow$

$$\sum_{j=1}^{N} 2\tan^{-1} \frac{c}{2(k_j - \Lambda_\alpha)} = 2\pi \cdot \text{integer} + \pi + \sum_{\beta=1}^{M} 2\tan^{-1} \frac{c}{\Lambda_\beta - \Lambda_\alpha}$$

$$\sum_{j=1}^{N} \pi - \sum_{j=1}^{N} 2\tan^{-1} \frac{2(k_j - \Lambda_\alpha)}{c} = 2\pi\, [\theta]\, [\text{half integer}] + \sum_{\beta=1}^{M} \pi - \sum_{\beta=1}^{M} 2\tan^{-1} \frac{\Lambda_\beta - \Lambda_\alpha}{c}$$

$$-\sum_{j=1}^{N} 2\tan^{-1}\frac{2(k_j-\Lambda_\alpha)}{c} = 2\pi\left(\text{half integer} + \frac{N-M}{2}\right) - \sum_{\beta=1}^{M} 2\tan^{-1}\frac{\Lambda_\beta-\Lambda_\alpha}{c}$$

$$\boxed{2\sum_{j=1}^{N}\tan^{-1}\frac{2(k_j-\Lambda_\alpha)}{c} = -2\pi J_\alpha + 2\sum_{\beta=1}^{M}\tan^{-1}\frac{\Lambda_\beta-\Lambda_\alpha}{c} \qquad \alpha=1,\cdots M}$$

where $J_\alpha = \begin{cases} \text{integer,} & \text{if } N-M \text{ is odd} \\ \text{half-integer} & \text{if } N-M \text{ is even} \end{cases}$

Ground state Solution:

① First consider the limit $c\to\infty$, $N=$ even, $M=$ odd

a) $k_j L = 2\pi I_j + 2\sum_{\beta=1}^{M}\tan^{-1} 2\Lambda_\beta/c \qquad j=1,\cdots N$

b) $-2N \tan^{-1}\frac{2\Lambda_\alpha}{c} = 2\pi J_\alpha + 2\sum_{\beta=1}^{M}\tan^{-1}\frac{\Lambda_\beta-\Lambda_\alpha}{c} \qquad \alpha=1,\cdots M$

I_j: half integer. J_α: integer

b) $\Rightarrow \quad -2N\sum_{\alpha=1}^{M}\tan^{-1}\frac{2\Lambda_\alpha}{c} = -2\pi\sum_{\alpha=1} J_\alpha \quad \leftarrow \quad \sum_{\alpha,\beta}\tan^{-1}\frac{\Lambda_\beta-\Lambda_\alpha}{c} = 0$

hence $\quad 2\sum_{\alpha=1}^{M}\tan^{-1}\frac{2\Lambda_\alpha}{c} = \frac{2\pi}{N}\sum_{\alpha=1}^{M} J_\alpha$

$$k_j = \frac{2\pi}{L}(I_j + a), \qquad \text{with } a = -1/N\sum_{\alpha=1}^{M} J_\alpha$$

$$E = \sum_{j=1}^{N} k_j^2 = \left(\frac{2\pi}{L}\right)^2 \sum_{j=1}^{N}(I_j+a)^2$$

Ground state: when $a=0$, and I_j is closely packed around 0 symmetrically.

* $a=0$ can be viewed as the center of mass momentum.

$(I_i+a)^2 \propto$ the ith particle kinetic energy.

$a=0 \Rightarrow J_\alpha$ takes vaule symmetrically with respect to zero

from b) $\Rightarrow$ Λ_α and $-\Lambda_\alpha$ appear in pairs.

2) $c \neq \infty$, in the ground state I_j still takes the value as the case of $c=\infty$. J_α distributes symmetrically around the origin, and Λ and $-\Lambda$ paired.

$$2\pi I_j = L k_j + \sum_\alpha 2\tan^{-1}\frac{2k_j - 2\Lambda_\alpha}{c} = L k_j + \sum_\alpha \tan^{-1}\frac{2k_j - 2\Lambda_\alpha}{c} + \tan^{-1}\frac{2k_j + 2\Lambda_\alpha}{c}$$

$$2\pi I_j = L k_j + \sum_\alpha \tan^{-1}\frac{4k_j c}{c^2 + 4\Lambda_\alpha^2}$$

where Λ_α^2 increases, k_j increases $\Rightarrow$ where Λ_α closely packed around zero, we arrive at the ground state

we consider the ground state in the case $N, L, M \to \infty$, but N/L, M/L fixed.

$I_{j+1} - I_j = 1$, $\quad J_{\alpha+1} - J_\alpha = 1$

define

$$\boxed{\begin{array}{l} L\,\rho(k)\,dk = \#\text{ of } k_j \text{ in the interval } k \to k+dk \\ L\,\sigma(\Lambda)\,d\Lambda = \#\text{ of } \Lambda_\alpha \text{ in the interval } \Lambda \to \Lambda + d\Lambda. \end{array}}$$

$f = I/L$ and $g = J/L$

$\Rightarrow df/dk = \rho(k)$ and $dg/d\Lambda = \sigma(\Lambda)$

From $k_j L = 2\pi I_j + \sum_{\alpha=1}^{M} \theta(2k_j - 2\Lambda)$ where $\theta(x) = -2\tan^{-1} x/c$

$$\rightarrow \boxed{k = 2\pi f + \int_{-B}^{B} \theta(2k - 2\Lambda)\,\sigma(\Lambda)\,d\Lambda}$$

$$\frac{d}{dk} \rightarrow \boxed{2\pi \rho(k) = 1 + \int_{-B}^{B} \frac{4c\,\sigma(\Lambda)}{c^2 + 4(k-\Lambda)^2}\, d\Lambda}$$

by using $d\theta/dx = -2\, c/(c^2+x^2)$

$$2\sum_{j=1}^{N} \tan^{-1} \frac{2(k_j - \Lambda_\alpha)}{c} = -2\pi J_\alpha + 2\sum_{\beta=1}^{M} \tan^{-1} \frac{\Lambda_\beta - \Lambda_\alpha}{c}$$

$$\sum_{j=1}^{N} \theta(2\Lambda_\alpha - 2k_j) = -2\pi J_\alpha + \sum_{\beta=1}^{M} \theta(\Lambda_\alpha - \Lambda_\beta)$$

$$\int_{-Q}^{Q} \theta(2\Lambda - 2k)\, \rho(k)\, dk = -2\pi g + \int_{-B}^{B} \theta(\Lambda - \Lambda')\, \sigma(\Lambda')\, d\Lambda'$$

$$\frac{d}{d\Lambda} \Rightarrow \boxed{2\pi \sigma(\Lambda) = -\int_{-B}^{B} \frac{2c\,\sigma(\Lambda')}{c^2 + (\Lambda - \Lambda')^2}\, d\Lambda' + \int_{-Q}^{Q} \frac{4c\,\rho(k)\, dk}{c^2 + 4(k-\Lambda)^2}}$$

