Lect 2. Off-diagonal long-range order

(Onsager)

1 Penrose's introduction to ODLRO

$$\lim_{r'-r\to\infty} \rho_1(r,r') \longrightarrow \lambda_0 \psi(r)\psi^*(r') + \cdots$$

λ_0/N finite, $\psi(r) \sim \frac{1}{\sqrt{V}}$

2. Yang's formulation

2-particle fermions. (pairing)

$$\rho_2(r_1r_2; r_1'r_2') \rightarrow \lambda_0 \chi_0(r_1r_2)\chi_0^*(r_1r_2) + \cdots$$

λ_0/N finite, $\chi_0(r_1r_2) \sim \frac{1}{\sqrt{V}} \frac{1}{\sqrt{2}} e^{-|r_1-r_2|/a_0}$

3. Flux quantization $\longleftrightarrow$ mesoscopic ring persistent current

Ref: ① O. Penrose Phil. Mag. 42, 1373 (1951) } boson

② O. Penrose, L. Onsager, PR 104, 576 (1956)

③ N. Bayers. C. N. Yang PRL 7, 46 (1961) } pairing ("2e")

④ C. N. Yang, RMP 34, 694 (1962)

⑤ C. Wu PRL 95, 266404 (2005)

↓ ODLRO of quartetting ("4e")

§1. Penrose's introduction to ODLRO

Hydrodynamic description of a quantum liquid: at low temperature the density $\rho(\vec{r})$, velocity $\vec{v}(\vec{r})$, and temperature $T(\vec{r})$ are not sufficient. A new potential of velocity is needed to be superimposed on the classical motion. This situation occurs where the de Broglie wavelength λ_T is large compared to the inter-particle distance.

For a many-body system, for simplification we use the density matrix description. In the coordinate representation

$$\rho(r_1 \cdots r_N ; r_1' \cdots r_N') = \sum_s p_s \psi_s(r_1 \cdots r_N)\, \psi_s^*(r_1' \cdots r_N')$$

where "s" refers to index of states, and p_s is the probability in that state. It satisfies

$$\mathrm{tr}\,\rho = \int dr_1 \cdots dr_N \sum_s p_s\, |\psi_s(r_1 \cdots r_N)|^2 = \sum_s p_s = 1.$$

The reduced density matrix is defined as

$$\rho_1(r_1, r_1') = N \int dr_2 \cdots dr_N\, \rho(r_1 r_2 \cdots r_N ; r_1' r_2 \cdots r_N)$$

$$= N \sum_s p_s \int dr_2 \cdots dr_N\, \psi_s(r_1 r_2 \cdots r_N)\, \psi_s^*(r_1' r_2 \cdots r_N)$$

$$\rho_2(r_1 r_2 ; r_1' r_2') = N(N-1) \int dr_3 \cdots dr_N\, \rho(r_1 r_2 r_3 \cdots r_N ; r_1' r_2' r_3 \cdots r_N)$$

$$\rho_k(r_1 \cdots r_k ; r_1' r_2' \cdots r_k') = \frac{N!}{(N-k)!} \int dr_{k+1} \cdots dr_N\, \rho(r_1 \cdots r_k ; r_{k+1} \cdots r_N ; r_1' \cdots r_k', r_{k+1} \cdots r_N).$$

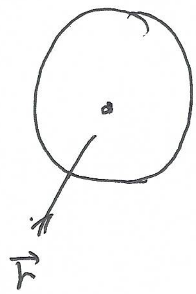
$$\rho_1(r_1; r_1') = \int dp \; F(r; p) \; e^{i p \cdot (r_1 - r_1')/\hbar}$$

Wigner distribution function

$$r = \frac{r_1 + r_1'}{2}$$

macroscopic small

microscopically large - Fourier



$$F(r; p) \xrightarrow[\text{result}]{\text{classic}} e^{-\frac{p^2}{2m}\beta} \quad \text{with } \beta = k_B T.$$

$$\rho_1(r_1; r_1') \propto \int dp \; e^{-\frac{p^2}{2m}\beta} \; e^{i p (r_1 - r_1')/\hbar}$$

$$\propto e^{-(r_1 - r_1')^2 \frac{m}{2\beta\hbar^2}}$$

define $k_B T = \frac{\hbar^2}{2m\lambda^2}$

$$\boxed{\rho_1(r_1; r_1') \propto e^{-\left(\frac{r_1 - r_1'}{2\lambda(T)}\right)^2}}$$

$$\lambda^2 = \frac{\hbar^2}{2m k_B T}$$

thermal wave length

Hence $\lim_{|r_1 - r_1'| \to \infty} \rho_1(r_1; r_1') = 0$ at $|r_1' - r_1| \to \infty$

in the classic regime.

non-classical part

How about $F(r; p) = f(p) + f_0 \, \delta(p)$? we will have

regular at $p \to 0$

$$\rho_1(r_1; r_1') = \int dp \left(f(p) + f_0 \, \delta(p) \right) e^{i p (r_1 - r_1')/\hbar}$$

$$\xrightarrow{r_1 - r_1' \to \infty} f_0$$

a constant.

this means a macroscopic occupation of a single-particle level.

Since $F(r,p)$ is the Wigner distribution function, $f_0\delta(p)$ is essentially the occupation on the single particle state $p=0$. Hence, this is only possible for a bosonic system, i.e. BEC.

Comment: ① BEC at a finite momentum / momenta are also possible. — p-band BEC into $\psi_{Q_x}(\vec{r}) + i\psi_{Q_y}(\vec{r})$

C. Wu Mod. Phys. Lett 23, 1 (2009).

② Pair density wave — Cooper pairing at finite momenta.

§2. ~~Extraction~~ the asymptotic behavior of $\rho(r,r')$ based on general consideration

Time evolution:

$$i\frac{\partial}{\partial t}\rho_1(r_1;r_1',t) = N\sum_s P_s\int dr_2 \cdot dr_N \sum_j -\frac{\hbar^2}{2m}\sum_j\Big[\nabla_j^2\psi_s(r_1\cdots r_N)\,\psi_s^*(r_1'\cdots r_N)$$

$$-\psi_s(r_1\cdots r_N)\,\nabla_j\psi_s^*(r_1'\cdots r_N)$$

$$+\sum_{i<j}\Big[u(r_i-r_j) - \sum_{i'<j'}u(r_{i'}-r_{j'})\Big]\psi_s(r_1\cdots r_N)\,\psi_s^*(r_1'\cdots r_N)$$

$$= N\sum_s P_s\int dr_2\cdots dr_N\Big\{\Big[\frac{-\hbar^2}{2m}(\nabla_{r_1}^2-\nabla_{r_1'}^2)\Big]\psi_s(r_1\cdots r_N)\,\psi_s^*(r_1'\cdots r_N)$$

$$+\Big(\sum_{j=2}^N u(r_1-r_j) - u(r_{1'}-r_j)\Big)\psi_s(r_1,\cdots r_N)\,\psi_s^*(r_1',\cdots r_N)$$

$$= \sum_s NP_s\int dr_2\cdot\cdot dr_N\Big(-\frac{\hbar^2}{2m}(\nabla_{r_1}^2-\nabla_{r_1'}^2)\Big)\psi_s(r_1\cdots r_N)\,\psi_s^*(r_1'\cdots r_N)$$

$$+ N(N-1)\int \underset{\widehat{dr_2}}{dr_3}\cdots dr_N\,\big[u(r_1-r_2)-u(r_{1'}-r_2)\big]\,\psi_s(r_1\cdots r_N)\,\psi_s^*(r_1'\cdots r_N)$$

$$\boxed{\begin{aligned} i\frac{\partial}{\partial t}\rho_1(r_1;r_1';t) &= -\frac{\hbar^2}{2m}(\nabla^2_{r_1}-\nabla^2_{r_1'})\,\rho_1(r_1;r_1';t) \\ &+\int dr_2\,[u(r_1-r_2)-u(r_1'-r_2)]\,\rho_2(r_1 r_2;r_1' r_2) \end{aligned}}$$

$$i\hbar\frac{\partial}{\partial t}\rho_1(r_1,r_1';t) = -\frac{\hbar^2}{2m}(\nabla^2_{r_1}-\nabla^2_{r_1'})\,\rho_1(r_1;r_1',t)$$

$$+\int dr\; u(r)\,[\rho_2(r_1,r_1+r;r_1' r_1+r] - \rho_2(r_1,r_1'+r,\; r_1',r_1'+r)]$$

