Lect 3: Bose gas of hard spheres

1. Scattering theory — resonance, bound state etc.

2. Pseudo-potential

3. Many-body problem

4. Lee-Huang-Yang $\frac{E}{N} = \frac{2\pi\hbar^2 a n}{m}\left(1 + \frac{128}{15}\sqrt{\frac{na^3}{\pi}}\right)$

5: Phonons

Ref: 1 K. Huang and C. N. Yang, PR 105, 767 (1957)

2. T.D. Lee, K. Huang, CN Yang PR 106, 1135 (1957)

3. K. Huang, "50 years of hard sphere bose gas 1957–2007

4. R. P. Feynman, PR 94, 262 (1954).

§. Scattering theory

Consider the Schrödinger Eq under the scattering boundary condition

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V(r)\right]\psi = E\psi, \qquad \psi(r) \xrightarrow{r\to+\infty} e^{ikz} + f(\theta)\frac{e^{ikr}}{r}.$$

$f(\theta)$ is the scattering amplitude, carrying the unit of length.

partial wave decomposition: $f(\theta) = \sum_l f_l Y_{l0}(\theta)$

$$\left\{ e^{ikz} = \sum_{l=0}^{\infty} \sqrt{4\pi(2l+1)}\, i^l Y_{l0}(\theta)\, j_l(kr) \right.$$

$$\xrightarrow{kr\to\infty} \sum_{l=0}^{\infty} \sqrt{4\pi(2l+1)}\, i^l \frac{1}{2ikr}\left(e^{i(kr-\frac{l}{2}\pi)} - e^{-i(kr-\frac{l\pi}{2})}\right) \underbrace{}_{Y_{l0}}$$

s-wave scattering:

$$\psi(\vec{r}) \xrightarrow{r\to\infty} j_0(kr) + \frac{f_0}{r} e^{ikr}$$

$$= \frac{1}{2ikr}(e^{ikr} - e^{-ikr}) + \frac{f_0}{r} e^{ikr}$$

$$= \frac{1}{2ikr}(1 + 2ikf_0)\, e^{ikr} - \frac{e^{-ikr}}{2ikr}$$

$$1 + 2ikf_0 = e^{2i\delta_0} \quad \Rightarrow \quad \psi(r) = \frac{e^{i\delta_0}}{2ikr}\, 2i \sin(kr+\delta_0)$$

$$\boxed{kf_0 = \frac{e^{2i\delta_0} - 1}{2i} = e^{i\delta_0}\sin\delta_0} \qquad \boxed{\psi(r) = \frac{e^{i\delta_0}}{kr}\sin(kr+\delta_0)}$$

if $k \to -k$, $\delta_0(k) \to -\delta(k)$

such that $\psi(r)$ remains up to a sign.

§ Zero energy scattering — S-wave

$$\psi = \frac{1}{\sqrt{4\pi}} R_0(r) = \frac{u(r)}{r}, \quad \text{where } u(r) \text{ satisfies}$$

$$\frac{d^2}{dr^2} u + \left(k^2 - \frac{2m}{\hbar^2} V(r)\right) u = 0, \quad \text{where } k = \sqrt{\frac{2mE}{\hbar^2}}$$

at $k \to 0$, at $r > R$ where R is the interaction range

$$\frac{d^2}{dr^2} u = 0 \longrightarrow u(r) = (1 - r/a_0) \cdot \text{const}$$

if $a_0 = 0 \Rightarrow u(r) \propto r \sim \sin kr$ (no-phase shift)

$a_0 \to \pm\infty$ $\quad u(r) \propto \sin(kr + \pi/2)$ (maximum phase shift)

at $r > R_0$ $\quad u(r) = A \sin(kr + \delta_0) = A(\sin\delta_0 + kr\cos\delta_0)$

$$\longrightarrow (1 + k\cot\delta_0\, r)$$

$$\Rightarrow \boxed{k\cot\delta_0 \Big|_{k=0} = -\frac{1}{a_0}}$$

Since δ_0 is an odd function of k, $\quad k\cot\delta_0(k) = -\frac{1}{a_0} + \frac{k^2}{2} R$ as $k \to 0$

