$(\eta\pi\chi)$

Lect 1: η-pairing and its development

1. Warm-up: Hubbard model — 2 sites

   spectrum and symmetries

2. η-pairing and pseudo-spin $SU(2)$ algebra

   $(SO(4))$

3: η-pairing as pseudo-Goldstone mode.

   ~~Competetion~~ Competition between SC and CDW

4: Futher development

   π-resonance of high T_c superconductivity

$SO(7)$ η and χ_a modes in spin-$3/2$ Hubbard model.

Ref: 1. C.N. Yang, PRL 63, 2144 (1989)

2. C.N. Yang and S.C. Zhang, Mod. Phys. Lett. 4, 759 (1990)

3. E. Demler, W. Hanke, S.C. Zhang, Rev. Mod. Phys 76, 909 (2004)

4. C. Wu, J.P. Hu, S.C. Zhang, PRL 91, 186402 (2003).

§1 Definition of the Hubbard mode and spin SU(2) symmetry

$$\boxed{H = -t\sum_{\langle ij\rangle}\left(C^{\dagger}_{i\sigma}C_{j\sigma} + h.c.\right) - \mu\sum_i C^{\dagger}_{i\sigma}C_{i\sigma} + U\sum_i\left(n_{i\uparrow}-\tfrac{1}{2}\right)\left(n_{i\downarrow}-\tfrac{1}{2}\right)}$$

for later convenience.

spin su(2) $\quad S^{+} = \sum_i C^{\dagger}_{i\uparrow}C_{i\downarrow}, \quad S^{-} = \sum_i C^{\dagger}_{i\downarrow}C_{i\uparrow},$

$$S^{z} = \frac{1}{2}\sum_i\left(C^{\dagger}_{i\uparrow}C_{i\uparrow} - C^{\dagger}_{i\downarrow}C_{i\downarrow}\right).$$

$n_{i\uparrow}n_{i\downarrow} = C^{\dagger}_{i\uparrow}C^{\dagger}_{i\downarrow}C_{i\downarrow}C_{i\uparrow}$, since $P^{\dagger}_{0} = C^{\dagger}_{i\uparrow}C^{\dagger}_{i\downarrow}$ is an spin SU(2) singlet, $n_{i\uparrow}n_{i\downarrow}$ is actually SU(2) invariant.

Hints from a few sites

		E	spin
• a single site	$\lvert\uparrow\rangle = C^{\dagger}_{\uparrow}\lvert vac\rangle, \quad \lvert\downarrow\rangle = C^{\dagger}_{\downarrow}\lvert vac\rangle$	$-U/4$	(doublet)
	$\lvert -\rangle = \lvert vac\rangle, \quad \lvert\uparrow\downarrow\rangle = C^{\dagger}_{\uparrow}C^{\dagger}_{\downarrow}\lvert vac\rangle$	$U/4$	(singlets)

• 2-site : $4^2 = 16$ states

8 - fermionic state

$$\lvert\psi^{p}_{\uparrow}\rangle_{b,ab} = \frac{1}{\sqrt{2}}\left(\lvert\uparrow\rangle\lvert-\rangle \pm \lvert-\rangle\lvert\uparrow\rangle\right)$$

$$\lvert\psi^{p}_{\downarrow}\rangle_{b,ab} = \frac{1}{\sqrt{2}}\left(\lvert\downarrow\rangle\lvert-\rangle \pm \lvert-\rangle\lvert\downarrow\rangle\right)$$

$$\lvert\psi^{h}_{\uparrow}\rangle_{b,ab} = \frac{1}{\sqrt{2}}\left(\lvert\uparrow\rangle\lvert\uparrow\downarrow\rangle \mp \lvert\uparrow\downarrow\rangle\lvert\uparrow\rangle\right)$$

$$\lvert\psi^{h}_{\downarrow}\rangle_{b,ab} = \frac{1}{\sqrt{2}}\left(\lvert\downarrow\rangle\lvert\uparrow\downarrow\rangle \mp \lvert\uparrow\downarrow\rangle\lvert\downarrow\rangle\right)$$

$E_b = -t$

$E_{\text{anti-bond (ab)}} = t$

bosonic state

① $|vac\rangle = |-\rangle|-\rangle$,

$|\psi_\eta\rangle = \frac{1}{\sqrt{2}}\left[|\uparrow\downarrow\rangle|-\rangle - |-\rangle|\uparrow\downarrow\rangle\right]$

$|\psi_{full}\rangle = |\uparrow\downarrow\rangle|\uparrow\downarrow\rangle$

$\Big\}$ spin singlet (3-fold degnerate)

$E = U/2$

"η"

② spin triplet

$|\psi_1^t\rangle = |\uparrow\rangle|\uparrow\rangle$, $|\psi_0^t\rangle = \frac{1}{\sqrt{2}}\left[|\uparrow\rangle|\downarrow\rangle + |\downarrow\rangle|\uparrow\rangle\right]$, $|\psi_{-1}^t\rangle = |\downarrow\rangle|\downarrow\rangle$.