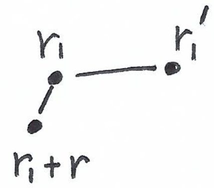
$$\swarrow$$

$$\int dr\; u(r)\,[\rho_2(r_1,r_1+r;r_1' r_1+r] - \rho_2^*(r_1',r_1'+r,\; r_1,r_1'+r]\}$$

Assume $\rho_2(r_1,r_1+r;r_1',r_1+r) \sim \rho_1(r_1;r_1')\,A(r_1,r)$ at $|r_1'-r_1|\to\infty$

Then

$$i\hbar\frac{\partial}{\partial t}\rho_1(r_1,r_1';t) = -\frac{\hbar^2}{2m}(\nabla^2_{r_1}-\nabla^2_{r_1'})\,\rho_1(r_1 r_1';t)$$



$$+\int dr\; u(r)\,[\rho_1(r_1;r_1')A(r_1,r) - \rho_1^*(r_1',r_1)A^*(r_1',r)]$$

$$\swarrow$$

$$\int dr\; u(r)\,[A(r_1,r)-A^*(r_1',r)]\,\rho_1(r_1,r_1']$$

real imaginary

Define $\boxed{X(r_1) = \int dr\; u(r)\,[A(r_1,r)] = V(r_1)+iW(r_1)}$

$$\Rightarrow \boxed{i\hbar\frac{\partial}{\partial t}\rho_1(r_1,r_1';t) = -\frac{\hbar^2}{2m}(\nabla^2_{r_1}-\nabla^2_{r_1'})\,\rho_1(r_1 r_1';t) + [X(r_1)-X^*(r_1')]\,\rho_1(r_1 r_1';}$$

Seperating variable $\rho_1 = \psi(r,t)\,\psi^*(r',t) \Rightarrow$

$$i\hbar\frac{\partial}{\partial t}\psi(r,t) = -\frac{\hbar^2}{2m}\nabla^2\psi(r,t) + X(r)\,\psi(r,t).$$

Then $\psi(r_1, \;) = |\psi(r_1)|\, e^{i\phi(r_1)}$

$$R_1(r_1, r_1') = \quad |\psi(r_1)| \cdot |\psi(r_1')|\, e^{i(\phi(r_1) - \phi(r_1'))}$$

$$i\hbar \frac{\partial}{\partial t}\psi(r_1,t) = i\hbar \frac{\partial}{\partial t}|\psi(r_1,t)|\, e^{i\phi(r_1,t)} + i\hbar |\psi(r_1,t)| e^{i\phi(r_1,t)} \left(i\frac{\partial\phi}{\partial t}\right)$$

$$i\hbar\, \psi^*(r_1,t) \frac{\partial}{\partial t}\psi(r_1,t) = i\hbar\, |\psi(r_1,t)| \frac{\partial}{\partial t}|\psi(r_1,t)| + i\hbar |\psi(r_1,t)|^2 i\left(\frac{\partial\phi}{\partial t}\right)$$

$$\nabla\psi = \nabla|\psi|\, e^{i\phi} + \quad \overset{\psi}{\;} i\nabla\phi(r)$$

$$\nabla^2\psi = \nabla^2|\psi|\, e^{i\phi} + \nabla|\psi|\, e^{i\phi}(i\nabla\phi(r)) + \underset{\nabla|\psi| e^{i\phi} + i\psi\nabla\phi(r)}{\underbrace{\nabla\psi}}\, (i\nabla\phi(r))$$

$$+\; i\psi\, \nabla^2\phi(r)$$

$$= \nabla^2|\psi| e^{i\phi} + 2\nabla|\psi| e^{i\phi}(i\nabla\phi(r)) + \quad \psi\left(-(\nabla\phi(r))^2 + i\nabla^2\phi(r)\right)$$

$$\psi^*\nabla^2\psi = |\psi|\nabla^2|\psi| + 2|\psi|\nabla|\psi|\, i(\nabla\phi(r)) + |\psi|^2\left(-(\nabla\phi(r))^2 + i\nabla^2\phi(r)\right)$$

$$\Rightarrow i\hbar \frac{1}{2}\frac{\partial}{\partial t}|\psi|^2 = -\frac{i\hbar^2}{2m}\left[2|\psi|\nabla|\psi|\, \nabla\phi(r) + |\psi|^2\, \nabla^2\phi(r)\right] + iW(r)|\psi|^2$$

$$\frac{1}{2}\frac{\partial}{\partial t}|\psi|^2 = \frac{-\hbar}{2m}\left[\nabla|\psi|^2 \cdot \nabla\phi(r) + |\psi|^2\nabla^2\phi(r)\right] + W(r)|\psi|^2$$

$$\boxed{\frac{\partial}{\partial t}|\psi|^2 + \frac{\hbar}{m}\nabla\left(|\psi|^2\nabla\phi(r)\right) = \frac{2|\psi|^2 W}{\hbar}}$$

$$-\hbar|\psi|^2\frac{\partial\phi}{\partial t} = -\frac{\hbar^2}{2m}\left[|\psi|\nabla^2|\psi| - |\psi|^2(\nabla\phi(r))^2\right] + V|\psi|^2$$

$$\hbar\frac{\partial\phi}{\partial t} + \underbrace{\frac{\hbar^2}{2m}(\nabla\phi)^2}_{\frac{m}{2}\left(\frac{\hbar\nabla\phi}{m}\right)^2} + V = \frac{\hbar^2}{2m}\frac{\nabla^2|\psi|}{|\psi|} \longrightarrow 0$$

for a slow varying $|\psi|$.

c.f. Euler eq $\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v} = -\frac{\nabla P}{\rho}$

if $\nabla \times \vec{v} = 0 \Rightarrow \vec{v} = \nabla \phi$

$(\vec{v}\cdot\nabla)\vec{v} = \frac{1}{2}\nabla \vec{v}^2 \Rightarrow$

$$\nabla\left(\frac{\partial \phi}{\partial t} + \frac{1}{2}v^2 + \frac{P}{\rho}\right) = 0$$

hence $\boxed{\frac{\partial \phi}{\partial t} + \frac{1}{2}v^2 + P/\rho = f(t)}$

(*) what's special of quantum condensate

- Single particle QM: "v" ~~has~~ is easiler disturbed by noise or detection: For example $\psi(x) = \sqrt{2/L}\,\sin(n\pi x/L)$ in potential well. $\Rightarrow v = 0$, since $\psi(x)$ is real, But actually if you measure momentum, it's $\pm h/mL$.
- Classical fluid: $\vec{v}$ has meaning with little fluctuation but $\oint \vec{v}\cdot d\vec{l}$ usually not quantized.
- Quantum condensate

$$\oint \vec{v}\cdot d\vec{l} = \frac{nh}{m}$$

quantized along a closed path — where the order parameter does not vanish.

§ Eigenvalues of density matrix

$$\rho_1 = N\, tr_{2,3,\cdots N}(\rho) \quad - \text{ partial trace over } 2,3,\cdots N$$

λ_M is the largest eigenvalue of ρ_1, if

λ_M / N is finite $\longleftrightarrow$ BEC or ODLRO.

- For system with translation symmetry, we can diagonalize

$$\langle p | \rho_1 | p' \rangle = \frac{1}{V} \int e^{-i(pr - p'r')/\hbar} \langle r | \rho_1 | r' \rangle \, dr\, dr'$$

$$= \lambda_p \, \delta_{p,p'}$$

$$\langle p | \rho_1 | p \rangle = N\, tr_{p_2 \cdots p_N} \langle p\, p_2 \cdots p_N | \rho | p\, p_2 \cdots p_N \rangle$$

\# of particles in the state of momentum p.