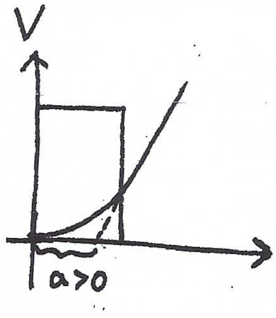
R: interaction range

(energy $\propto k^2$
Schröding eq takes k^2 as a variable)

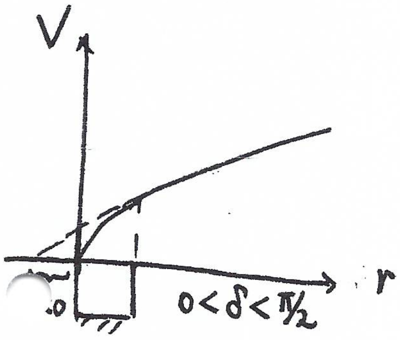
then
$$f_0 = \frac{1}{k} e^{i\delta_0} \sin\delta_0 = \frac{1}{k} \frac{\sin\delta_0}{\cos\delta_0 - i\sin\delta_0} = \frac{1}{k\cot\delta_0 - ik}$$

$$f_0 = \frac{1}{-\frac{1}{a_0} - ik + \frac{k^2}{2} R_0} \xrightarrow{a_0 \to \pm\infty} \frac{i}{k} \quad \text{(reasonace)}$$

repulsive potential

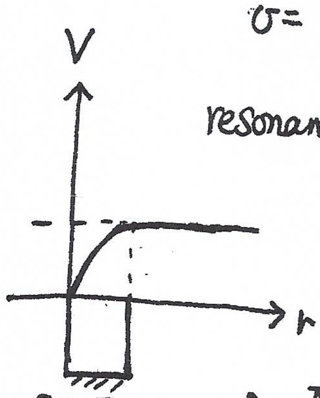


attractive potential



$\sigma = \frac{4\pi}{k^2}$

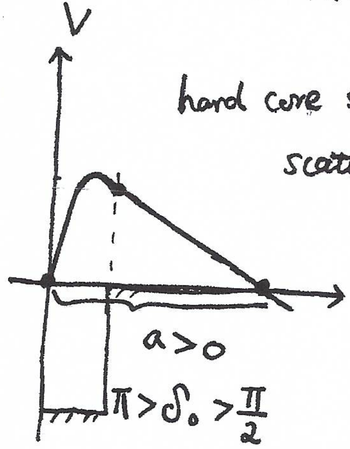
resonance scattering

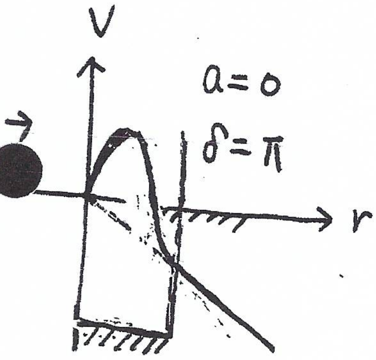


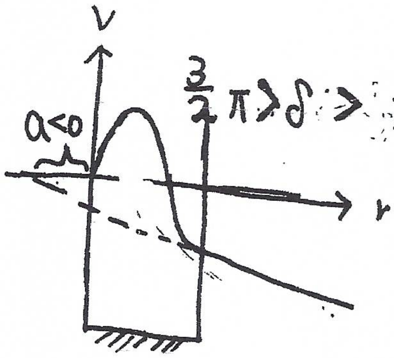
$a = \mp\infty$, $\delta_0 = \frac{\pi}{2} \mp 0^+$