$\boxed{E = -U/2}$

③ Valence-bond - molecular orbit mixture

$|\psi_{VB}\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle\right)$

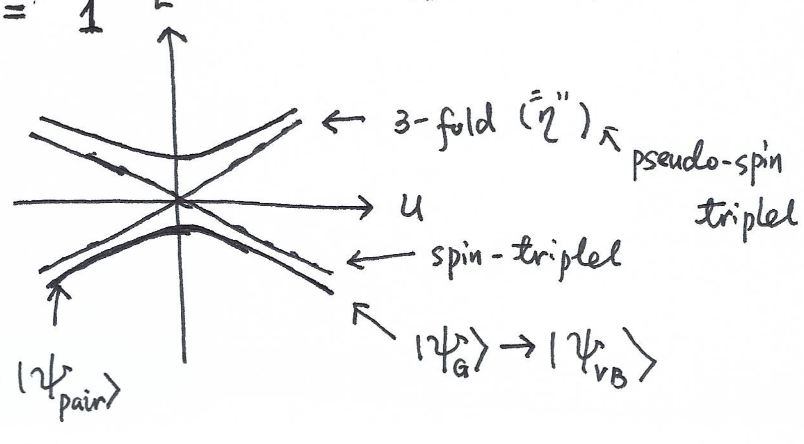
$|\psi_{Pair}\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle|-\rangle + |-\rangle|\uparrow\downarrow\rangle\right)$

$$H: \begin{array}{c} |\psi_{VB}\rangle \\ |\psi_P\rangle \end{array} \begin{bmatrix} -U/2, & -t \\ -t, & U/2 \end{bmatrix} \Rightarrow E = \pm\sqrt{\left(\frac{U}{2}\right)^2 + t^2}$$

$\quad |\psi_{VB}\rangle, \ |\psi_P\rangle$

$$= \begin{cases} \pm\frac{|U|}{2}\left(1 + \frac{4t^2}{|U|}\right) & U \to \pm\infty \\ \pm\left(t + \frac{U^2}{8t}\right) & U \to 0 \end{cases}$$

$\frac{E^2}{t^2} - \left(\frac{U}{2t}\right)^2 = 1$



§ More symmetry properties

(*) particle-hole

$$C_{i\sigma} \to (-)^i R_{\sigma\sigma'} C^+_{i\sigma'} \qquad R = -i\sigma_2 = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$$

$$\begin{pmatrix} C_{i\uparrow} \\ C_{i\downarrow} \end{pmatrix} \to (-)^i \begin{pmatrix} -C^+_{i\downarrow} \\ C^+_{i\uparrow} \end{pmatrix}$$

then $H \to$

$$H' = -t \sum_{\langle ij \rangle} \left(C^+_{i\sigma} C_{j\sigma} + h.c. \right) + \mu \sum_i C^+_{i\sigma} C_{i\sigma}$$

$$+ U \sum_i \left(n_{i\uparrow} - 1/2 \right) \left(n_{j\uparrow} - 1/2 \right)$$

At $\mu = 0$, $H = H'$, and then particle-hole symmetric

$$n_{i\uparrow} + n_{i\downarrow} \longrightarrow 1 - n_{i\uparrow} + 1 - n_{i\downarrow} = 2 - (n_{i\uparrow} + n_{i\downarrow})$$

$$\mu = 0 \Rightarrow \langle | n_{i\uparrow} + n_{i\downarrow} | \rangle = 1$$

degeracy between $|-\rangle$ and $|\uparrow\downarrow\rangle$.

(*) mapping from $U \to -U$ (partial particle-hole transf)

$$C_{i\uparrow} \to C_{i\uparrow}, \quad C_{i\downarrow} \to (-)^i C^+_{i\downarrow}$$

$$\left. C^+_{i\downarrow} C_{j\downarrow} \to (-)^{i-j} C_{i\downarrow} C^+_{j\downarrow} = C^+_{j\downarrow} C_{i\downarrow}, \quad C^+_{i\uparrow} C_{j\uparrow} \to C^+_{i\uparrow} C_{j\uparrow} \right\} \Rightarrow$$

$$n_\downarrow \to 1 - n_\downarrow, \quad n_\uparrow \to n_\uparrow$$

$$H' = -t \sum_{\langle ij \rangle} \left(C^+_{i\sigma} C_{j\sigma} + h.c. \right) - 2h \sum_i S_z(i) - U \sum_i (n_{i\uparrow} - 1/2)(n_{i\downarrow} - 1/2)$$

μ

§ Map spin SU(2) to pseduo-spin SU(2) (η-pairing)

We know H possesses the SU(2) symmetry, but H' after the partial particle-hole symmetry apparently breaks it.
Then what's the symmetry of H' ?