Also the constraints

$$\boxed{\frac{N}{L} = \int_{-Q}^{Q} \rho(k)\, dk, \quad \frac{M}{L} = \int_{-B}^{B} \sigma(\Lambda)\, d\Lambda}$$

we can solve ρ, σ, Q, B, then

$$\frac{E}{L} = \int_{-Q}^{Q} k^2 \rho(k)\, dk$$

Lect 9: Hubbard model : Lieb-Wu solution

The BA equation for the continuous model can be straightforwardly generalized to the lattice Hubbard model by the substitution

$$\boxed{k_i \to \sin k_i, \quad c \to U/2}$$ ← t is set to 1.

Then

$$e^{ik_jL} = \prod_{\alpha=1}^{M} \frac{k_j - \Lambda_\alpha + ic/2}{k_j - \Lambda_\alpha - ic/2} \to$$

$$e^{ik_jL} = \prod_{\alpha=1}^{M} \frac{\sin k_j - \Lambda_\alpha + iU/4}{\sin k_j - \Lambda_\alpha - iU/4} \quad (1)$$

$$\prod_{j=1}^{N} \frac{k_j - \Lambda_\alpha - ic/2}{k_j - \Lambda_\alpha + ic/2} = -\prod_{\beta=1}^{M} \frac{\Lambda_\beta - \Lambda_\alpha - ic}{\Lambda_\beta - \Lambda_\alpha + ic} \to$$

$$\prod_{j=1}^{N} \frac{\sin k_j - \Lambda_\alpha - iU/4}{\sin k_j - \Lambda_\alpha + iU/4} = -\prod_{\beta=1}^{M} \frac{\Lambda_\beta - \Lambda_\alpha - iU/2}{\Lambda_\beta - \Lambda_\alpha + iU/2} \quad (2)$$

define $\Theta(x) = -2\tan^{-1}\left(\frac{2x}{U}\right)$, Eq (1) →

$$k_j L = 2\pi \cdot I_j + \sum_{\alpha=1}^{M} \Theta(2\sin k_j - 2\Lambda_\alpha) \quad (a) \quad j = 1, 2, \cdots N$$

N # of total particles

Eq(2) →

$$\sum_{j=1}^{N} \Theta(2\sin k_j - 2\Lambda_\alpha) = 2\pi J_\alpha + \sum_{\beta=1}^{M} \Theta(\Lambda_\beta - \Lambda_\alpha) \quad (b) \quad \alpha = 1, \cdots M$$

M # of spin down

or

where $I_j = \begin{cases} \text{integer} & M = \text{even} \\ \text{half integer} & M = \text{odd} \end{cases}$

$J_\alpha = \begin{cases} \text{integer} & N-M = \text{odd} \\ \text{half integer} & N-M = \text{even} \end{cases}$

In the ground state, I_j and J_α symmetrically distribute on both sides of zero. $I_{j+1} - I_j = 1$, $J_{\alpha+1} - J_\alpha = 1$.

Set $L \to \infty$, $N \to \infty$, $M \to \infty$, and N/L, M/L fixed.

Again set $L\,\rho(k)\,dk = \#$ of k_j from $k \to k+dk$

$L\,\sigma(\Lambda)\,d\Lambda = \#$ of Λ_α from $\Lambda \to \Lambda + d\Lambda$

define $f = I/L$, $g = J/L$ $\Rightarrow$ $df/dk = \rho(k)$, $dg/d\Lambda = \sigma(\Lambda)$

$$\frac{d\theta}{dx} = -2 \cdot \frac{u/2}{(u/2)^2 + x^2} = -\frac{4u}{u^2+4x^2}$$

$$\boxed{\begin{array}{l} \text{Eq (a)} \Rightarrow \quad k = 2\pi f + \displaystyle\int_{-B}^{B} \theta(2\sin k - 2\Lambda)\,\sigma(\Lambda)\,d\Lambda \\ \text{Eq (b)} \Rightarrow \\ \quad -\displaystyle\int_{-Q}^{Q} \theta(2\Lambda - 2\sin k)\,\rho(k)\,dk = 2\pi g - \int_{-B}^{B} \theta(\Lambda - \Lambda')\,\sigma(\Lambda')\,d\Lambda' \end{array}}$$

$$1 = 2\pi \frac{df}{dk} + \int_{-B}^{B} \frac{-4u}{u^2 + 4(2\sin k - 2\Lambda)^2} \cdot 2\cos k\, \sigma(\Lambda)\,d\Lambda$$

$\rightarrow$

$$\boxed{2\pi\rho(k) = 1 + \cos k \int_{-B}^{B} \frac{8u}{u^2 + 16(\sin k - \Lambda)^2}\,\sigma(\Lambda)\,d\Lambda \quad (*)}$$

$\rightarrow$

$$-\int_{-Q}^{Q} \frac{-4u}{u^2+16(\Lambda - \sin k)^2} \cdot 2\, \rho(k)\,dk = 2\pi \frac{dg}{d\Lambda} - \int_{-B}^{B} \frac{-4u}{u^2+4(\Lambda-\Lambda')^2}\,\sigma(\Lambda')\,d\Lambda'$$

$$\boxed{\int_{-Q}^{Q} \frac{8u}{u^2+16(\Lambda-\sin k)^2}\,\rho(k)\,dk = 2\pi\sigma(\Lambda) + \int_{-B}^{B} \frac{4u}{u^2+4(\Lambda-\Lambda')^2}\,\sigma(\Lambda')\,d\Lambda' \quad (**)}$$

Recap BA equation

(*) $\rho(k) = \dfrac{d(I/L)}{dk} = \dfrac{1}{2\pi} + \dfrac{4}{\pi}\cos k \displaystyle\int_{-B}^{B} a(4\sin k - 4\Lambda)\,\sigma(\Lambda)\,d\Lambda$

(**) $\sigma(\Lambda) = \dfrac{d(J/L)}{dk} = \dfrac{4}{\pi}\displaystyle\int_{-Q}^{Q} a(4\sin k - 4\Lambda)\,\rho(k)\,dk - \dfrac{2}{\pi}\int_{-B}^{B} a(2\Lambda - 2\Lambda')\,\sigma(\Lambda')\,d\Lambda'$

where $a(x) = \dfrac{u}{u^2 + x^2}$,

and $\displaystyle\int_{-Q}^{Q} dk\, \rho(k) = N/L$, $\displaystyle\int_{-B}^{B} d\Lambda\, \sigma(\Lambda) = \dfrac{N_\downarrow}{L}$.