- If spatially non-uniform, we approximate

$$\rho_1(r\, r') = \psi(r)\psi^*(r') + \cdots$$

Such that $\frac{1}{N} | \rho_1(r\, r') - \psi(r)\psi^*(r') | \le \frac{1}{V} \gamma(|r-r'|)$

and $\quad \Gamma(r) \equiv \frac{1}{V} \int \gamma(|r-r'|)\, d^3r' = o(1)$.

then define $n_\psi \equiv \int |\psi(r)|^2 d^3r$, which is a good approximation of $\lambda_M \approx n_\psi$. Intuitively, $\rho_1 \approx \lambda_M |\psi\rangle_{nm} \langle \psi|_{nm} + \cdots$

where $\langle r | \psi \rangle = \frac{1}{\sqrt{n_\psi}} \psi(r)$ is the normalized wavefunction.

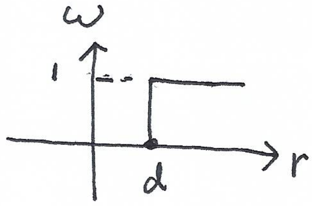
and $\frac{1}{\sqrt{n_\psi}} \psi(r)$ is a good approx of the eigenfunction of $\rho_1(r\, r')$.

(*) Example : trial WF for ^4He

$$\psi(x_1 .. x_N) = \Big(\overbrace{\prod_j \mu(x_j) \prod_{i<j} \omega|x_i - x_j|}^{F_N(x_1 .. x_N)}\Big) \frac{1}{\Omega_N^{1/2}} \leftarrow \text{normalization}$$

$$\rho(x_1 .. x_N ; x_1' .. x_N') = \psi(x_1 .. x_n)\, \psi^*(x_1' .. x_N')$$

$$\begin{cases} \mu = 1 \\ \omega(r) = \begin{cases} 0 & \text{for } r < d \\ 1 & \text{for } r \geq d \end{cases} \end{cases}$$



$$\frac{\Omega_N}{N!} = \int \frac{dx_1 .. dx_N}{N!} \prod_{i<l} \omega|x_i - x_j| \leftarrow \text{Configuration of N - noninteracting hard sphere}$$

$$\rho(x, x') = \frac{N}{\Omega_N} \int dx_2 .. dx_N\, \omega(x-x_2) .. \omega(x-x_N) \prod_{2 \leq i \neq j \leq N} \omega(x_i - x_j)\, \omega(x'-x_2) .. \omega(x'-x_N)$$

$$= \frac{1}{(N+1)\Omega_N}\, \Omega_{N+1}\, \frac{N(N+1)}{\Omega_{N+1}} \int dx_2 .. dx_N\, F_{N+1}(x, x', x_2, .. x_N)$$

$$= n_2(x, x')/z, \quad \text{where } z = (N+1)\Omega_N / \Omega_{N+1} \leftarrow \text{activity of hard sphere.}$$

as $|x - x'| \to +\infty$, $n_2(x, x') \longrightarrow (N/V)^2$

$\uparrow$ pair-distribution function

hard core $r \simeq 2.6$Å density is 28% of the closest packed case!

Hence, we can take

$$\rho(x, x') \xrightarrow{|x-x'| \to \infty} \left(\frac{N}{V} \cdot z^{-1/2}\right)^2, = \psi(x)\,\psi(x')$$

$$\text{Hence } \psi(x) = \frac{N}{V} z^{-1/2}. \Rightarrow \lambda_M \simeq n_\Psi = \left(\frac{N}{V}\right)^2 z^{-1}\, V$$

$$\frac{\lambda_M}{N} \simeq \frac{N}{Vz} = \frac{N}{N+1} \frac{\Omega_{N+1}}{V \Omega_N} \xrightarrow{N \to \infty} 8\%$$

§ C.N. Yang RMP 1962.

Define normalized ρ with $\mathrm{tr}\,\rho = 1$ for a fixed particle number N. $a_i\, a_j$ as the annilation operators for single particle state $i, j, \cdots$, then

$$\rho^{(1)}_{ij} = \mathrm{tr}(a_i \rho a_j^+) = \mathrm{tr}[\rho a_j^+ a_i].$$

$$\rho^{(2)}_{i_1 i_2, j_1 j_2} = \mathrm{tr}(a_{i_1} a_{i_2} \rho\, a^+_{j_2} a^+_{j_1}) = \mathrm{tr}[\rho\, a^+_{j_2} a^+_{j_1} a_{i_1} a_{i_2}].$$

A few properties:

① All ρ_n's are positive-semidefinit.

Proof: Define $\psi = \sum_{i=1}^{n} C_i^* a_i$ where $\sum |C_i|^2 = 1$

then $\mathrm{tr}[\psi \hat{\rho} \psi^+] = C_i^* (\mathrm{tr}\, a_i \hat{\rho} a_j^+) C_j = \sum_{ij} C_i^* \rho_{ij} C_j$

$$\mathrm{tr}[\hat{\rho} \psi^+ \psi] = \sum_s \rho_s \langle s | \psi^+ \psi | s \rangle \geq 0 \Rightarrow \rho_{ij} \text{ positive-semidefinit.}$$

② $\mathrm{tr}\, \rho_1 = N$

$\mathrm{tr}\, \rho_2 = N(N-1)$

$\mathrm{tr}\, \rho_3 = N(N-1)(N-2)$

$\vdots$

Proof: $a^+_{i_1} a^+_{i_2} a^+_{i_3} a_{i_3} a_{i_2} a_{i_1}$

$= a^+_{i_1} a^+_{i_2} a_{i_2} (a^+_{i_3} a_{i_3} - \delta_{i_2 i_3}) a_{i_1}$

$= a^+_{i_1} a_{i_1} (a^+_{i_2} a_{i_2} - \delta_{i_1 i_2})$

$(a^+_{i_3} a_{i_3} - \delta_{i_2 i_3} - \delta_{i_1} \delta_{i_3})$

$$\rightarrow \sum_{i_1 i_2 i_3} a^+_{i_1} a^+_{i_2} a^+_{i_3} a_{i_3} a_{i_2} a_{i_1} = N(N-1)(N-2)$$

$\Rightarrow$ The largest eigenvalues λ_i for ρ_i satisfying $\begin{cases} \lambda_1 \leq N \\ \lambda_2 \leq N(N-1) \\ \lambda_3 \leq N(N-1)(N-2). \end{cases}$

③ A few theorems on the relations ~~&~~ among λ's.

① $\lambda_2 \geq \lambda_1^2 - \lambda_1$: for bosons

Proof. Let f_i be the normalized eigenvector of $\rho_{i'i}$, with the eigenvalue λ_1;

Define $F = \sum f_i^* a_i$, then $tr(F^\dagger F \rho) = \sum_{ji} a_j^* a_i \rho_{ji} = \lambda_1$

use $f_i f_j$ as a trial state for $\langle i'j' | \rho_2 | ij \rangle$

$$\lambda_2 \geq tr(F F \rho F^\dagger F^\dagger) = tr[F^\dagger F^\dagger F F \rho)$$

$$F^\dagger F = F F^\dagger - 1$$

$$\lambda_2 \geq tr[(F^\dagger F F^\dagger F - F^\dagger F)\rho] = tr[(F^\dagger F)^2 \rho] - tr[(F^\dagger F)\rho]$$

$$= tr[(F^\dagger F)^2 \rho] - \lambda_1$$

since $tr[(F^\dagger F - \lambda_1)^2 \rho] = tr[(F^\dagger F)^2 - 2\lambda_1 (F^\dagger F)\rho + \lambda_1^2 \rho] \geq 0$

$$\Rightarrow tr[(F^\dagger F)^2 \rho] - \lambda_1^2 \geq 0$$

$$\Rightarrow \boxed{\lambda_2 \geq \lambda_1^2 - \lambda_1}$$

④ $\lambda_3 \geq \lambda_1^3 - 2\lambda_1^2 - \lambda_2$ for a system of bosons.