The generators of the SU(2) algebra

$$S^+ = \sum_i C_{i\uparrow}^+ C_{i\downarrow} \qquad S^- = \sum_i C_{i\downarrow}^+ C_{i\uparrow}$$

$$S^z = \frac{1}{2}\sum_i \left(C_{i\uparrow}^+ C_{i\uparrow} - C_{i\downarrow}^+ C_{i\downarrow}\right)$$

$\downarrow$ partial particle hole transformation

$$S^+ \to \eta^+ = \sum_i (-)^i C_{i\uparrow}^+ C_{i\downarrow}^+ , \qquad S^- \to \eta = \sum_i (-)^i C_{i\downarrow} C_{i\uparrow}$$

$$S^z \to \eta_0 = \frac{1}{2}\sum_i (n_\uparrow + n_\downarrow - 1) = \frac{N - M}{2}$$

N ← # of sites

M ← # of particles

the particle-hole transformation maintains the fermion anti-commutation law

$$\{c_{i\downarrow}, c_{i\downarrow}^{+}\} \to ((-)^{i})^{2} \{c_{i\downarrow}^{+}\ c_{i\downarrow}\} = 1$$

Hence, we expect that η^{+}, η, η_{0} follow the same $SU(2)$ algebra as $(S^{\pm}, S_z)$ do.

$$[\eta^{+}, \eta] = \sum_i [c_{i\uparrow}^{+} c_{i\downarrow}^{+}, c_{i\downarrow} c_{i\uparrow}] = \sum_i c_{i\uparrow}^{+} \{c_{i\downarrow}^{+}, c_{i\downarrow}\} c_{i\uparrow} - c_{i\downarrow} \{c_{i\uparrow} c_{i\uparrow}^{+}\} c_{i\downarrow}^{+}$$

$$= \sum_i (c_{i\uparrow}^{+} c_{i\uparrow} + c_{i\downarrow}^{+} c_{i\downarrow} - 1) = 2\eta_0$$

$$[\eta_0, \eta^{+}] = \frac{1}{2} \sum_i [c_{i\uparrow}^{+} c_{i\uparrow} + c_{i\downarrow}^{+} c_{i\downarrow}, (-)^{i} c_{i\uparrow}^{+} c_{i\downarrow}^{+}]$$

$$= \frac{1}{2} \sum_i (-)^{i} (c_{i\uparrow}^{+} c_{i\downarrow}^{+} - c_{i\downarrow}^{+} c_{i\uparrow}^{+}) = \eta^{+}$$

$$[\eta_0, \eta^{-}] = -\eta^{-}$$

Hence, in addition to the usual spin $SU(2)$ algebra, the Hubbard model defined on a bipartite lattice also possesses a pseudo-spin $SU(2)$ algebra.

These two sets of $SU(2)$ algebra actually commute with each other

$[S_\mu, \eta_a] = 0$, since all η-operators are invariant under spin $SU(2)$ operations.

$S^{\pm}, S_z, \eta^{\pm}, \eta_0$ form the $SO(4) = SU(2) \otimes SU(2)/Z_2$ group symmetry.

(*) Check the Hamitonian's commutation relation with η-algebra

(a) $[C^+_{i\sigma} C_{j\sigma},\ C_{i'\uparrow} C_{i'\downarrow}] = [C^+_{i\sigma},\ C_{i'\uparrow} C_{i'\downarrow}] C_{j\sigma} = \{C^+_{i\sigma}, C_{i'\uparrow}\} C_{i'\downarrow} C_{j\sigma}$