Formula

$$\int_{-\infty}^{+\infty} \frac{e^{-i\omega x}\,dx}{a^2 + x^2} = \frac{\pi}{a} e^{-a|\omega|} \quad \text{for } a > 0,$$

$$\int_{-\pi}^{\pi} \frac{dk}{2\pi}\, e^{-i\omega\sin k} = J_0(\omega), \quad \int_{-\pi}^{\pi} dk/2\pi \,\cos^2 k\, \cos(\omega \sin k) = \frac{J_1(\omega)}{\omega}$$

$$\int_{-\infty}^{+\infty} d\Lambda\, e^{-i\omega\Lambda}\, a(4\sin k - 4\Lambda) = \frac{\pi}{4} e^{-u|\omega|/4}\, e^{-i\omega \sin k}$$

$$\int_{-\infty}^{+\infty} d\Lambda\, e^{-i\omega\Lambda}\, a(2\Lambda - 2\Lambda') = \frac{\pi}{2} e^{-u|\omega|/2}\, e^{-i\omega\Lambda'}$$

$$\frac{E}{L} = -2\int_{-Q}^{Q} \rho(k) \cos k \, dk$$

(*) Solution at half-filling

$N = L, \quad N_\uparrow = L/2, \quad M = N_\downarrow = L/2$

It can be proved that as $L \to \infty$, the integral boundaries $Q = \pi$, $B = \infty$ at half-filling. And the normalization condition is $\int_{-\pi}^{\pi} \rho(k) dk = 1$ and $\int_{-\infty}^{+\infty} \sigma(\Lambda) d\Lambda = 1/2$.

Define Fourier transform:

$$\sigma(\Lambda) = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \tilde{\sigma}(\omega) e^{i\omega\Lambda}, \quad \tilde{\sigma}(\omega) = \int_{-\infty}^{+\infty} \sigma(\Lambda) e^{-i\omega\Lambda} d\Lambda$$

1. Do Fourier transformation $\int_{-\infty}^{+\infty} d\Lambda\, e^{-i\omega\Lambda}$ to Eq (**).

$$\tilde{\sigma}(\omega) = \frac{4}{\pi} \cdot \frac{\pi}{4} \int_{-\pi}^{\pi} e^{-u|\omega|/4} e^{-i\omega \sin k} \rho(k) dk - \frac{2}{\pi} \cdot \frac{\pi}{2} \int_{-\infty}^{+\infty} e^{-\frac{u|\omega|}{2}} e^{-i\omega\Lambda'} \sigma(\Lambda') d\Lambda'$$

$$\tilde{\sigma}(\omega)\left[1 + e^{-u|\omega|/2}\right] = e^{-u|\omega|/4} \int_{-\pi}^{\pi} dk\, \rho(k)\, e^{-i\omega \sin k}$$

$$\boxed{\tilde{\sigma}(\omega) = \frac{1}{2} \operatorname{sech} \frac{u\omega}{4} \int_{-\pi}^{\pi} dk\, \rho(k)\, e^{-i\omega \sin k} = \frac{1}{2} \operatorname{sech} \frac{u\omega}{4} J_0(\omega)}$$

see below

2. Then apply $\int_{-\pi}^{\pi} \frac{dk}{2\pi} e^{-i\omega \sin k}$ to Eq (*)

$$\int_{-\pi}^{\pi} \frac{dk}{2\pi} e^{-i\omega \sin k} \rho(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{dk}{2\pi} e^{-i\omega \sin k} + \frac{4}{\pi} \int_{-\infty}^{+\infty} d\Lambda\, \sigma(\Lambda) \int_{-\pi}^{\pi} \frac{dk}{2\pi} \cos k\, a(4\sin k - 4\Lambda)\, e^{-i\omega \sin k}$$

$$\int_{-\pi}^{\pi} d\sin k\, e^{-i\omega \sin k}\, a(4\sin k - 4\Lambda) = 0$$

$$\boxed{\int_{-\pi}^{\pi} \frac{dk}{2\pi} e^{-i\omega \sin k} \rho(k) = J_0(\omega)}$$

$$\tilde{\sigma}(\omega=0) = \int_{-\infty}^{\infty} \sigma(\Lambda)\, d\Lambda = 1/2$$ — consistent with the normalization.

$$\sigma(\Lambda) = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi}\, e^{i\Lambda\omega}\, \tilde{\sigma}(\omega) = \int_{-\infty}^{+\infty} \frac{d\omega}{4\pi}\, e^{i\Lambda\omega}\, J_0(\omega)\, \mathrm{sech}\, \frac{u\omega}{4}$$

plug in $\sigma(\Lambda)$ into Eq $(*)$

$$p(k) = \frac{1}{2\pi} + \frac{4}{\pi}\cos k \int_{-\infty}^{+\infty} d\Lambda\, e^{i\Lambda\omega}\, a(4\sin k - 4\Lambda) \int_{-\infty}^{+\infty} \frac{d\omega}{4\pi}\, J_0(\omega)\, \mathrm{sech}\, \frac{u\omega}{4}$$

$$= \frac{1}{2\pi} + \frac{4}{\pi}\cos k \cdot \frac{\pi}{4} \int_{-\infty}^{+\infty} \frac{d\omega}{4\pi}\, J_0(\omega)\, \mathrm{sech}\, \frac{u\omega}{4}\, e^{-u|\omega|/4}\, e^{i\omega\sin k}$$

even on ω.