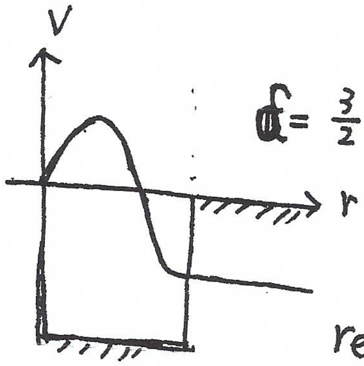
$\sigma = 4\pi a^2$

hard core sphere scattering



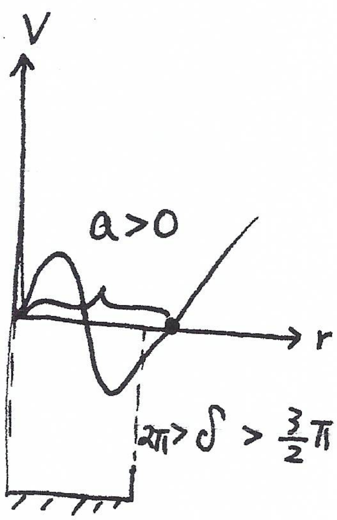


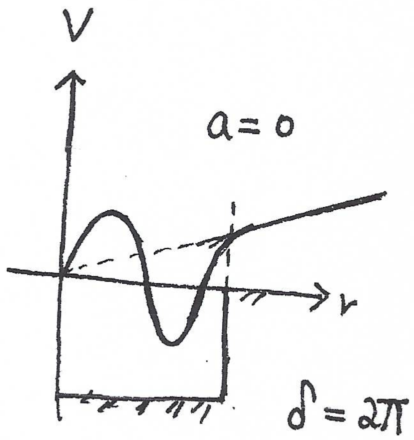




$a = \mp\infty$

resonance scattering



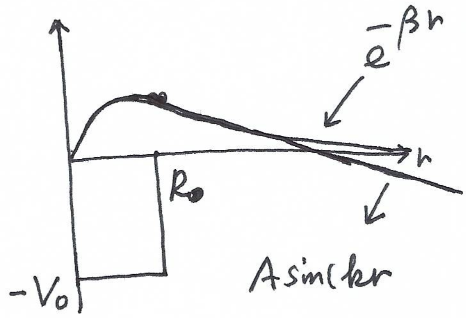


§ Bound state

Bound state appears at $\delta_0 \to \frac{\pi}{2} + 0^+$, when a new node appear in the WF. This means there's a bound state appears just below the zero energy. For such a bound state with $E \to 0^-$, at

$$\begin{cases} r>R. & u_b(r) \sim e^{-\beta r} = 1-\beta r \\ r<R & u_b(r) = \sin k_0 r \end{cases}$$

$$\frac{\hbar^2 k_0^2}{2m} = V_0 - \frac{\hbar^2\beta^2}{2m}$$



Nevertheless for a scattering wave state with k outside the well.

$$r>R, \quad u_s \sim A\sin(kr+\delta_0) \longrightarrow (1 + k\cot\delta_0\, r)$$

$$r<R \quad u_s(r) = \sin k_0' r$$

where $$\frac{\hbar^2 k_0'^2}{2m} = V_0 + \frac{\hbar^2 k^2}{2m} \simeq \frac{\hbar^2(k_0^2+\beta^2+k^2)}{2m}$$

here $$k_0'^2 = k_0^2 + \beta^2 + k^2 \simeq k_0^2.$$

math boundary condition

$$\left.\frac{u_s'}{u_s}\right|_{r=R^-} = \left.\frac{u_b'}{u_b}\right|_{r=R^-} = \boxed{\left.\frac{u_s'}{u_s}\right|_{r=R^+} = \left.\frac{u_b'}{u_b}\right|_{r=R^+}}$$

$$\swarrow$$

$$\frac{\cos k_0 R}{\sin k_0 R}\cdot k_0$$

$$-\beta = k\cot\delta_0 = -\frac{1}{a_0}$$

$$\Rightarrow \boxed{a_0 = \frac{1}{\beta}}$$

$$f_0 = \frac{1}{k\cot k - ik} = \frac{1}{\frac{-1}{a_0} - ik} = \frac{-1}{\beta + ik} = \frac{-1}{\beta + i\sqrt{\frac{2mE}{\hbar^2}}}$$

hence $E_b = -\frac{\hbar^2\beta^2}{2m}$ can be obtained as the pole of $f_0(E)$

in the 1st Riemann sheet.

~~bound state~~ $\rightarrow$ pole of $f_0(E)$, or $f_0(k) \xrightarrow[k=i\beta]{} \infty$.

even without an incoming wave, there's still something!

§ Pseudo-potential

In most case, we do not know the micro-scopic potential.

Experimentally, what we can measure σ — cross section.

$$\sigma = \frac{4\pi}{k^2}\sin^2\delta_0 = 4\pi a^2\cos^2\delta_0 \xrightarrow[k\to 0]{} 4\pi a^2$$

off-resonance, $\delta_0 \to 0$.