$- C_{i'\uparrow} \{C_{i'\downarrow} C^+_{i\sigma}\} C_{j\sigma}$

$= C_{i'\downarrow} C_{j\sigma}\, \delta_{ii'} \delta_{\sigma\uparrow} - C_{i'\uparrow} C_{j\sigma}\, \delta_{i'i} \delta_{\downarrow\sigma}$

$$\sum_{\langle ij\rangle,\sigma} \sum_{i'} [C^+_{i\sigma} C_{j\sigma},\ (-)^{i'} C_{i'\uparrow} C_{i'\downarrow}] = \sum_{\langle ij\rangle} (-)^i (C_{i\downarrow} C_{j\uparrow} - C_{i\uparrow} C_{j\downarrow})$$

$$\sum_{\langle ij\rangle,\sigma} \sum_{i} [C^+_{j\sigma} C_{i\sigma},\ (-)^{i} C_{i'\uparrow} C_{i'\downarrow}] = \sum_{\langle ij\rangle} (-)^j [C_{j\downarrow} C_{i\uparrow} - C_{j\uparrow} C_{i\downarrow})$$

since i, j belong to difference sublattices, $(-)^i + (-)^j = 0$

$\Rightarrow [H_t, \eta] = 0$, similarly $[H_t, \eta^+] = 0$

of course that $\boxed{[H_t, \eta_z] = 0.}$

The U-term also commutes with η-algebra

$$(n_{i\uparrow} - 1/2)(n_{i\downarrow} - 1/2) = n_{i\uparrow} n_{i\downarrow} - \tfrac{1}{2}(n_{i\uparrow} + n_{i\downarrow}) + 1/4$$

$$= -(n_{i\uparrow} - n_{i\downarrow})^2/2 + 1/4 = -2 S_z^2 + \text{const}$$

$$\Rightarrow [\sum_i (n_{i\uparrow} - 1/2)(n_{i\downarrow} - 1/2),\ \eta^\pm] = [\sum_i (n_{i\uparrow} - 1/2)(n_{i\downarrow} - 1/2),\ \eta_z] = 0$$

- chemical potential $H_\mu = -2\mu\, \eta_0$

$[H_\mu, \eta_0] = 0$, $\quad [H_\mu, \eta^+] = -2\mu [\eta_0, \eta^+] = -2\mu\, \eta^+$

$[H_\mu, \eta^-] = 2\mu\, \eta^-$

$H = H_t + H_\mu + H_u$

$$\boxed{\begin{aligned} [H, \eta^+] &= -2\mu\, \eta^+ \\ [H, \eta^-] &= 2\mu\, \eta^- \\ [H, \eta_0] &= 0 \end{aligned}}$$

$\rightarrow$ $\eta^\pm$ are eigen-operators of the Hubbard model defined on the bipartite lattice.

Hence, the pseudo-spin SU(2) symmetry is exact at $\mu = 0$, i.e. half-filling, in which $n_i = 1$.

$$\left.\begin{aligned} \eta^+ &= \sum_i (-)^i c^+_{i\uparrow} c^+_{i\downarrow} \\ c^+_{i\uparrow} &= \frac{1}{\sqrt{M}} \sum_k e^{i\vec{k}\vec{R}_i} c_{\vec{k}} \end{aligned}\right\} \Rightarrow \eta^+ = \sum_{\vec{k}} c^+_{\vec{k}\uparrow} c^+_{-\vec{k}+\vec{Q}\downarrow}$$

where $\vec{Q} = (\pi, \pi)$

similarly $\eta^- = \sum_{\vec{k}} c^+_{\vec{k}\downarrow} c^+_{-\vec{k}+Q\uparrow}$

$$\eta_0 = \frac{1}{2} \sum_k \left[c^+_{k\uparrow} c_{k\uparrow} + c^+_{k\downarrow} c_{k\downarrow} - 1 \right]$$

It means that $\eta^\pm$ carry the momentum $\vec{Q} = (\pi, \pi)$.

§ The SO(4) structure — $\eta^\pm$ generators.

① single site

$$|\downarrow\rangle \xleftrightarrow{S^\pm} |\uparrow\rangle \qquad (\tfrac{1}{2};0)$$

$$|-\rangle \xleftrightarrow{\eta^\pm} |\uparrow\downarrow\rangle \qquad (0;\tfrac{1}{2})$$

SO(4) Rep $(j_1 ; j_2)$

② 2-site

fermionic:

$$\begin{tikzcd}
|\psi^{p}_{b\uparrow}\rangle \arrow[r, leftrightarrow, "\eta^\pm"] \arrow[d, leftrightarrow, "S^\mp"'] & |\psi^{h}_{b\uparrow}\rangle \arrow[d, leftrightarrow, "S^\mp"] \\
|\psi^{p}_{b\downarrow}\rangle \arrow[r, leftrightarrow, "\eta^\pm"] & |\psi^{h}_{b\downarrow}\rangle
\end{tikzcd}$$

SO(4) ~~quartet~~ $(\tfrac{1}{2};\tfrac{1}{2})$

the anti-bonding states are similar.

bosonic:

① η - triplet

$$|vac\rangle \xleftrightarrow{\eta^\pm} |\psi^{\eta}\rangle \xleftrightarrow{\eta^\pm} |\psi_{full}\rangle \qquad (0;1)$$

② spin - triplet

$$|\downarrow\rangle|\downarrow\rangle \xleftrightarrow{S^\pm} \frac{|\uparrow\rangle|\downarrow\rangle + |\downarrow\rangle|\uparrow\rangle}{\sqrt{2}} \xleftrightarrow{S^\pm} |\uparrow\rangle|\uparrow\rangle \quad (1;0)$$

③ $|\psi_{VB}\rangle$ and $|\psi_{pair}\rangle$ are ~~both~~ SO(4) singlets $(0;0)$

Hence 16 states decoupled to

$$\boxed{2(\tfrac{1}{2};\tfrac{1}{2}) \oplus (0;1) \oplus (1;0) \oplus 2(0;0)}$$

§ Consequence of many-body physics — "η"-states

① construction of many-body eigenstate (meta-stability)

Consider the case of $\mu < 0$, i.e. $n < 1$. Apply $\eta^\dagger$ to $|\Omega\rangle$,

then $$H(\eta^\dagger|\Omega\rangle) = ([H, \eta^\dagger] + \eta^\dagger H)|\Omega\rangle = (-2\mu + E_0)\eta^\dagger|\Omega\rangle$$

$$H(\eta^\dagger|\Omega\rangle) = (E_0 - 2\mu)(\eta^\dagger|\Omega\rangle)$$

Hence, the excitation energy is $\omega = -2\mu > 0$ if $\eta^\dagger|\Omega\rangle \neq 0$.

we can construct a ladder of states

$$\boxed{\eta^\dagger|\Omega\rangle, \quad (\eta^\dagger)^2|\Omega\rangle, \quad \cdots\cdots (\eta^\dagger)^n|\Omega\rangle} \leftarrow \text{excitation}$$

If we start with the particle vaccum state $|vac\rangle$, creat

$$\psi^\eta_n = (\eta^\dagger)^n |vac\rangle \quad \longrightarrow \quad [H, (\eta^\dagger)^n] = -2n\mu(\eta^\dagger)^n$$

$$H|\psi^\eta_n\rangle = ([H, (\eta^\dagger)^n] + (\eta^\dagger)^n H)|vac\rangle$$

$$= \left[-2n\mu + \left(\frac{U}{4}\right)\cdot M\right]|\psi_n\rangle$$

but is $|\psi^\eta_n\rangle$ the ground state? No!

Constract $$|\psi^n_p\rangle = \left(\sum_i c^\dagger_{i\uparrow} c^\dagger_{i\downarrow}\right)^n |vac\rangle$$

Such a state only contains either empty or doubly occupied site.

$$H_u |\psi_p^n\rangle = \frac{u}{4} M |\psi_p^n\rangle, \quad \text{each site contributes } \frac{u}{4}.$$

$$H_\mu |\psi_p^n\rangle = -2n\mu |\psi_p^n\rangle.$$

But $|\psi_p^n\rangle$ is not the eigenstate of H_t. Intuitvely, we would expect its momentum distribution covers the entire Brouillin Zone,

$$\sum_i C_{i\uparrow}^\dagger C_{i\downarrow}^\dagger = \sum_k C_{k\uparrow}^\dagger C_{-k\downarrow}^\dagger, \quad \text{Hence } \langle\psi_p^n| H_t |\psi_p^n\rangle = 0$$

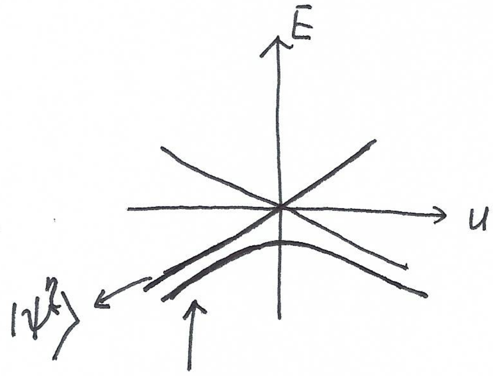
More precisely, we introduce a chiral transformation $C_{i\sigma} \to (-)^i C_{i\sigma}$

then $H_t \to -H_t$, but $|\psi_p^n\rangle = |\psi_p^n\rangle$. Hence

$$\langle\psi_p^n| H_t |\psi_p^n\rangle = -\langle\psi_p^n| H_t |\psi_p^n\rangle = 0$$

$\Rightarrow$ The energy expectation value $\langle\psi_p^n| H |\psi_p^n\rangle = (-2n\mu + \frac{Mu}{4})$