Proof: use $f_i f_j f_k$ as the trial state for ρ_3.

$$\lambda_3 \geq tr[F^\dagger F^\dagger F^\dagger FFF\rho] = tr[F^\dagger F^\dagger (F^\dagger F - 1) FF\rho]$$
$$= tr[F^\dagger F^\dagger FF^\dagger FF\rho] - tr[F^\dagger F^\dagger FF\rho]$$
$$\geq tr(F^\dagger F^\dagger FF^\dagger FF\rho) - \lambda_2$$

consider $tr[(F^\dagger(F^\dagger F - \lambda_1)^2 F)\rho] \geq 0 \to \sum_s \rho_s \langle s|F^\dagger|n\rangle (F^\dagger F - \lambda_1)^2_{nn} \langle n|F|s\rangle \geq 0$

$$tr[F^\dagger(F^\dagger F)^2 F\rho] - 2\lambda_1 tr[F^\dagger F^\dagger FF\rho] + \lambda_1^3$$
$$= tr[F^\dagger F^\dagger FF^\dagger FF\rho] \; -2\lambda_1 tr[F^\dagger FF^\dagger F\rho] + 2\lambda_1 tr[F^\dagger F\rho] + \lambda_1^3 \geq 0$$

$$\Rightarrow tr[F^\dagger F^\dagger F F^\dagger FF\rho] \geq 2\lambda_1 tr[F^\dagger FF^\dagger F\rho] - 2\lambda_1^2 - \lambda_1^3$$
$$\geq 2\lambda_1^3 - 2\lambda_1^2 - \lambda_1^3 \quad \uparrow \text{ see before}$$

$$\Rightarrow \boxed{\lambda_3 \geq \lambda_1^3 - 2\lambda_1^2 - \lambda_2}$$

⑤ $\lambda_4 \geq \lambda_2^2 - 4\lambda_3 - 2\lambda_2$ for bosons ~~(unfinished!)~~.

Proof: Let f_{12} be the eigenstate of ρ_2 with the eigenvalue λ_2.

i.e. $F = \sum_{i_1 i_2} f^*_{i_1 i_2} a_{i_1} a_{i_2} \Rightarrow tr[F^\dagger F\rho_2] = f^*_{i_1 i_2} \rho_{2; i_1 i_2, i_1' i_2'} f_{i_1' i_2'}$
$$= \lambda_2$$

$$0 \leq tr[(F^\dagger F - \lambda_2)^2 \rho_2] = tr[(F^\dagger F)^2 \rho_2] - 2\lambda_2 tr[(F^\dagger F)\rho_2] + \lambda_2^2$$
$$= tr[(F^\dagger F)^2 \rho] - \lambda_2^2.$$

Again we use trial state $FF = \sum_{i_1 i_2 i_3 i_4} f^*_{i_1 i_2} f^*_{i_3 i_4} a_{i_1} a_{i_2} a_{i_3} a_{i_4}$

For ρ_4, we have $\lambda_4 \geq tr[FF\rho F^\dagger F^\dagger] = tr[F^\dagger F^\dagger FF\rho]$

$$F^\dagger F^\dagger F F = F^\dagger F F^\dagger F + F^\dagger(F^\dagger F - F F^\dagger)F$$

$$\lambda_4 \geq tr[F^\dagger F^\dagger F F\rho] = tr[(F^\dagger F)^2\rho] + tr[F^\dagger [F^\dagger, F] F \rho]$$

$$\geq \lambda_2^2 + tr(F^\dagger [F^\dagger, F] F\rho]$$

$$[F^\dagger, F] = \sum_{i_1 i_2, i_1' i_2'} f_{i_1 i_2} f^*_{i_1' i_2'} [a^\dagger_{i_1} a^\dagger_{i_2}, a_{i_1'} a_{i_2'}]$$

$$[a^\dagger_{i_1} a^\dagger_{i_2}, a_{i_1'} a_{i_2'}] = -a^\dagger_{i_1} a_{i_2'} \delta_{i_2 i_1'} - a^\dagger_{i_1} a_{i_1'} \delta_{i_2 i_2'}$$

$$- \delta_{i_1 i_1'} a_{i_2'} a^\dagger_{i_2} - \delta_{i_1 i_2'} a_{i_1'} a^\dagger_{i_2}$$

$$[AB, CD] = A[B, CD] + [A, CD]B$$
$$= A[B,C]D + AC[B D]$$
$$+ [A,C]DB + C[A,D]B$$

$$= -(\delta_{i_1 i_1'} a^\dagger_{i_2} a_{i_2'} + \delta_{i_2 i_2'} a^\dagger_{i_1} a_{i_1'}$$
$$+ \delta_{i_1 i_2'} a^\dagger_{i_2} a_{i_1'} + \delta_{i_2 i_1'} a^\dagger_{i_1} a_{i_2'})$$
$$-(\delta_{i_1 i_1'} \delta_{i_2 i_2'} + \delta_{i_1 i_2'} \delta_{i_1' i_2})$$

$$[F^\dagger, F] = -\sum_{i_2 i_2'} \left(\sum_a f_{a i_2} f^*_{a i_2'}\right) a^\dagger_{i_2} a_{i_2'}$$
$$- \sum_{i_1 i_1'} \left(\sum_a f_{i_1 a} f^*_{i_1' a}\right) a^\dagger_{i_1} a_{i_1'}$$
$$- \sum_{i_1' i_2} \left(\sum_a f_{a i_2} f^*_{i_1' a}\right) a^\dagger_{i_2} a_{i_1'}$$
$$- \sum_{i_1 i_2'} \left(\sum_a f_{i_1 a} f^*_{a i_2'}\right) a^\dagger_{i_1} a_{i_2'}$$

$$- \underbrace{\sum_{i_1 i_2} \left(f_{i_1 i_2} f^*_{i_1 i_2} + f_{i_1 i_2} f^*_{i_2 i_1}\right)}_{= 2}$$

symmetry $f_{i_1 i_2} = f_{i_2 i_1}$, define $g_{ij} = \sum_a f_{ia} f^*_{ja}$

$$\Rightarrow [F^\dagger, F] = -4 \sum_{ij} g_{ij} a^\dagger_i a_j - 2$$

$$tr[F^\dagger[F^\dagger F]F\rho] = -4\sum_{ii'} g_{ii'}\, tr[F^\dagger a_i^\dagger a_{i'} F\rho] - 2\, tr[F^\dagger F\rho]$$

$$\sum_{ii'} g_{ii'}\, tr[F^\dagger a_i^\dagger a_{i'} F\rho] = \sum_a f_{ia}^* f_{i'a} f_{i_1i_2}^* f_{i_1'i_2'}\ \rho_{3,\, i_1i_2i;\, i_1'i_2'i'}$$

$$= \sum_a (f_{ia} f_{i_1i_2})^* \rho_{3,\, i_1i_2i;\, i_1'i_2'i'} (f_{i'a} f_{i_1'i_2'}) \leq \lambda_3 .$$

$$\Rightarrow \quad tr[F^\dagger[F^\dagger, F]F\rho] \geq -4\lambda_3 - 2\lambda_2$$