$$\boxed{p(k) = \frac{1}{2\pi} + \frac{\cos k}{\pi} \int_0^{\infty} d\omega\, \frac{J_0(\omega)\cos(\omega \sin k)}{1 + e^{u\omega/2}}}$$

$$\int_{-\pi}^{\pi} p(k)\,dk = 1 + \int_0^{\infty} d\omega\, \frac{J_0(\omega)}{1+e^{u\omega/2}} \int_{-\pi}^{\pi} \frac{dk}{\pi} \cos k \cdot \cos(\omega \sin k)$$

$$\int_{-\pi}^{\pi} d\sin k\, \cos(\omega \sin k) = 0$$

$$\boxed{\int_{-\pi}^{\pi} p(k)\,dk = 1}$$ consistent with the normalization.

plug in

$$\frac{E}{L} = -2\int_{-Q}^{Q} p(k)\cos k\, dk = -2\int_{-\pi}^{\pi} dk\, \frac{\cos^2 k}{\pi} \int_0^{\infty} \frac{J_0(\omega)\cos(\omega \sin k)}{1+e^{u\omega/2}}\, d\omega$$

$$= -2\int_0^{\infty} d\omega\, \frac{J_0(\omega)}{1+e^{u\omega/2}} \int_{-\pi}^{\pi} \frac{dk}{\pi} \cos^2 k\, \cos(\omega \sin k)$$

$$\boxed{\frac{E}{L} = -4\int_0^{\infty} d\omega\, \frac{J_0(\omega) J_1(\omega)}{\omega(1+e^{u\omega/2})}}$$

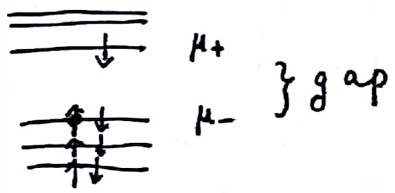
(*) Gap

SU(2) symmetry, only change $N_\downarrow$

Definition: $\mu_+ = E(N_\uparrow, N_\downarrow+1) - E(N_\uparrow, N_\downarrow)$ where $N_{\uparrow,\downarrow}$ are

$\mu_- = E(N_\uparrow, N_\downarrow) - E(N_\uparrow, N_\downarrow - 1)$, particle #'s of spin $\uparrow, \downarrow$.

if $\mu_+ - \mu_- = O(1/L)$, then it's a gapless metal



if $\mu_+ - \mu_- = \Delta \to$ finite

insulator

particle-hole symmetry:

Hubbard model when expressed as

$$H' = -t \sum_{\langle ij \rangle} (C^\dagger_{i\sigma} C_{j\sigma} + h.c) + \mathcal{U} \sum_i (n_{i\uparrow} - 1/2)(n_{i\downarrow} - 1/2)$$

$$= H - \frac{1}{2}(N_\uparrow + N_\downarrow)\,\mathcal{U} + \text{const}$$

H' has a particle-hole symmetry, that under $n_{i\sigma} \to 1 - n_{i\sigma}$

or $C_{i,\sigma} \to (-)^i C_{i\sigma}$, we have $H' \to H'$.

Hence $\quad E'(N_\uparrow, N_\downarrow) = E'(L-N_\uparrow, L-N_\downarrow)$

$$E(N_\uparrow, N_\downarrow) = E(N'_\uparrow, N'_\downarrow) + \frac{1}{2}(N_\uparrow + N_\downarrow)\,\mathcal{U}$$

$$= E'(L-N_\uparrow, L-N_\downarrow) + 1/2\,(N_\uparrow + N_\downarrow)\,\mathcal{U}$$

$$= E(L-N_\uparrow, L-N_\downarrow) - \frac{1}{2}(L-N_\uparrow + L - N_\downarrow)\,\mathcal{U} + 1/2\,(N_\uparrow + N_\downarrow)\,\mathcal{U}$$

$$= E(L-N_\uparrow, L-N_\downarrow) - (L - N_\uparrow - N_\downarrow)\,\mathcal{U}$$

$$E(N_\uparrow = L/2, N_\downarrow = L/2 - 1) = E(N_\uparrow = L/2, N_\downarrow = L/2 + 1) - (L - L + 1)\,\mathcal{U}$$

$$= E(N_\uparrow = L/2, N_\downarrow = L/2 + 1) - \mathcal{U} \quad \Rightarrow \quad \boxed{\mu_+ + \mu_- = \mathcal{U} \text{ when } N_\uparrow = N_\downarrow = L/2.}$$

Define the spectral density function ρ_0 for the case $N=L$

and ρ for the case of $\boxed{\begin{matrix} N_\uparrow = L/2 - 1 \\ N_\downarrow = \frac{L}{2} - 1 \end{matrix}}$. We should have ← spin up and down symmetrically

$$\int_{-Q=-\pi}^{Q=\pi} \rho_0(k)\, dk = 1, \quad \int_{-Q^-}^{Q^-} \rho(k)\, dk = 1 - 2/L.$$

when $N_\downarrow$ changes by one, I_j's pattern changes for integers ⟷ half integers.

i.e. # of I's increases by one. we express

$$\rho_0 - \rho = \frac{1}{L}\left[\delta(k-\pi) + \delta(k+\pi)\right] + \delta\rho(k) \leftarrow \quad |k| \le Q^- < Q$$

Count the extra ~~two~~ particles

then $\mu_- = \frac{1}{2}\left[E(N=L) - E(N=L-2)\right] = -\frac{2}{2}L\left[\int_{-Q}^{Q} \cos k\, \rho_0(k)\, dk - \int_{-Q^-}^{Q^-} \cos k\, \rho(k)\, dk\right]$

$$\boxed{\mu_- = -2\cos Q - L\int_{-Q^-}^{Q^-} \cos k\, \delta\rho\, dk}$$

(*) $\rho_0 = \frac{1}{2\pi} + \frac{4}{\pi}\cos k \int_{-B}^{B} a(4\sin k - 4\Lambda)\, \sigma_0(\Lambda)\, d\Lambda$

$$\rho(k) = \frac{1}{2\pi} + \frac{4}{\pi}\cos k \int_{-B}^{B} a(4\sin k - 4\Lambda)\, \sigma(\Lambda)\, d\Lambda$$

$$\Rightarrow \boxed{\delta\rho(k) + 1/L\left[\delta(k-Q) + \delta(k+Q)\right] = \frac{4}{\pi}\cos k \int_{-B}^{B} a(4\sin k - 4\Lambda)\, \delta\sigma(\Lambda)\, d\Lambda}$$

where $\delta\sigma(\Lambda) = \sigma_0(\Lambda) - \sigma(\Lambda)$ (***)