This means that the ground state $E_G < -2n\mu + \frac{Mu}{4}$, since $|\psi_p^n\rangle$ is not an eigenstate.



$$|\psi_G\rangle = |\psi_p\rangle + o\left(\frac{t}{u}\right)|\psi_{VB}\rangle$$

pesudo-Goldstone mode

(2)

(*) positive-U Hubbard model (2D) — antiferromagnetic long-range order

$$\vec{N} = \sum_i c^+_{i\alpha} (\vec{\sigma})_{\alpha\beta} c_{i\beta} (-)^i = \sum_k c^+_{k,\alpha} (\vec{\sigma})_{\alpha\beta} c_{k+Q,\beta}$$

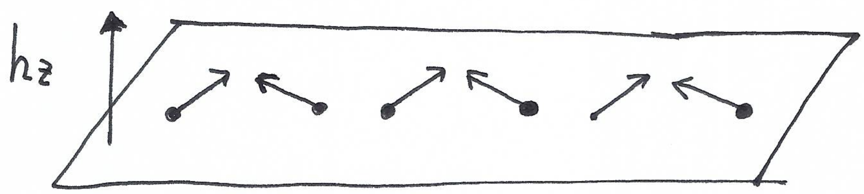
$$[S_\alpha \; N_\beta] = i \epsilon_{\alpha\beta\gamma} N_\gamma , \qquad [S_\alpha, S_\beta] = i \epsilon_{\alpha\beta\gamma} S_\gamma ,$$

$$[N_\alpha, N_\beta] = i \epsilon_{\alpha\beta\gamma} S_\gamma .$$

At zero-field (N_x, N_y, N_z) — isotropic

Applying magnetic field along z-direction $\Delta H = -h_z S_z$,

then neel-order flips into the xy-plane, canted.



(*) Negative-U Hubbard model (2D) — CDW and SC

$$\Delta^+ = \sum_i c^+_{i\uparrow} c^+_{i\downarrow} = \mathrm{Re}\Delta + i \mathrm{Im}\Delta$$

$$\Delta = \sum_i c_{i\downarrow} c_{i\uparrow}$$

$$O_{CDW} = \frac{1}{2} \sum_i (n_{i\uparrow} + n_{i\downarrow}) (-)^i$$

Then $(\mathrm{Re}\Delta, \mathrm{Im}\Delta, O_{CDW})$ form a 3-vector representation of the pseudo-spin

$$[\eta^\dagger, \Delta] = -2O_{CDW}, \quad [\eta^\dagger, O_{CDW}] = -\Delta^\dagger, \quad [\eta, O_{CDW}] = \Delta.$$

At $\mu = 0$, i.e. half-filling, superconductivity and CDW are degenerate under the pesudo-spin $SU(2)$. Upon doping, $\mu \neq 0$, superconductivity wins over CDW.

$$e^{i[\eta \cdot \alpha^* + \alpha \eta^\dagger]} |\psi_G^{SC}\rangle \xrightarrow[\mu<0]{\alpha\to 0} (1 + i\alpha\eta^\dagger)|\psi_G^{SC}\rangle$$

$$H(\eta^\dagger|\psi_G^{SC}\rangle) = (E_0 - 2\mu)(\eta^\dagger|\psi_G^{SC}\rangle)$$

$\rightarrow$ pesudo-Goldstone mode small gap due to the weak symmetry breaking (explicitly).

Can this mode be detected?

Suppose that we have an experiment tool to measure the response to the O_{CDW}. The dynamic structure factor spectral function

$$\mathrm{Im}\, \chi(Q, \omega) = \sum_n |\langle n| O_{CDW} |\psi_G^{SC}\rangle|^2 \delta(\omega - \omega_n)$$

$Q = (\pi, \pi)$

Using the single mode approximation: $\frac{1}{A}\eta^\dagger|\psi_G^{SC}\rangle$

Contributes the major spectra weight, where $\frac{1}{A}$ is the normalization factor. $\Rightarrow \mathrm{Im}\,\chi(Q,\omega) \approx \frac{1}{A^2}|\langle\psi_G^{SC}|\eta\, O_{CDW}|\psi_G^{SC}\rangle|^2 \delta(\omega - \omega_n)$

$$\eta|\psi_G^{sc}\rangle = 0 \Rightarrow \langle\psi_G^{sc}|\eta\, O_{cdw}|\psi_G^{sc}\rangle = \langle\psi_G^{sc}|[\eta, O_{cdw}]|\psi_G^{sc}\rangle$$