Hence, $\lambda_4 \geq tr[FF\rho F^\dagger F^\dagger] = tr[(F^\dagger F)^2\rho] + tr[F^\dagger[F^\dagger, F]F\rho]$

$$\boxed{\lambda_4 \geq \lambda_2^2 - 4\lambda_3 - 2\lambda_2}$$

(6) If for fermions, we need to recalculate $[AB, CD]$

$$[AB, CD] = A[B, CD] + [A, CD]B = A\{B,C\}D - AC\{B,D\} + \{A,C\}DB - C\{A,D\}B$$

$$[a_{i_1}^\dagger a_{i_2}^\dagger, a_{i_1'} a_{i_2'}] = a_{i_1}^\dagger \{a_{i_2}^\dagger, a_{i_1'}\} a_{i_2'} - a_{i_1}^\dagger a_{i_1'} \{a_{i_2}^\dagger, a_{i_2'}\} + \{a_{i_1}^\dagger, a_{i_1'}\} a_{i_2'} a_{i_2}^\dagger - a_{i_1'} \{a_{i_1}^\dagger, a_{i_2'}\} a_{i_2}^\dagger$$

$$= a_{i_1}^\dagger a_{i_2'} \delta_{i_2i_1'} - a_{i_1}^\dagger a_{i_1'} \delta_{i_2i_2'} - a_{i_1'} a_{i_2}^\dagger \delta_{i_1i_2'} + a_{i_2'} a_{i_2}^\dagger \delta_{i_1i_1'}$$

$$= -\delta_{i_1i_1'} a_{i_2}^\dagger a_{i_2'} - \delta_{i_2i_2'} a_{i_1}^\dagger a_{i_1'} + \delta_{i_1i_2'} a_{i_2}^\dagger a_{i_1'} + \delta_{i_2i_1'} a_{i_1}^\dagger a_{i_2'}$$

$$+ (\delta_{i_2i_2'} \delta_{i_1i_1'} - \delta_{i_1i_2'} \delta_{i_1'i_2})$$

$$[F^\dagger, F] = -\sum_{i_2 i_2'} \left(\sum_a f_{ai_2} f^*_{ai_2'}\right) a^\dagger_{i_2} a_{i_2'}$$

$$- \sum_{i_1 i_1'} \left(\sum_a f_{i_1 a} f^*_{i_1' a}\right) a^\dagger_{i_1} a_{i_1} \quad - \sum_{i_1 i_2} \left(f_{i_1 i_2} f^*_{i_1 i_2} - f_{i_1 i_2} f^*_{i_2 i_1} \right)$$

$$+ \sum_{i_1' i_2} \left(\sum_a f_{ai_2} f^*_{i_1' a}\right) a^\dagger_{i_2} a_{i_1'}$$

$$+ \sum_{i_1, i_2'} \left(\sum_a f_{i_1 a} f^*_{a i_2'}\right) a^\dagger_{i_1} a_{i_2'}$$

Consider the anti-sym $f_{i_1 i_2} = -f_{i_2 i_1}$

$$\Rightarrow [F^\dagger, F] = -4 \sum_{ij} \left(\sum_a f_{ai} f^*_{aj}\right) a^\dagger_i a_j \; -2$$

~~(It seems I cannot arrive Yang's result that λ_3 should vanish.~~

It seems that I cannot arrive at Yang's result of Theorem 4.

I cannot see why the coefficient of λ_3 should vanish here.

$\boxed{\lambda_4 \geq \lambda_2^2 - 2\lambda_2}$ → why λ_3 is absent ?

⑦ For fermions. $\lambda_1 \leq 1$. This is obvious: In the eigenbasis of ρ_1,

$$\rho_{1,ii} = tr[a_i \rho a_i^+] = \langle a_i^+ a_i \rangle \leq 1.$$

⑧ For a system with N fermions in M states, we have

$$\lambda_2 \leq N(M-N+2)/M$$

$\Rightarrow \quad \lambda_2 \leq N \,, \quad \rightarrow \lambda_2$ reaches maximum around $N \sim \frac{M}{2}$.

Proof: to be added!

Moore interpretation on pairing in Fermi system

Again $\rho_2(r_1 r_2; r_1' r_2') = \lambda_0 \chi_0(r_1 r_2)\, \chi_0^*(r_1' r_2') + \cdots\cdots$ → vanishes at $r_1' r_2' \xleftrightarrow[+\infty]{} r_1 r_2$

$\frac{\lambda_0}{N}$ finite, and λ_0 is the number of pairs of the 2-particle state $\chi_0(r_1 r_2)$. It satisfies the normalization

$$\int dr_1 dr_2 \, |\chi_0(r_1, r_2)|^2 = 1 .$$

$R = \frac{r_1 + r_2}{2}$; $r = r_1 - r_2$

For translation invariant system $\chi_0(r_1 r_2) = \chi(R, r) = \frac{1}{\sqrt{V}} e^{i\vec{P}_0 \cdot \vec{R}/\hbar} \cdot \frac{1}{\sqrt{\Omega_0}} e^{-r/\lambda}$

hence as $|r_2 - r_1| \to \infty$, $\chi_0 \to 0$.

But as $r_2 - r_1$ and $r_2' - r_1'$ finite, with $R' - R \to \infty$, $\chi_0 \sim \frac{1}{(V \cdot \Omega_0)^{1/2}}$

$$\rho_2(r_1 r_2; r_1' r_2') \xrightarrow[R'-R \to \infty]{} \frac{\lambda_0}{V \cdot \Omega_0} = \frac{\lambda_0}{N} \frac{\Omega}{\Omega_0}$$ remains finite.

This is the ODLRO.

But $\rho_2(r_1 r_2; r_1' r_2') \longrightarrow 0$ if $|r_2 - r_1| \to \infty$ or $|r_2' - r_1'| \to \infty$.

$$|\psi\rangle = \left[\int dr dr' \, \chi_0(r, r')\, \psi^\dagger(r)\, \psi^\dagger(r') \right]^{N/2} |vac\rangle$$

or $\psi(r_1 \ldots r_N) = \mathcal{A}\left\{ \chi_0(r_1 - r_2)\, \chi_0(r_3 - r_4) \cdots \chi_0(r_{n-1} - r_n) \right\}$

Similarly, we can consider 4-fermion (quartetty)

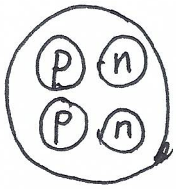
$$\rho_4(r_1 r_2 r_3 r_4 ; r_1' r_2' r_3' r_4') = \lambda_0 \chi_0(r_1 r_2 r_3 r_4) \chi_0^*(r_1' r_2' r_3' r_4') + \cdots\cdots$$

$$\chi_0(r_1 r_2 r_3 r_4) = \chi_0(R, r, r'' r''') = \frac{1}{\sqrt{V}} \frac{1}{\Omega^{3/2}} e^{-r/\lambda} e^{-r''/\lambda} e^{-r'''/\lambda}$$

hence $$\rho_4(r_{1234} ; r_{1234}') \xrightarrow{R-R' \doteq \infty} \frac{\lambda_0}{V} \frac{}{\Omega_0^3} = \frac{\lambda_0}{N} (n \Omega_0^{-3}) .$$

Quartetting:

① alpha-partice



: deuteron v.s. alpha condensation

② 4-component fermions pairing of pairs. C. Wu PRL 95, 266404 (2005)

$$\Delta_1 = \langle | c_\uparrow^\dagger c_\downarrow^\dagger | \rangle . \qquad \Delta_2 = \langle | d_\uparrow^\dagger d_\downarrow^\dagger | \rangle$$
$$= |\Delta_1| e^{i\varphi_1} \qquad\qquad = |\Delta_2| e^{i\varphi_2}$$

Define $\varphi_0 = \dfrac{\varphi_1 + \varphi_2}{2}$. $\qquad \varphi_r = \varphi_2 - \varphi_1$

Then if ① φ_0 is pinned, but φ_r ~~rear~~ disordered $\Rightarrow$

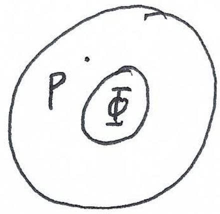
$$\Delta_1 \Delta_2 = |\Delta_1 \Delta_2| e^{i2\varphi_0} \quad \Longleftrightarrow \text{ quartetting}$$

② φ_r pinned but φ_0 disordered

$$\Delta_1^* \Delta_2 = |\Delta_1 \Delta_2| e^{i\varphi_r} \quad \Longleftrightarrow \text{ 4-fermion normal state}$$

The phase transitin between quartetting and pairing is Ising. It breaking the Z_2 sym : $\varphi_0 \to \varphi_0 + \pi$ in the pairing phase not in the quartetting phase.