For a physical theory, it should be built up on observable quantity "a". The detailed microscopic potential is unimportance. As long as it reproduces the correct a, it should be ok. We can use a "δ"-like potential to reproduce the long-rang behavior of WF.

3D case

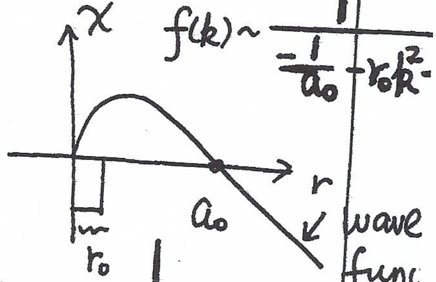
§1.*) The physical meaning of $\delta^{(3)}_{reg}(\vec{r}) = \delta^{(3)}(\vec{r})\frac{\partial}{\partial r}r$

Rember that in the 3D zero energy scattering theory, we introduced

$\chi(r) = rR(r)$, and the boundary condition $\chi(r) \to r^{l+1}$.

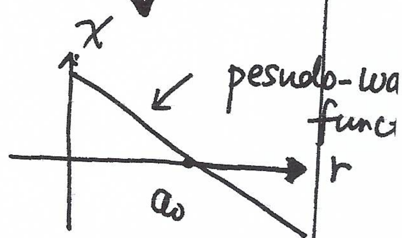
In other words, $\chi(r)=0$, at $r=0$. If so, we cannot use $\delta^{(3)}(\vec{r})$ becau[se] it has no effect on $\chi(r)$ at all since $\delta^3(\vec{r})$ is only nonzero at $r=0$. We have to use finite range potential such as attractive potential well with a width $\sim r_0$. For example, in the s-wave channel,

$f(k) \sim \frac{1}{-\frac{1}{a_0} - r_0 k^2 \ldots}$



$$\chi(r) = 1 - \frac{r}{a_0} \quad \text{for } (r \gg r_0), \text{ with } E=0.$$

If we want to neglect the behavior as $r \to r_0$, and only keep $\chi_{ps}(r) = 1 - \frac{r}{a_0}$, or $R_{ps}(r) = \frac{1}{r} - \frac{1}{a_0}$,



we need to regularize $\delta^{(3)}(\vec{r})$ such that,

we do can use zero-range potential. The advantage is that, only "a_0" is needed, "r_0"- interaction range is not needed.

Let's plug in $R_{ps}(r)$ into the Schröding Eq.

$$-\frac{\hbar^2}{2m}\nabla^2\left(R_{ps}(r)\right) = \frac{\hbar^2}{2m}\cdot 4\pi\delta^{(3)}(\vec{r})$$

$$\delta^{(3)}(\vec{r})\,R_{ps}(r) = \underset{\uparrow}{\frac{1}{r}\delta^{(3)}(\vec{r})} - \frac{1}{a_0}\delta^{(3)}(\vec{r})$$

this term causes trouble, and we want to elimini[ate] it.

uch that
ve use

$$\underbrace{\delta^{(3)}(\vec{r})\frac{\partial}{\partial r}}_{\text{pseudo-potential}}\left(rR_{ps}(r)\right) = -\frac{1}{a_0}\delta^{(3)}(\vec{r})$$