$$= \langle\psi_G^{sc}|\Delta|\psi_G^{sc}\rangle = \Delta_0 N$$

$$A^2 = \langle\psi_G|\eta\,\eta^\dagger|\psi_G\rangle = \langle\psi_G|[\eta,\eta^\dagger]|\psi_G\rangle$$

$$= 2\langle\psi_G|\eta_0|\psi_G\rangle = N - M$$

$$\Rightarrow \boxed{\frac{\mathrm{Im}\,\chi(Q,\omega)}{N} \approx \frac{|\Delta_0|^2}{1-\frac{M}{N}}\,\delta(\omega-\omega_\eta)}$$

single mode approximation to the dynamic structure factor

§ Further development

① π- resonance of SO(5) theory of high T_c superconductivity

$$\vec{\pi}^{+}_{Q} = \sum_{k} d(k)\, C^{+}_{k\alpha} (\vec{\sigma}\, i\sigma_2)_{\alpha\beta}\, C^{+}_{-k+Q,\beta}\,, \qquad d(k) = \cos k_x - \cos k_y .$$

$$\vec{\pi}_{Q} = \sum_{k} d(k)\, C_{-k+Q,\alpha} (-i\sigma_2 \vec{\sigma})_{\alpha\beta}\, C_{k,\beta}\,,$$

$$\Delta = \sum_{k} d(k)\, C_{-k} (-i\sigma_2)_{\alpha\beta}\, C_{k\beta} .$$

$$[\vec{\pi}^{+}_{Q}, \Delta] = \sum_{kk'} d(k)\, d(k') \Big[C^{+}_{k\alpha} (\vec{\sigma}\, i\sigma_2)_{\alpha\beta} \{ C^{+}_{-k+Q,\beta}\ C_{-k'\gamma} \} (-i\sigma_2)_{\gamma\delta}\, C_{k'\delta}$$

$$- C_{-k'\alpha} (-i\sigma_2)_{\alpha\beta} \{ C_{k'\beta}\ C^{+}_{k\gamma} \} (\vec{\sigma}\, i\sigma_2)_{\gamma\delta}\, C^{+}_{-k+Q,\delta} \Big]$$

$$= \sum_{kk'} d(k)\, d(k') \Big[C^{+}_{k}\, \vec{\sigma}\, C_{k'}\, \delta_{k'=k-Q} - C_{-k'} \underbrace{(\sigma_2 \vec{\sigma} \sigma_2)}_{\vec{\sigma}^{T}} C^{+}_{-k+Q,}\, \delta_{kk'} \Big]$$

$$= -\sum_{k} |d(k)|^2 \Big[C^{+}_{k}\, \vec{\sigma}\, C_{k-Q} + C^{+}_{k+Q}\, \vec{\sigma}\, C_{k} \Big]$$

$$\overline{|d(k)|^2} = \overline{\cos^2 k_x} + \overline{\cos^2 k_y} = \frac{1}{2} + \frac{1}{2} = 1$$

$$\approx -2 \sum_{k} C^{+}_{k}\, \vec{\sigma}\, C_{k-Q}$$

$$= -4\, \vec{N}(Q)$$

π-operator rotates superconductivity into anti-ferromagnetism.

→ relate to the 41 meV resonance mode observed in neutron scattering experiments. Nevertheless, the algebra is not exact!

② spin-3/2 Hubbard model

$$H = -t \sum_{\sigma = \pm 3/2, \pm 1/2} \left(c^\dagger_{i\sigma} c_{j\sigma} + h.c \right) - \mu \sum_i n_i + 2U_0 \sum_i P_0^\dagger(i) P_0(i)$$

$$+ U_2 \sum_{i,m = \pm 2, \pm 1, 0} P^\dagger_{2m}(i) P_{2m}(i)$$

$$\boxed{\Gamma^1 = \begin{pmatrix} 0 & -iI \\ iI & 0 \end{pmatrix} \quad \Gamma^5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \quad \Gamma^{2\sim 4} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & -\vec{\sigma} \end{pmatrix}}$$

Hidden $Sp(4)$ symmetry:

$$\{\Gamma^a, \Gamma^b\} = 2\delta^{ab}$$

$$\Gamma^1 = \frac{1}{\sqrt{3}}(S_x S_y + S_y S_x) \qquad \Gamma^4 = (S_z^2 - 5/4)$$

$$\Gamma^2 = \frac{1}{\sqrt{3}}(S_z S_x + S_x S_z) \qquad \Gamma^5 = \frac{1}{\sqrt{3}}(S_x^2 - S_y^2)$$

$$\Gamma^3 = \frac{1}{\sqrt{3}}(S_z S_y + S_y S_z)$$

$$\Gamma^{ab} = -\frac{i}{2}[\Gamma^a, \Gamma^b] \quad (1 \le a, b \le 5).$$

Charge conjugation

$$R = \Gamma^1 \Gamma^3 = \begin{pmatrix} 0 & -i\sigma_2 \\ -i\sigma_2 & 0 \end{pmatrix}$$