Flux quantization



If a flux Φ is applied to a multi-connected superconducting ring, the system would response to generate super current in the inner surface.

The screening current would generate a flux ϕ', such that $\Phi_{tot} = \Phi + \Phi'$ satisfies the quantization condition. In this case, the current in the bulk vanishes to minimize the free energy.

$$\vec{j} = \rho_s\left(\nabla\phi - \frac{e}{\hbar c}\vec{A}\right) \qquad \oint \vec{j}\cdot d\vec{l} = 0 \Rightarrow \oint \nabla\phi = \frac{e^*}{\hbar c}\oint \vec{A}\, dl$$

$$\Rightarrow \frac{e^*}{hc}\cdot 2\pi\,\Phi_{tot} = n\cdot 2\pi \Rightarrow \boxed{\Phi_{tot} = n\frac{hc}{e^*}} \leftarrow \Phi_0 = hc/e^*$$

In order to exhibit such a phenomenon, the bulk free energy must depend on the flux Φ_{tot}, i.e. $F(\Phi_{tot})$ reaches energy minimal at $\Phi_{tot} = \frac{nhc}{e^*} = n\Phi_0$. otherwise, the system would not care about Φ_{tot}, and no need for flux quantization. If $B = 0$, inside the bulk, i.e. Φ_{tot} is a constant, the flux is equivalent to a twisted boundary condition.

We consider a $L_1 \times L_2 \times L_3$ system, with periodical boundary condition

$$\begin{cases} \psi(y+L_2) = \psi(y) \\ \psi(z+L_3) = \psi(z) \end{cases} \qquad \psi(x+L) = \psi(x)\, e^{i\cdot 2\pi\,\Phi/\Phi_0}$$

Consider at a ~~temperature~~ T, define $R = e^{-\beta H}$.

Then $Z = \mathrm{tr}\, R = \mathrm{tr}\, e^{-\beta H}$, and then $\rho = \frac{1}{Z} R$

and $Z \rho_n = R_n$ (following the same definition).

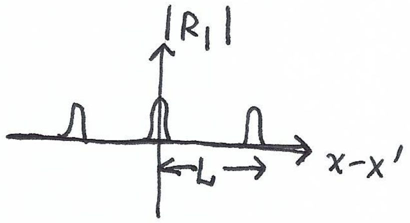
$\Rightarrow Z\, \mathrm{tr}\, \rho_k = \mathrm{tr}\, R_k \Rightarrow \boxed{Z = [N(N-1)\cdots(N-k+1)]^{-1}\, \mathrm{tr}\, R_k.}$

$F = -1/\beta \ln Z =$

Consider

$$\langle x | R_1 | x' \rangle \longrightarrow \langle x+L | R_1 | x' \rangle = \langle x | R_1 | x-L \rangle$$
$$= \langle x | R_1 | x' \rangle e^{i 2\pi \Phi/\Phi_0}$$

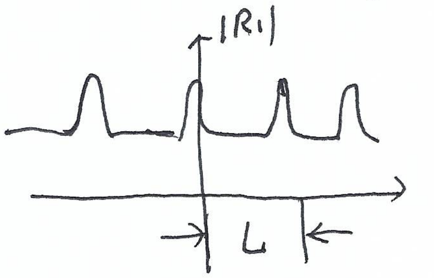
If without ODLRO, $R(x, x') \to 0$ at $|x - x'|$ goes large.



since the large region where $R(x, x') \to 0$, the phase change ~~at~~ between $x'-x=0$ and L, can be locally multiplicate a phase factor of $e^{i2\pi\Phi/\Phi_0}$.

~~This will not~~ Basically these peaks are independent, and F has no dependence on Φ.

But with ODLRO, the situation changes



in this case, when impose the flux $e^{i 2\pi \Phi/\Phi_0}$, or, the periodic boundary condition

The entire profile are affected.

Basically, the eigenvalues and the eigenfunctions are significantly changed!

In this, we do expect F's dependence on Φ, hence, flux quantization.

- For fermions, $R(x, x')$ does not show ODLRO, no flux quantization at $\Phi = hc/e$.

$$\langle x_1+L, x_2 | R_2 | x_1' x_2' \rangle = \langle x_1, x_2+L | R_2 | x_1', x_2' \rangle$$

$$= \langle x_1' x_2' | R_2 | x_1'-L, x_2' \rangle = \langle x_1 x_2 | R_2 | x_1', x_2'-L \rangle$$

$$= \langle x_1 x_2 | R_2 | x_1' x_2' \rangle e^{i 2\pi \Phi/\Phi_0}$$

But as $x_1 \to x_1+L$, while maintaining x_2, x_1' x_2' invariant. R_2 goes to zero, then the phase can be applied without affect free energy.

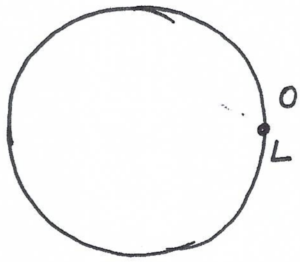
But $\langle x_1+L, x_2+L | R_2 | x_1' x_2' \rangle = \langle x_1 x_2 | R_2 | x_1'+L \; x_2'+L \rangle$

$$= \langle x_1 x_2 | R_2 | x_1' x_2' \rangle e^{i 2\pi \Phi / \frac{hc}{e^*}} \qquad e^* = 2e.$$

as $x_1 x_2 \to x_1+L, x_2+L$, if with ODLRO $R(x_1 x_2 ; x_1' x_2')$ are finite during the process. Hence the phase Φ indeed affect the free energy, unless it's quantized at $\Phi = nhc/e^*$.

⊛ Flux penetrating a mescoscopic ring — persistent current ①

§ Energy v.s. magnetic flux (phase coherent-length $L_{ring} < L_{pha}$



① Single particle spectrum

$$H = \frac{(P - \frac{e}{c}A)^2}{2M} \qquad \psi(x) = \frac{1}{\sqrt{L}} e^{ikx}$$

periodical boundary $k \cdot L = 2n\pi \Rightarrow k = \frac{2\pi}{L} n \quad (n = 0, 1, \cdots L-1).$

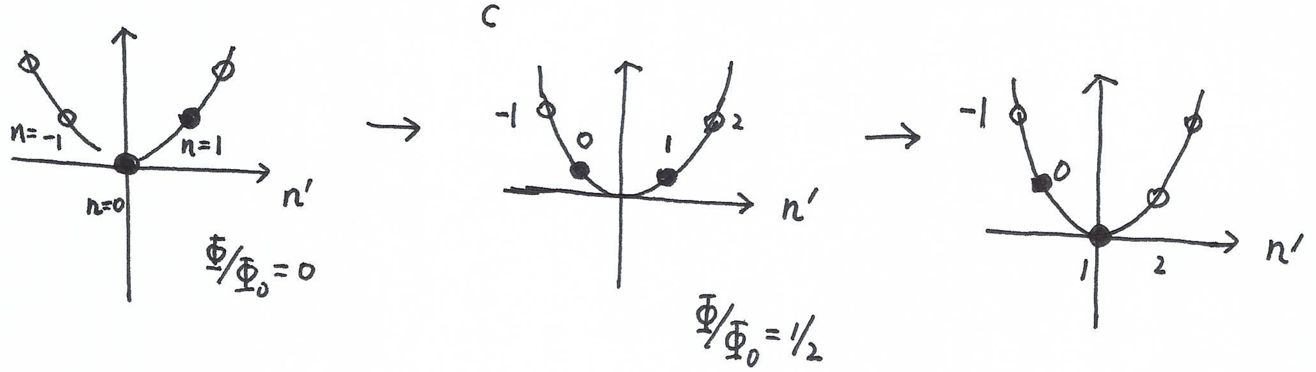
$$H = \frac{\hbar^2}{2M}\left(k - \frac{e}{\hbar c}A\right)^2 = \frac{\hbar^2}{2M}\left(k - \frac{2\pi}{L}\frac{\Phi}{\Phi_0}\right)^2 \qquad \text{where } \Phi_0 = \frac{hc}{e}$$

$$\Phi = \oint A \cdot dr = \quad L \cdot 2\pi$$

$$= \frac{\hbar^2}{2M}\left(\frac{2\pi}{L}\right)^2 \left[n - \Phi/\Phi_0\right]^2$$

define $\varepsilon_0 = \frac{\hbar^2}{2m}\left(\frac{2\pi}{L}\right)^2$ as the energy unit.

$$H = \varepsilon_0 (n - \Phi/\Phi_0)^2 = \varepsilon_0 n'^2 \quad \text{where } n' = n - \Phi/\Phi_0$$



Apparently, there exist a periodicity at $\Phi \to \Phi + \Phi_0$.