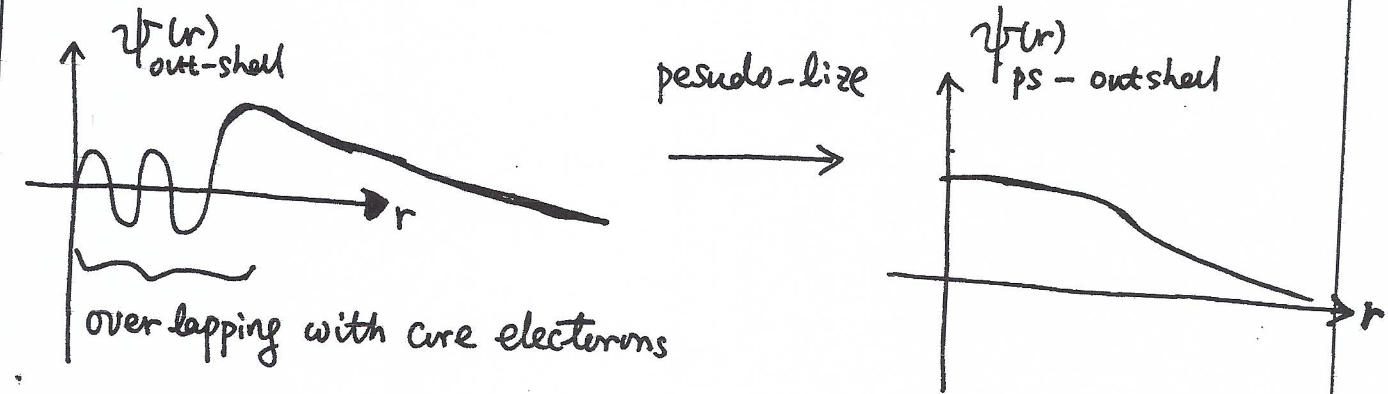
$\Rightarrow$ ~~we can define~~ $H = -\frac{\hbar^2}{2m}\nabla^2 + \frac{2\pi\hbar^2 a_s \delta^{(3)}_{reg}(\vec{r})}{m}$,

and $\delta^{(3)}_{reg}(\vec{r}) = \delta(\vec{r})\frac{\partial}{\partial r}(r$ ← this is essentially a projection, only $R(r) \sim r^{l=0}$ enter interact

⊗ This is the simplest version of the general method of pesudo-poten

we are not interested/are not able to figure out the short-range physics at r_0-scale, thus we simplify the potential, and also smooth the wavefunction. The long range behavior and the eigenvalue are not changed.

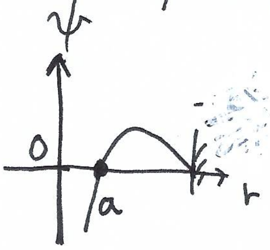
⊗ in solid state band structure calculation, we use a similar trick. The real wavefunctions for outer shell electrons are also complicated as $r \to 0$.



The potential is also modified by taking into account the orthogona requirement to inner-shell electrons. This is a process of renormalization.

§ Illustrating example

We replace the hard core sphere with radius "a", by the scattering length a, or, reversely a scattering length as a hard sphere. The hard sphere potential is replaced by the boundary condition $\psi(r)\Big|_{r=a}=0$.



$$\boxed{-\frac{\hbar^2}{2m}\nabla^2\psi=\frac{\hbar^2k^2}{2m}\psi \quad \text{with } \psi=0 \text{ at } r=a \text{ and } r=R>>a.}$$

boundary problem

$$\Rightarrow\ (\nabla^2+k_n^2)\psi=0$$

$$\Rightarrow\ \frac{d^2}{dr^2}(r\psi)=-k_n^2(r\psi) \quad \Rightarrow \quad \left.\begin{aligned} &r\psi=\sin(k_n(r-a)) \\ &k_n(R-a)=n\pi,\ n=1,2,3,\cdots \end{aligned}\right\}$$

After normalization

$$\psi_n=\frac{1}{\sqrt{2\pi(R-a)}}\,\frac{\sin k_n(r-a)}{r}, \qquad E_n=\frac{\hbar^2}{2m}\,\frac{n^2\pi^2}{(R-a)^2}.$$

If a is small, we expand in terms of perturbation theory, but

$$\psi_n=\frac{1}{\sqrt{2\pi R}}\left(1-\frac{a}{R}\right)^{-1/2}\frac{1}{r}\sin k_n\left(1-\frac{a}{R}\right)^{-1}(r-a)$$

$$=\frac{1}{r\sqrt{2\pi R}}\sin\left(k_n r-ak_n\left(1-\frac{r}{R}\right)\right)\left(1+\frac{a}{2R}\right)=\frac{1}{r\sqrt{2\pi R}}\left(1+\frac{a}{2R}\right)\left[\sin k_n r-\cos k_n r\; a k_n\left(1-\frac{r}{R}\right)\right]$$

$$=\underbrace{\frac{1}{\sqrt{2\pi R}}\,\frac{\sin k_n r}{r}}_{\psi_0}\ \underbrace{-\frac{k_n a}{\sqrt{2\pi R}\,r}\left[\left(1-\frac{r}{R}\right)\cos k_n r-\frac{\sin k_n r}{2k_nR}\right]}_{\psi_1}$$