Fermion bilinears:

$$n(i) = c^\dagger_{i\sigma} c_{i\sigma}$$

$$n_a(i) = \frac{1}{2} c^\dagger_{i\alpha} \Gamma^a_{\alpha\beta} c_{i\beta}$$

$$L_{ab}(i) = -\frac{1}{2} c^\dagger_{i\alpha} \Gamma^{ab}_{\alpha\beta} c_{i\beta}$$

$$\eta^\dagger(i) = \frac{1}{2} c^\dagger_{i\alpha} R_{\alpha\beta} c^\dagger_{i\beta}$$

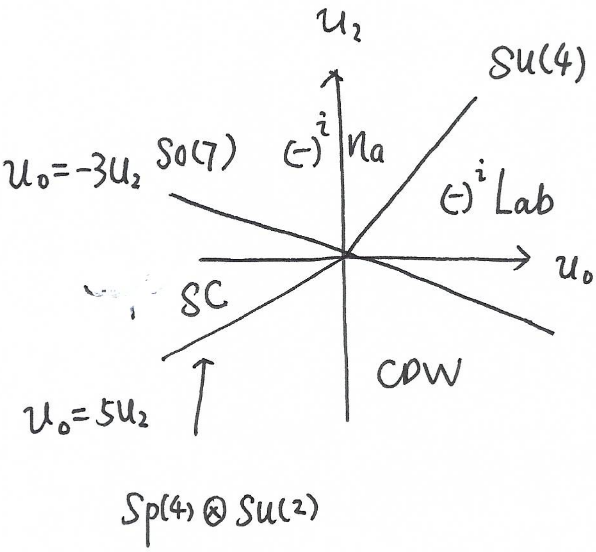
$$\chi^\dagger_a(i) = -\frac{i}{2} c^\dagger_{i\alpha} (\Gamma^a R)_{\alpha\beta} c^\dagger_{i\beta}$$

$$H_{int} = -\sum_i \left\{ \frac{V}{2}(n_i - 2)^2 + \frac{W}{2} n^2_{a,i} \right\}$$

explicit $Sp(4)$ invariant form

$$V = \frac{3U_0 + 5U_2}{8}, \quad W = \frac{U_2 - U_0}{2}$$

① pesudo-spin SU(2) generater



$$\eta^{\dagger}(i)=\frac{1}{2}\sum_i (-)^i C^{\dagger}_{i,\alpha} R_{\alpha\beta} C^{\dagger}_{i\beta}$$

$$=\sum_i (-)^i\left[C^{\dagger}_{i,3/2}\, C^{\dagger}_{i,-3/2} - C^{\dagger}_{i,1/2}\, C^{\dagger}_{i-1/2}\right]$$

$$\eta(i)=\frac{1}{2}\sum_i (-)^i C_{i\alpha} R_{\alpha\beta} C_{i\beta}$$

$$Q=\frac{1}{2}\sum_{i,\alpha}(n_{i,\alpha}-2)$$

Along the line of $u_0=5u_2$, $H_I=\sum_i -u_2 L^2_{ab}(i)-(\mu-\mu_0)\, n(i)$

$$[H,\eta^{\dagger}]=-(\mu-\mu_0)\eta^{\dagger},\quad [H,\eta]=(\mu-\mu_0)\eta.$$

② SO(7) - line

$$\chi^{\dagger}_a=\frac{1}{2}\sum_i (-)^i C^{\dagger}_{i\alpha}(\Gamma^a R)_{\alpha\beta} C^{\dagger}_{i\beta}$$

along the SO(7) line — superconductivity degenerate with spin qudrapole.

$$[H,\chi^{\dagger}_a]=-(\mu-\mu_0)\chi^{\dagger}_a,\quad [H,\chi_a]=(\mu-\mu_0)\chi_a$$

① $\langle\Omega|\Delta|\Omega\rangle\neq 0$

$\chi^{\dagger}_a|\Omega\rangle$ — Rotate superconductivity to SDW mode.

② $\langle\Omega_a|\Delta^a|\Omega_a\rangle\neq 0$ $\quad\chi^{\dagger}_a|\Omega_b\rangle$ rotate Δ^a into $(-)^i L_{ab}$ if $a\neq b$

CDW if $a=b$.