(**)

$$\sigma_0(\Lambda) = \frac{d(J/L)}{dk} = \frac{4}{\pi}\int_{-Q}^{Q} a(4\sin k - 4\Lambda)\rho_0(k)dk - \frac{2}{\pi}\int_{-B}^{B} a(2\Lambda - 2\Lambda')\sigma_0(\Lambda')d\Lambda'$$

$$\sigma(\Lambda) = \frac{4}{\pi}\int_{-Q^-}^{Q^-} a(4\sin k - 4\Lambda)\rho(k)dk - \frac{2}{\pi}\int_{-B}^{B} a(2\Lambda - 2\Lambda')\sigma(\Lambda')d\Lambda'$$

$$\rightarrow \boxed{\delta\sigma(\Lambda) = \frac{4}{\pi}\cdot\frac{2}{L}\,a(4\Lambda) + \frac{4}{\pi}\int_{-Q^-}^{Q^-} a(4\sin k - 4\Lambda)\,\delta\rho(k)\,dk - \frac{2}{\pi}\int_{-\infty}^{\infty} a(2\Lambda - 2\Lambda')\,\delta\sigma(\Lambda')\,d\Lambda' \quad (\text{****})}$$

Apply Fourier transform $\int_{-\infty}^{\infty} d\Lambda\, e^{-i\Lambda w}$ to Eq (****)

$$\delta\tilde{\sigma}(\omega) = \frac{8}{\pi L}\cdot\frac{\pi}{4}\, e^{-u|\omega|/4} + \frac{4}{\pi}\int_{-Q^-}^{Q^-} dk\,\delta\rho(k)\cdot\frac{\pi}{4}\, e^{-u|\omega|/4}\, e^{-i\omega\sin k} - \frac{2}{\pi}\cdot\frac{\pi}{2}\int_{-\infty}^{\infty} e^{-u|\omega|/2}\, e^{-i\omega\Lambda'}\,\delta\sigma(\Lambda')d\Lambda'$$

$$\delta\tilde{\sigma}(\omega) = \frac{2}{L}e^{-u|\omega|/4} + \int_{-Q^-}^{Q^-} dk\;\delta\rho(k)\, e^{-i\omega\sin k}\, e^{-u|\omega|/4} - \delta\tilde{\sigma}(\omega)\, e^{-u|\omega|/2}$$

$$\delta\tilde{\sigma}(\omega)\left[e^{-u|\omega|/4} + e^{u|\omega|/4}\right] = \frac{2}{L} + \int_{-Q^-}^{Q^-} dk\;\delta\rho(k)\, e^{-i\omega\sin k} \rightarrow 0 \text{ (see below)}$$

Based on (**) since $\delta(k\pm Q)$ is out of $[-Q^-, Q^-]$, we plug in Eq (***) without the δ-function.

$$\frac{4}{\pi}\int_{-\pi}^{\pi} dk \underbrace{\int_{-\infty} \cos k}_{d\Lambda}\; a(4\sin k - 4\Lambda)\,\delta\sigma(\Lambda)\, e^{-i\omega\sin k} = \frac{4}{\pi}\underbrace{\int_{-\infty}^{+\infty} d\Lambda \int_{-\pi}^{\pi} d\sin k\; a(4\sin k - 4\Lambda)\, e^{-i\omega\sin k}}_{\delta\sigma(\Lambda)}$$

$$= 0$$

$$\delta\tilde{\sigma}(\omega)\, 2\, \mathrm{ch}\, u\omega/4 = 2/L$$

$$\boxed{\delta\tilde{\sigma}(\omega) = \frac{1}{L}\,\mathrm{sech}\,\frac{u\omega}{4}}$$

$$\Rightarrow \int_{-\infty}^{+\infty} \delta\sigma(\Lambda)\, d\Lambda = \delta\tilde{\sigma}(0) = 1/L$$

consistent with the normalization

$$\delta\sigma(\Lambda) = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi}\, \delta\tilde{\sigma}(\omega)\, e^{i\omega\Lambda} = \frac{1}{2\pi L}\int_{-\infty}^{+\infty} d\omega\ \mathrm{sech}\,\frac{u\omega}{4}\ e^{i\omega\Lambda}$$

plug in Eq (***)

$$\delta\rho(k) + \frac{1}{L}[\delta(k+\pi) + \delta(k-\pi)] = \frac{\cos k\, 4}{\pi L}\int_{-\infty}^{+\infty} \frac{d\omega}{2\pi}\ \mathrm{sech}\,\frac{u\omega}{4}\int_{-\infty}^{+\infty} d\Lambda\ a(\sin k - 4\Lambda)\, e^{-i\omega\Lambda}$$

$$= \frac{\cos k}{L}\cdot\frac{4}{\pi}\cdot\frac{\pi}{4}\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\ \mathrm{sech}\,\frac{u\omega}{4}\ e^{-u|\omega|/4}\, e^{-i\omega\sin k}$$

$$= \frac{\cos k}{L}\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\ \frac{2e^{-i\omega\sin k}}{1+e^{u|\omega|/2}}$$

$$= \frac{2\cos k}{L}\int_0^{+\infty}\frac{d\omega}{\pi}\ \frac{\cos(\omega\sin k)}{1+e^{u\omega/2}}$$ ← plug to the expression of μ_- on page 7

$$\Rightarrow \mu_- = 2 - \frac{2L}{L}\int_{-\pi^-}^{\pi^-} dk\, \cos^2 k\int_0^{+\infty}\frac{d\omega}{\pi}\ \frac{\cos(\omega\sin k)}{1+e^{u\omega/2}}$$ ← δ-function no contribution since it's outside the integrand.

$$= 2 - \int_0^{+\infty} d\omega\left[\int_{-\pi}^{\pi}\frac{dk}{2\pi}\cos^2 k\,\cos(\omega\sin k)\right]\frac{4}{1+e^{u\omega/2}}$$

$$\mu_- = 2 - 4\int_0^{\infty}\frac{J_1(\omega)\, d\omega}{\omega(1+e^{u\omega/2})}$$

$$\boxed{\mu_+ - \mu_- = u - 2\mu_- = u - 4 + 8\int_0^{\infty}\frac{J_1(\omega)\, d\omega}{\omega(1+e^{u\omega/2})}}$$