The energy: $\epsilon_n = \frac{\hbar^2 k_n^2}{2m} = \frac{\hbar^2 \pi^2 n^2}{2m(R-a)^2} = \frac{\hbar^2 \pi^2 n^2}{2m\, R^2}\left(1-\frac{a}{R}\right)^{-2}$ ⑨

$$= \frac{\hbar^2 x_n^2}{2m}\left(1 + \frac{2a}{R} + 3\left(\frac{a}{R}\right)^2 + \dots\right)$$

Can we perturbatively arrive at the above result?

☆ We first use the regularized δ- potential to replace the boundary condition

$$\Big(-\frac{\hbar^2}{2m}\nabla^2 + \underbrace{\frac{2\pi\hbar^2}{m} a\, \delta^{(3)}_{reg}(\vec r)}_{\text{an effective potential to mimic the boundary condition}}\Big)\psi = \frac{\hbar^2 k^2}{2m}\psi$$

$$\psi \sim \frac{\cos kr}{r} - \frac{\sin kr}{a(kr)} = \frac{1}{rka}\left[ka\cos kr - \sin kr\right] \sim \frac{-1}{rka}\sin(k(r-a))$$

boundary condition

$$\rightarrow (\nabla^2 + k^2)\psi = 4\pi a\, \delta^{(3)}_{reg}(\vec r)\psi$$

$\nabla^2\psi = -k^2\psi$, at $r \neq 0$.

$$\nabla^2\psi\Big|_{r\to 0} = \nabla^2\frac{1}{r} = -4\pi\delta^{(3)}(\vec r)$$

$$4\pi a\, \delta^{(3)}(r)\frac{\partial}{\partial r}(r\psi) = 4\pi a\delta^{(3)}(r)\frac{\partial}{\partial r}\left(\cos kr - \frac{\sin kr}{ka}\right)\Bigg|_{r\to 0}$$

$$= -\frac{4\pi a}{a}\delta^{(3)}(\vec r)$$

Hence the boundary condition $\psi(r=a)=0$, is replaced by $V(r) = \frac{2\pi\hbar^2}{m} a\,\delta^{(3)}(\vec r)$.

We will use $V(r) = \dfrac{2\pi\hbar^2}{m} a\, \delta^{(3)}_{reg}(\vec{r})$

$$\left\{ \psi_n = \frac{1}{\sqrt{2\pi R}} \frac{\sin k_n r}{r} \right.$$

matrix elements

$$\langle n_1 | V(r) | n_2 \rangle = \frac{2\pi\hbar^2}{m} a \frac{1}{2\pi R} \int d^3\vec{r}\, \frac{\sin k_{n_1} r}{r} \frac{\sin k_{n_2} r}{r} \delta^3(r)$$

$$= \frac{\hbar^2 a}{mR} k_{n_1} k_{n_2} = \frac{\hbar^2 \pi^2}{mR^3} a\, n_1 n_2$$

$$\Rightarrow \quad \varepsilon_n^{(1)} = \langle n | V(r) | n \rangle = \frac{\hbar^2}{m} \left(\frac{n\pi}{R}\right)^2 \frac{a}{R}$$

$$\cancel{\varepsilon_n^{(2)} = \sum_{n'} \frac{|\langle n | V | n' \rangle|^2}{\frac{\hbar^2}{2m}(K_n^2 - K_{n'}^2)} = \sum_{n'} \frac{h}{\frac{\hbar^2}{2m}\left(\frac{\pi}{R}\right)^2 (n^2 - n'^2)}}$$

⊛ 1st order correction to ψ

$$\psi_n^{(1)} = \sum_{n'} \frac{V_{n'n}}{\frac{\hbar^2}{2m}(K_n^2 - K_{n'}^2)} \psi_{n'}^{(0)}$$

$$= \sum_{n' \neq n} \frac{\frac{\hbar^2\pi^2}{mR^3} a\, n n'}{\frac{\hbar^2}{2m} \frac{\pi^2}{R^2} (n^2 - n'^2)} \frac{1}{\sqrt{2\pi R}} \frac{\sin K_{n'} r}{r}$$

姬扬老师指点

$$n \sum \frac{1}{2}\left(\frac{1}{n'-n} + \frac{1}{n'+n}\right) \sin n'\theta$$

利用 $\displaystyle\sum_{n=1}^{\infty} \frac{\sin n\theta}{n} = \frac{\pi - \theta}{2}$

for $0 < \theta < 2\pi$.