Assume the total fermion number $N = 2m$, or $2m+1$.

① $$\frac{E(2m, \Phi/\Phi_0)}{\varepsilon_0} = \sum_{n=\pm 1}^{\pm(m-1)} (n - \Phi/\Phi_0)^2 + (\Phi/\Phi_0)^2 + (m - \Phi/\Phi_0)^2$$

$$= 2(1^2 + 2^2 + \cdots (m-1)^2) + 2m\left[\left(\frac{\Phi}{\Phi_0}\right)^2 - \left(\Phi/\Phi_0\right)\right] \quad \text{for } 0 \le \Phi/\Phi_0 \le 1/2.$$

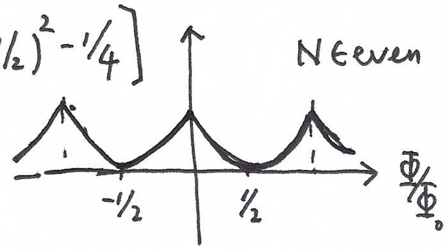
$$\frac{E(2m, \Phi/\Phi_0)}{\mathcal{E}_0} = \frac{1}{3}(m-1)(m)(2m-1) + 2m\left[\left(\frac{\Phi}{\Phi_0} - 1/2\right)^2 - 1/4\right]$$

Density $n = 2m/L \Rightarrow E(\Phi/\Phi_0)\,L^2 = \frac{\hbar^2}{2m}(2\pi)^2\left[\frac{1}{3}\left(\frac{nL}{2}-1\right)\left(\frac{nL}{2}\right)(nL-1)\right.$

$$\left. + nL\left[\left(\frac{\Phi}{\Phi_0} - 1/2\right)^2 - 1/4\right]\right]$$

$$E(\Phi/\Phi_0)/L = \frac{\hbar^2}{2M}(2\pi)^2\left[n^3 \cdot \frac{1}{6}\left(\frac{1}{2} - \frac{1}{N}\right)\left(1 - \frac{1}{N}\right) + \frac{n}{L^2}\left\{\left(\Phi/\Phi_0 - 1/2\right)^2 - 1/4\right\}\right]$$

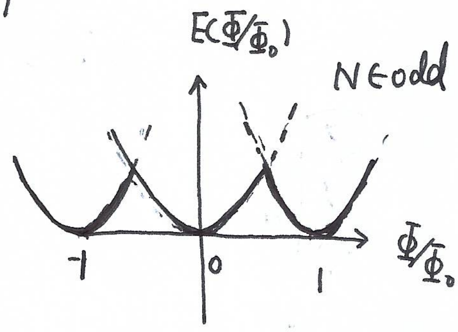
$$\longrightarrow \frac{\hbar^2}{6M}\pi^2 n^3 + n\frac{\hbar^2}{2M}\left(\frac{2\pi}{L}\right)^2\left[\left(\Phi/\Phi_0 - 1/2\right)^2 - 1/4\right]$$



② $$\frac{E(2m+1, \Phi/\Phi_0)}{\mathcal{E}_0} = \sum_{n=\pm 1}^{\pm m}\left(n - \Phi/\Phi_0\right)^2 + \left(\Phi/\Phi_0\right)^2$$

$$= 2(1^2 + 2^2 + \cdots m^2) + (2m+1)\left(\Phi/\Phi_0\right)^2$$

$$= \frac{1}{3}\,\frac{N-1}{2}\,\frac{N+1}{2}\cdot N + N\left(\Phi/\Phi_0\right)^2$$



$$E(\Phi/\Phi_0)\cdot L^2 = \frac{\hbar^2}{2M}(2\pi)^2\left[\frac{1}{12}(N^2-1)N + N\left(\Phi/\Phi_0\right)^2\right]$$

$$\frac{E}{L} \xrightarrow[N\to\infty]{} \frac{\hbar^2}{6M}\pi^2 n^3 + n\frac{\hbar^2}{2M}\left(\frac{2\pi}{L}\right)^2\left(\Phi/\Phi_0\right)^2$$

$$\longrightarrow \frac{1}{2}Mn\,v^2 \text{ with } v = \begin{cases} \frac{\hbar}{M}\,2\pi\left(\Phi/\Phi_0\right). & N \in \text{odd} \\ \frac{\hbar}{M}\,2\pi\left[\Phi/\Phi_0 - 1/2\right] & N \in \text{even} \end{cases}$$

$$\boxed{\Delta E = \frac{1}{2}MNv^2}$$

§ 2- body problem — boson

$$H = \frac{(P_1 - \frac{e}{c}A)^2}{2M} + \frac{(P_2 - \frac{e}{c}A)^2}{2M} + c\,\delta(x_1 - x_2)$$

$$= \frac{-\hbar^2}{2M}\left\{\left[\partial_{x_1} - \frac{ie}{\hbar c}A\right]^2 + \left[\partial_{x_2} - \frac{ie}{\hbar c}A\right]^2\right\} + c\,\delta(x_1 - x_2)$$

$$\psi = \Theta(x_2 - x_1)\,\psi_{12}(x_1, x_2) + \Theta(x_1 - x_2)\,\psi_{21}(x_1, x_2)$$

Boso statistics $\psi(x_1, x_2) = \psi(x_2, x_1)$

$$\Rightarrow \boxed{\psi_{12}(x_1, x_2) = \psi_{21}(x_2, x_1)}$$

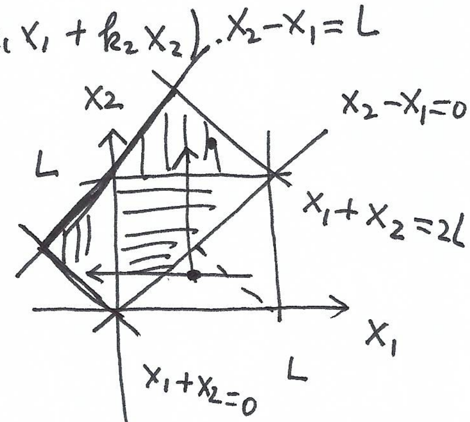
we only need to know the WF of $x_1 < x_2$, then the WF of $x_2 < x_1$ can be obtained via exchange.