⊛ Mott gap at $U>0$

$$\Delta = \mu_+ - \mu_- = U - 4 + 8\int_0^\infty \frac{J_1(\omega)\,d\omega}{\omega(1+e^{U\omega/2})}$$

$$J_1(\omega) = \sum_{m=0}^{\infty} \frac{(-)^m (\omega/2)^{2m+1}}{m!(m+1)!} = \frac{\omega}{2} - \frac{\omega^3}{16} + \frac{\omega^5}{384} + \cdots$$

$$\int_0^\infty \frac{\omega^{n}\,d\omega}{1+e^{U\omega/2}} = \left(\frac{2}{U}\right)^{2n+1}\int_0^\infty \frac{x^n\,dx}{1+e^x} = \left(\frac{2}{U}\right)^{2n+1}(1-2^{-n})\,n!\,\zeta(1+n)$$

$$\boxed{\begin{aligned}&\int_0^\infty \frac{x^s}{e^x-1}dx = \Gamma(s+1)\zeta(s+1)\\ &\int_0^\infty \frac{x^s\,dx}{e^x+1} = (1-2^{-s})\Gamma(s+1)\zeta(1+s)\end{aligned}}$$

where Zeta function $\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}$

$$\int_0^\infty \frac{d\omega}{1+e^{U\omega/2}} = \frac{2}{U}\ln 2$$

$$\Rightarrow \int_0^\infty \frac{J_1(\omega)\,d\omega}{\omega(1+e^{U\omega/2})} = \sum_{m=0}^{\infty} (-)^m \frac{2^{-2m-1}}{m!(m+1)!}\int_0^\infty \frac{\omega^{2m}\,d\omega}{1+e^{U\omega/2}}$$

$$= \sum_{m=0}^{\infty}(-)^m \frac{1}{(m+1)!}\left(\frac{1}{U}\right)^{2m+1}(1-2^{-m})\,\zeta(1+n)$$

by comparing Taylor series

$$\int_0^\infty \frac{J_1(\omega)\,d\omega}{\omega(1+e^{U\omega/2})} = \sum_{n=1}^{\infty}(-)^{n+1}\left(\sqrt{1+\frac{n^2U^2}{4}} - \frac{nU}{2}\right)$$

$$\Rightarrow \boxed{\Delta = \mu_+ - \mu_- = U - 4 - 8\sum_{n=1}^{\infty}(-)^n\left[\sqrt{1+\frac{n^2U^2}{4}} - \frac{nU}{2}\right]}$$

$$\mu_- = 2 - 4\sum_{n=1}^{\infty}(-)^n\left[\left(1+\frac{n^2U^2}{4}\right)^{1/2} - \frac{nU}{2}\right]$$

It can be proved that $\boxed{\Delta = \frac{16}{U}\int_1^\infty dy\,\frac{\sqrt{y^2-1}}{\sinh\frac{2\pi y}{U}} > 0, \text{ for any } U>0}$

① strong coupling limit $u \to \infty$

$$\left(1+\frac{n^2u^2}{4}\right)^{1/2} = \frac{nu}{2}\left(1+\left(\frac{2}{nu}\right)^2\right)^{1/2} \simeq \frac{nu}{2}\left[1+\frac{2}{(nu)^2}\right] = \frac{nu}{2}+\frac{1}{nu}$$

$$\Rightarrow \Delta = u - 4 - 8/u \sum_{n=1}^{\infty} (-)^n/n = u - 4 + \frac{8\ln 2}{u} + O\left(\frac{1}{u^2}\right)$$

$$\sim u - 4t + (8\ln 2)t^2/u + O(1/u^2)$$ — linear to u

② weak coupling $u \to 0$

$$\Delta \propto \frac{16}{u} e^{-2\pi/u} \int_0^\infty dx \frac{\sqrt{2x}\cdot 2}{e^{2\pi x/u}} \propto \underline{\frac{t^2}{u} e^{-2\pi t/u}}$$

exponetially small!

★ Hubbard model in 1D $\quad H = -t \sum_{\langle ij \rangle \sigma} (c_{i\sigma}^{+} c_{j\sigma} + h.c.) + U \sum_i n_{i\uparrow} n_{i\downarrow}$

N: number of particles ; $x_1 \cdots x_N$: coordinates.

$\underbrace{x_1, \cdots x_M}_{M}$ sites with spin down ; $\underbrace{x_{M+1}, \cdots x_N}_{N-M}$ sites with spin up.

The Bethe ansatz wavefunction

$\Theta(x_{Q_1} < x_{Q_2} < \cdots x_{Q_N})$

$$f(x_1, \cdots x_N) = \sum_Q \sum_P C(Q,P)\, e^{i \sum_{j=1}^{N} k_{P_j} x_{Q_j}} .$$

permutation on momenta.

where $\quad P = (P1, P2, \cdots PN), \quad Q = (Q1, Q2, \cdots QN)$ ↔ permutation on coordinates.

are two permutations of $(1\,2, \cdots N)$. $C(Q,P)$ is the coefficient.

The values of $(k_1, \cdots k_N)$ need to be determined from Bethe-ansatz equation (BA). And the energy $E = -2t \sum_{j=1}^{N} \cos k_j$.

k_j satisfy

$$\boxed{e^{i k_j L} = \prod_{\beta=1}^{M} \frac{it \sin k_j - i\Lambda_\beta - U/4}{it \sin k_j - i\Lambda_\beta + U/4}} \qquad j = 1, \cdots N$$

where $\Lambda_1, \cdots \Lambda_M$ are a set of unequal numbers satisfy another set of Eqs

$$- \prod_{j=1}^{N} \frac{it \sin k_j - i\Lambda_\alpha - U/4}{it \sin k_j - i\Lambda_\alpha + U/4} = \prod_{\beta=1}^{M} \frac{-i\Lambda_\beta + i\Lambda_\alpha + U/2}{-i\Lambda_\beta + i\Lambda_\alpha - U/2} \qquad \alpha = 1, \cdots M.$$

acting on Q

The coefficients satisfy $\boxed{C(Q,P) = y^{i,i+1}_{n,m}\, C(Q,P')}$, P, P' are fixed.

where P and P' satisfy
$$P = (P_1, \cdots P_i = m,\ P_{i+1} = n, \cdots P_N)$$
$$P' = (P'_1 \cdots P'_i = n,\ P'_{i+1} = m, \cdots P'_N)$$
$$P'_1 = P_1, \quad P'_N = P_N$$

(P', P) only differ by a nearest neighbour exchange.

$$y^{i,i+1}_{nm} = \frac{P_{i,i+1} - X_{nm}}{1 + X_{nm}}, \quad \text{with } X_{nm} = \frac{i\,U/(2t)}{\sin k_n - \sin k_m},$$

and $P_{i,i+1}$ is a permutation for exchange between Q_i and Q_{i+1}.