留做HW

$$= -\frac{1}{r} \frac{na}{R\sqrt{2\pi R}} \boxed{\sum_{n' \neq n} \frac{n'}{n'^2 - n^2} \sin n'\theta} \leftarrow \theta = \frac{\pi r}{R}$$

$$\longrightarrow \psi_1 = -\frac{K_n a}{\sqrt{2\pi R}\, r} \left[\left(1 - \frac{r}{R}\right) \cos K_n r - \frac{\sin K_n r}{2 K_n R} \right]$$

$$\mathcal{E}_n^{(2)} = \int d^3\vec{r}\; \psi_n^{(0)}(r)\, \frac{2\pi\hbar^2}{m} a\, \delta^{(3)}(\vec{r})\, \frac{\partial}{\partial r}\left(r\, \psi_n^{(1)}(r)\right)$$

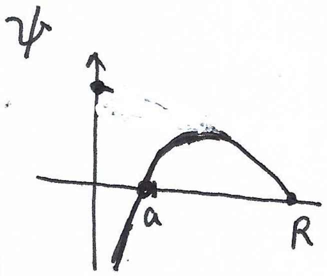
$$\frac{\partial}{\partial r}\left(r\, \psi_n^{(1)}(r)\right)\Big|_{r\to 0} = \frac{K_n a}{\sqrt{2\pi R}}\left(\frac{1}{R} + \frac{1}{2R}\right) = \frac{3K_n a}{2\sqrt{2\pi}}\, R^{-3/2}$$

$$\psi_n^{(0)}(r)\Big|_{r\to 0} = \frac{K_n}{\sqrt{2\pi R}}$$

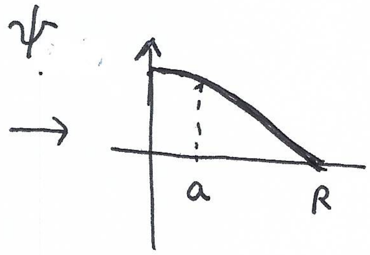
$$\Rightarrow \mathcal{E}_n^{(2)} = \frac{2\pi\hbar^2}{m} a^2\, \frac{3K_n^2}{4\pi R^2} = \frac{\hbar^2 K_n^2}{2m}\, \frac{3a^2}{R^2} = \mathcal{E}_n \cdot 3\left(\frac{a}{R}\right)^2$$

Hence in the perturbative process, the regularization means ·

$$\psi(r) \underset{r\to 0}{\rightarrow} -\frac{1}{r} + \frac{1}{a} \xrightarrow{\delta^{(3)}(r)} \frac{1}{a}$$



$$-\frac{\cos kr}{r} + \frac{\sin kr}{a(kr)} \longrightarrow \frac{\sin kr}{a(kr)}$$



§ Many-body problem

$$H = -\frac{\hbar^2}{2m}\sum_i \nabla_i^2 + \frac{4\pi\hbar^2 a}{m}\sum_{i<j} \delta^{(3)}_{reg}(\vec{r}_i - \vec{r}_j)$$

reduced mass $\mu = m/2$.

For an unperturbated system, $\psi_n^{(0)} = (a^{+}_{k_1})^{n_1}(a^{+}_{k_2})^{n_2}\cdots(a^{+}_{k_i})^{n_i}\cdots|vac\rangle$

$$\Rightarrow E_n^{(0)} = \sum n_i \frac{\hbar^2 k_i^2}{2m} \qquad \psi_k = \frac{1}{\sqrt{V}} e^{i\vec{k}\cdot\vec{r}}$$