For $x_1 < x_2$

$$\psi = A_{12}\, e^{i(k_1 x_1 + k_2 x_2)} + A_{21}\, e^{i(k_2 x_1 + k_1 x_2)},$$

For $x_2 < x_1$

$$\psi = A_{12}\, e^{i(k_2 x_1 + k_1 x_2)} + A_{21}\, e^{i(k_1 x_1 + k_2 x_2)}$$



set the relative coordinate $y = x_2 - x_1$,

the center of mass coordinate $X = \frac{x_1 + x_2}{2}$

equivalently

$$\begin{array}{ccc} (x_1, x_2) & \rightarrow & (x_1 + L, x_2) \\ \downarrow & & \downarrow \\ (x_1, x_2 + L) & \rightarrow & (x_1 + L, x_2 + L) \end{array}$$

$$\begin{array}{ccc} (X, y) & \rightarrow & (X + L/2, y - L) \\ \downarrow & & \downarrow \\ (X + L/2, y + L) & \rightarrow & (X + L, y) \end{array}$$

$$\psi = \int e^{iKX} \begin{cases} A_{12}\, e^{i(k_2-k_1)y/2} + A_{21}\, e^{-i(k_2-k_1)y/2} & y>0 \\ A_{12}\, e^{-i(k_2-k_1)y/2} + A_{21}\, e^{i(k_2-k_1)y/2} & y<0 \end{cases}$$

$$H = \frac{1}{4M}\left(-i\hbar\partial_X - \frac{2e}{\hbar c}A\right)^2 + \frac{1}{M/2}\left(-i\hbar\partial_y\right)^2 + c\delta(y)$$

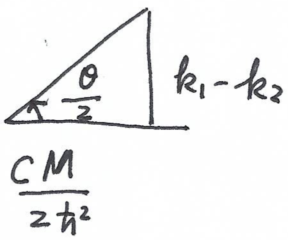
$$\frac{-\hbar^2}{M/2}\frac{d^2}{dy^2}\psi + c\delta(y)\psi = E\psi$$

$$\frac{-\hbar^2}{M/2}\left[\psi'(0^+) - \psi'(0^-)\right] + c\psi(0) = 0$$

$$\frac{\hbar^2}{M/2}\, i(k_2-k_1)(A_{12}-A_{21}) = c(A_{12}+A_{21})$$

$$\Rightarrow \frac{A_{21}}{A_{12}} = - \cancel{\frac{\frac{\hbar^2}{M/2} ic + (k_2-k_1)}{\frac{\hbar^2}{M/2} ic - (k_2-k_1)} = - \frac{c + \frac{iM}{2\hbar^2}(k_1-k_2)}{c - \frac{iM}{\hbar^2}(k_1-k_2)}}$$

$$= - \frac{c\left(\frac{M}{2\hbar^2}\right) + i(k_1-k_2)}{c\left(\frac{M}{2\hbar^2}\right) - i(k_1-k_2)} = - e^{i\theta(k_2-k_1)}$$



$$\frac{1}{2}\theta(k_2-k_1) = \tan^{-1}\left[\frac{2\hbar^2}{CM}(k_1-k_2)\right]$$

$$\begin{cases} \theta(k_2-k_1) = -2\tan^{-1}\left[\frac{2\hbar^2}{CM}(k_2-k_1)\right] \\ \frac{A_{21}}{A_{12}} = -e^{i\theta(k_2-k_1)} \end{cases}$$

⑤

$$\psi(x_1=0,\ x_2) = \psi(x_1=L,\ x_2)$$
$x_1 < x_2$, $\quad x_2 < x_1$

$$\psi_{12}(0, x_2) = \psi_{21}(\ , L, x_2)$$

$$A_{12} e^{i(k_1 0 + k_2 x_2)} + A_{21} e^{i(k_2 \cdot 0 + k_1 x_2)}$$
$$= A_{12} e^{i(k_2 L + k_1 x_2)} + A_{21} e^{i(k_1 \cdot L + k_2 x_2)}$$

$$\Rightarrow \begin{cases} A_{12} = A_{21} e^{ik_1 L} \\ A_{21} = A_{12} e^{ik_2 L} \end{cases}$$

$$\rightarrow \begin{cases} e^{ik_1 L} = \dfrac{A_{12}}{A_{21}} = -\dfrac{g \quad -i(k_1-k_2)}{g \quad +i(k_1-k_2)} \\ e^{ik_2 L} = \dfrac{A_{21}}{A_{12}} = -\dfrac{g \quad -i(k_2-k_1)}{g \quad +i(k_2-k_1)} \end{cases}$$

$g = c\left(\frac{M}{2\hbar^2}\right)$

$$\begin{cases} k_1 L = 2\pi I_1 + \theta(k_1-k_2) \\ k_2 L = 2\pi I_2 + \theta(k_2-k_1) \end{cases}$$

I_1, I_2 here are half-integers

$$\boxed{E = \frac{\hbar^2}{2M}(k_1 - \tfrac{e}{c}A)^2 + \frac{\hbar^2}{2M}(k_2 - \tfrac{e}{c}A)^2 = \frac{\hbar^2}{4M}(k_1+k_2-\tfrac{2e}{c}A)^2 + \frac{\hbar^2}{M/2}\left(\frac{k_1-k_2}{2}\right)^2}$$

$\rightarrow$ N particles

$$\boxed{\begin{gathered} k_i L = 2\pi I_i + \sum_{j=1}^{N} \theta(k_i - k_j), \text{ with } I_i = \begin{cases} \text{integer} & N=\text{odd} \\ \text{half integer} & N=\text{even} \end{cases} \\ e^{ik_i L} = (-)^{N-1} e^{i\sum_{j=1}^{N} \theta(k_i-k_j)} \end{gathered}}$$

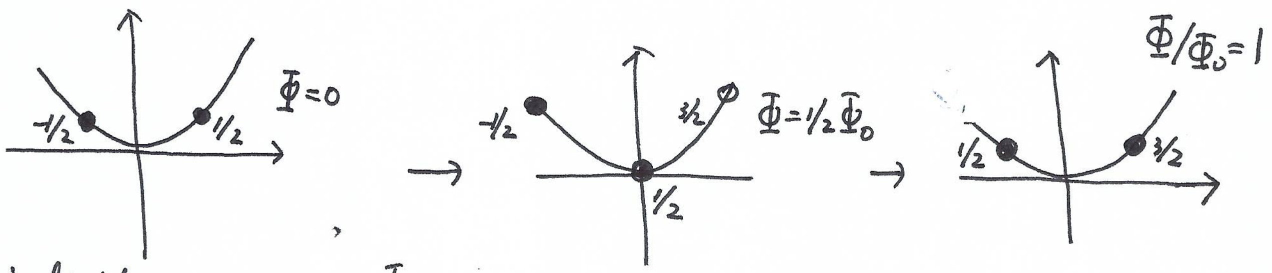
Analysis on 2-body problem

① Repulsive interaction: $g>0$ as $k \to 0$

$$e^{ik_1L} \approx -1, \quad e^{ik_2L} \approx -1, \quad k_1 \neq k_2 \Rightarrow \quad k_i = \frac{2\pi}{L} I_i$$

$$I_1, I_2 \in \{\cdots -\tfrac{3}{2}, -\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{3}{2} \cdots\}$$

$$E = \frac{\hbar^2}{2M}\left(\frac{2\pi}{L}\right)^2 \left[\left(I_1 - \Phi/\Phi_0\right)^2 + \left(I_2 - \Phi/\Phi_0\right)^2\right] \rightarrow \text{periodicity remain } \Phi_0$$



the periodicity remains Φ_0

② Attrative interaction $g<0$, consider bound state

set $A_{12} = 0$, then $k_2 - k_1 = -i\beta$, then

$$\psi = e^{iKX} \cdot e^{-\beta|y|/2} \Rightarrow \quad k_2 = \frac{K}{2} - i\beta/2, \quad k_1 = \frac{K}{2} + i\beta/2$$

$$e^{ik_1L} = e^{i\frac{K}{2}L} e^{-\beta L/2} \xrightarrow{L\to+\infty} 0 \Rightarrow g = i(+i)\beta$$

$$e^{ik_2L} \xrightarrow{L\to\infty} \infty$$

$\boxed{\text{periodicity changes to } \frac{1}{2}\Phi_0}$

$$g = i(k_1 - k_2) = -\beta \Rightarrow \beta = |g|$$

$$\boxed{E = \frac{\hbar^2}{4M}\left(K - \frac{2e}{c}A\right)^2 - \frac{\hbar^2}{2M}\beta^2}$$

$$K = \frac{2n\pi}{L}, \ (n = 0, \pm1, \pm2, \cdots)$$