In other words, for fixed permuations on momenta P and P', which only differ from each other by an exchange of NN at i and $i+1$ $m \leftrightarrow n$.

we treat $C(Q,P)$ as a column vector of Q, where Q runs for all the permutation on coordinates,
$C(Q,P')$

$y^{i,i+1}_{nm}$ depends on $P_{i,i+1}$, which is a matrix acting on the space of different Qs.

Suppose that we have the information of $C(Q, P=1)$ for all Qs, then we have all the information of $C(Q,P)$ for an arbitary P.

we can convert the relation $C(Q,P) = Y^{i,i+1}_{n,m}\, C(Q,P')$ to the same $P=I$, then $C(Q,1)$ satisfies

$$e^{ik_jL}\, C(Q,1) = \underbrace{X_{j+1,j} \cdots X_{N,j}}_{\text{the first index} > j}\ \underbrace{X_{1,j} \cdots X_{j-1,j}}_{\text{the first index} < j}\ C(Q,1)$$

where $X_{ij} = \dfrac{1 - x_{ij} P_{ij}}{1 + x_{ij}}$, then we face an eigenvalue equation of $C(Q,1)$.

For convenience, define $\chi(Q) = (-)^Q\, C(Q,1)$, then $\chi(Q)$ satisfy

$$e^{ik_jL}\,\chi(Q) =$$

$$= X'_{j+1,j} \cdots X'_{N,j}\, X'_{1j} \cdots X'_{j-1,j}\, \chi(Q),$$ where

$$X'_{ij} = \frac{1 + x_{ij} P_{ij}}{1 + x_{ij}} .$$

It seems the $\chi(Q)$ has $N!$ components, however it can be proved it only has $\dfrac{N!}{M!(N-m)!}$ independent configuration, Because M spin down $N-M$ spin up particles are identical. We can get $M!\,(N-M)!$ copies through Fermi statistics. In other words, χ is chracterized by the location of spin down particles ($y_1 < y_2, < \cdots y_M$) in the permution of ($Q1, Q2, \cdots QN$).

Please note $y_1, \cdots y_M$ are not actual site-coordinate of spin down particles.

Say for $Q = (1, 2, \cdots N) \Rightarrow y_1 = 1,\ y_2 = 2, \cdots\ y_M = M$

$Q = (2, 1, \cdots N) \nearrow$

$Q = (1, 2, \cdots M-1, M+1, M, M+2, \cdots N) \Rightarrow y_1 = 1, \cdots y_{M-1} = M-1$

$\downarrow \downarrow \quad \downarrow \quad \uparrow \quad \downarrow \qquad y_M = M+1$.

C.N.Yang reduces the eigenvalue problem of $\chi(Q)$ to a Heisenberg chain problem

$$\Rightarrow \quad \chi = \phi(y_1 \cdots y_M) = \sum_P A_P \, F(\Lambda_{P1}, y_1) \cdots F(\Lambda_{PM}, y_M)$$

where P is a permutation among $(1, \cdots M) \leftarrow$ spin down particles.

where functions

$$F(\Lambda, y) = \prod_{j=1}^{y-1} \frac{it \sin k_j - i\Lambda - U/4}{it \sin k_{j+1} - i\Lambda + U/4}$$

and
$$A_P = (-)^P \prod_{i<j} \left[\Lambda_{P_i} - \Lambda_{P_j} - \frac{iU}{2} \right].$$

So far we presented the Bethe-ansartz solution without proof.

Now let us study the limit of $U/t \to +\infty$, and see how the Bethe-ansartz WF simplifies.

As $U/t \to \infty$, $y^{i,i+1}_{n,m} = -1$ just a number $\Rightarrow$

$$C(Q,P) = (-)\,C(Q,P') = (-)^P\, C(Q,1)$$

$$\Rightarrow f(x_1 \cdots x_N) = \sum_Q \Theta(x_{Q_1} < x_{Q_2} < \cdots x_{Q_N}) \sum_P C(Q,P)\, e^{i \sum_{j=1}^{N} k_{P_j} x_{Q_j}}$$

$$= \sum_Q \Theta(x_{Q_1} < x_{Q_2} < \cdots x_{Q_N}) \sum_P (-)^P\, C(Q,1)\, e^{i \sum_{j=1}^{N} k_{P_j} x_{Q_j}}$$

$$C(Q,1) = (-)^Q\, \phi(y_1 \cdots y_M)$$ determined by Q $\Rightarrow$

$$f(x_1 \cdots x_N) = \sum_Q \Theta(x_{Q_1} < x_{Q_2} < \cdots x_{Q_N})\, (-)^Q\, \phi(y_1 \cdots y_M) \sum_P (-)^P\, e^{i \sum_{j=1}^{N} k_{P_j} x_{Q_j}}$$

$$= \sum_Q \Theta(x_{Q_1} < x_{Q_2} < \cdots x_{Q_N})\, (-)^Q\, \phi(y_1 \cdots y_M)\, \boxed{\det M_{ij}}$$

i, and j run $1, 2, \cdots N$

with $M_{ij} = e^{i k_i x_{Q_j}}$

$$= \begin{bmatrix} e^{ik_1 x_{Q_1}} & e^{ik_1 x_{Q_2}} & \cdot \\ e^{ik_2 x_{Q_1}} & e^{ik_2 x_{Q_2}} & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}$$

Thus for each domain of $\Theta(x_{Q_1} < x_{Q_2} < \cdots x_{Q_N})$, the BA wave function factorizs into a Slater determinant of spinless fermions. and a Heisenberg spin chain WF $\phi(y_1, \cdots y_M)$.

without proof, Say Ogatta & Shiba.

Further, the 2-sets of BA Eqs decouple. We neglect sink terms

$$-\left[\frac{-i\Lambda_\alpha - U/4}{-i\Lambda_\alpha + U/4}\right]^N = \prod_{\beta=1}^{M} \frac{-i\Lambda_\beta + i\Lambda_\alpha + U/2}{-i\Lambda_\beta + i\Lambda_\alpha - U/2} \quad (*) \text{ for each } \alpha = 1, \cdots M$$

and
$$e^{ik_jL} = \prod_{\beta=1}^{M} \frac{-i\Lambda_\beta - U/4}{-i\Lambda_\beta + U/4} \qquad (**)$$

From $** \Rightarrow$
$$k_j L = 2\pi I_j' + \sum_{\beta=1}^{M}\left[\arg\left(\frac{\frac{U}{4} + i\Lambda_\beta}{\frac{U}{4} - i\Lambda_\beta}\right) + \pi\right]$$

$$= 2\pi I_j + \sum_{\beta=1}^{M} 2\tanh^{-1}\frac{4\Lambda_\beta}{U}$$

where I_j takes integer for M even

half integer for M odd.

from $* \Rightarrow$ take Arg

$$\pi + N\left[\pi + 2\tan^{-1}\frac{4\Lambda_\alpha}{U}\right] = \sum_{\beta=1}^{M} \pi + \left[2\tan^{-1}\frac{2(\Lambda_\alpha - \Lambda_\beta)}{U}\right] + 2\pi J_\alpha'$$

$$\Rightarrow 2N\tan^{-1}\frac{4\Lambda_\alpha}{U} = 2\sum_{\beta=1}^{M}\tan^{-1}\frac{2(\Lambda_\alpha - \Lambda_\beta)}{U} + 2\pi J_\alpha, \quad \alpha = 1, \cdots M$$

where $J_\alpha = J_\alpha' + (M-N+1)/2$ = integer for $N-M$ odd

half integer for $N-M$ even

$$\rightarrow \sum_{\beta=1}^{M} 2\tan^{-1}\frac{4\Lambda_\beta}{u} = \frac{2}{N}\sum_{\alpha\beta}\tan^{-1}\frac{2(\Lambda_\alpha-\Lambda_\beta)}{u} + \frac{2\pi}{N}\sum_\alpha J_\alpha = \frac{2\pi}{N}\sum_\alpha J_\alpha$$

$$\Rightarrow \boxed{k_j L = 2\pi\left[I_j + \frac{1}{N}\sum_{\alpha=1}^{M} J_\alpha\right]}$$

I_j : integer for M even
half integer for M odd

J_α : integer for $N-M$ odd
half integer $N-M$ even

① if N = even, M = odd

J_α : integer, I_j : half integer. the ground state corresponds to the distribution of k_j closest to $k=0$.

$$J_\alpha = -\frac{M-1}{2}, \cdots -1, 0, 1, \cdots \frac{M-1}{2} \Rightarrow \sum_{\alpha=1}^{M} J_\alpha = 0.$$

$$I_j = \left\{-\frac{N-1}{2}, \cdots -\frac{1}{2}, \frac{1}{2}, \cdots \frac{N-1}{2}\right\},$$

k_j: take free $\underbrace{\text{fermion}}_{\text{spinless}}$ momenta with a twist boundary condition.

independent of spin polarization $M = 1, 3, 5, \cdots N-1$.

how about the other sector M = even. in the case of N: even.

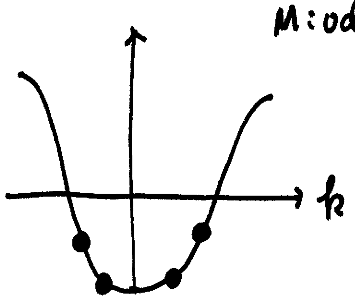
$\Rightarrow J_\alpha$: half integer, I_j : integer

$$J_\alpha = \left\{-\frac{M-1}{2}, \cdots -\frac{1}{2}, \frac{1}{2}, \cdots \frac{M-1}{2}\right\} \Rightarrow \underline{\sum_{\alpha=1}^{M} J_\alpha = 0 \text{ independent of } M}$$

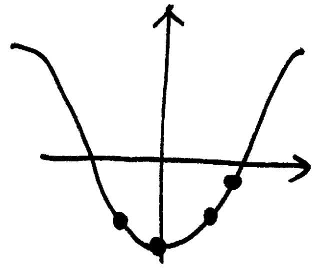
$$I_j = \left\{-\frac{N}{2}, -\frac{N}{2}+1, \cdots -1, 0, 1, \cdots \frac{N}{2}-1\right\}, \quad M = 0, 2, 4, \cdots N.$$

N: even

M: odd



M: even



$$E_1 = -2 \sum_{j=1}^{\frac{N}{2}} \cos \frac{2\pi}{L} \cdot \frac{2j-1}{2}$$

$$E_2 = -2 \sum_{j=1}^{\frac{N}{2}-1} \cos \frac{2\pi}{L} j - 2 - 2\cos\frac{2\pi}{L}\frac{N}{2}$$

Consider low density limit $\longrightarrow$ parabolic

$$E_1 \propto 2\left[\left(\frac{1}{2}\right)^2 + \cdots \left(\frac{N-1}{2}\right)^2\right] = \frac{2}{4}\left[\frac{N^3}{6} - \frac{2}{3}N + 1\right] = \frac{N^3}{12} - \frac{N}{12}$$

$$E_2 \propto 0 + 2\left[1^2 + 2^2 + \cdots + \left(\frac{N}{2}-1\right)^2\right] + \left(\frac{N}{2}\right)^2$$

$$= 2\,\frac{1}{6}\left(\frac{N-1}{2}\right)\left(\frac{N}{2}\right)(N-1) + \frac{N^2}{4} =$$

$$= \frac{N^3}{12} + \frac{N}{6}$$

$$\Delta E \propto \left(\frac{2\pi}{L}\right)^2 \cdot \frac{N}{4} \propto \underset{\substack{\uparrow \\ \text{density}}}{n} \cdot \frac{1}{L} \longrightarrow 0$$

Total energy difference $\to 0$, as $L \to +\infty$, while keep $n = \frac{N}{L}$ const.

$$\Delta E \propto A^2 \rho_s \qquad A = \frac{\pi}{L}, \; \rho_s \propto N$$

Thus this is a general result, regardless of quadratic spectra.