How about $E^{(1)}$

$$H_{int} = \frac{1}{2}\left(\frac{4\pi a\hbar^2}{m}\right)\int dr_1\, dr_2\, \psi^*(r_1)\psi^*(r_2)\,\delta(r_1 - r_2)\frac{\partial}{\partial r_{12}}\left[r_{12}\,\psi(r_2)\psi(r_1)\right]$$

$$\rightarrow \frac{2\pi a\hbar^2}{m}\frac{1}{V}\sum_{\substack{k_1 k_2\\ k_3 k_4}} a^{+}_{k_1} a^{+}_{k_2} a_{k_3} a_{k_4}\,\delta(k_1 + k_2 - k_3 - k_4)$$

when $\langle n | H_{int} | n\rangle = \frac{2\pi a\hbar^2}{mV}\Big(\sum a^{+}_{k_1} a^{+}_{k_2} a_{k_2} a_{k_1} + a^{+}_{k_1} a^{+}_{k_2} a_{k_1} a_{k_2} - \sum_k a^{+}_{k} a^{+}_{k} a_k a_k\Big)$

$$= \frac{2\pi a\hbar^2}{mV}\Big[2\sum_{\substack{k_1\\ k_2}} a^{+}_{k_1} a_{k_1}(a^{+}_{k_2} a_{k_2} - \delta_{k_1 k_2}) - \Big(\sum_k (a^{+}_k a_k a^{+}_k a_k) - \sum_k a^{+}_k a_k\Big)\Big]$$

$$= \frac{2\pi a\hbar^2}{m}\frac{1}{V}\Big[2N(N-1) + N - \sum_k n_k^2\Big]$$

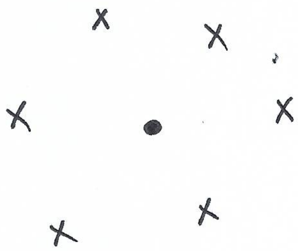
Since N is fixed, the lowest energy for partition is that one state of k_0, $n_{k_0} = N$ and all other state $n_k = 0$

$$\Rightarrow \boxed{\langle n | H_{int} | n\rangle = \frac{4\pi a\hbar^2}{m}\frac{1}{V}\frac{N(N-1)}{2}}$$

\# of pairs

Hence, at the HF level. all particles go to $k=0$

$$E_0 \xrightarrow{N\to\infty} N \cdot \frac{2\pi\hbar^2}{m} a n$$



$$\frac{E_0}{N} \to \underbrace{\frac{1}{L^2}}_{K_0} \underbrace{\left(\frac{a}{L}\right)}_{\text{perturbation}} N \leftarrow \text{fix all other particles}$$

if $|N-2, k1, -k1\rangle$, it's energy $E_1 = \frac{2\hbar^2 k^2}{2m}$

$$+ \frac{2\pi a\hbar^2}{mV}\{+N^2 - [(N-2)^2 + 1^2 + 1^2]\}$$

$$\to E_1 = 2\varepsilon_k + \frac{4\pi a\hbar^2}{m} n \cdot 2 \cdot 1$$

if $|N-2l, kl, -kl\rangle$ $\quad E_l = 2l\,\varepsilon_k + \frac{2\pi a\hbar^2}{mV}[N^2 - ([N-l\cdot 2]^2 + l^2 + l^2)]$

$$= 2l\,\varepsilon_k + \frac{4\pi a\hbar^2}{m} n \cdot 2l$$

$$\langle N-2(l-1); k(l-1); -k(l-1) | H_{int} | N-2l, kl, -kl\rangle$$

$$= \frac{2\pi a\hbar^2}{m} \cdot \frac{1}{V} (n_0 - 2l)^{1/2} (n_0 - 2(l-1))^{1/2} l^{1/2} l^{1/2} \times 2$$

$$\xrightarrow[\substack{V\to\infty \\ N\to\infty}]{} \frac{4\pi a\hbar^2}{m} n \cdot l$$

$$a_0^\dagger a_0^\dagger a_{-k} a_k$$
$$+ a_0^\dagger a_0^\dagger a_k a_{-k}$$

Hence: $\begin{array}{l} \quad\ 0,\ k,\ -k \\ |N,\ 0\ \ 0\rangle \\ \left.\begin{array}{l} |N-2,\ 1\ 1\rangle \\ \quad \vdots \\ |N-l,\ l\ l\rangle \end{array}\right\} \end{array}$ basis, we have the Hamitonian

$$\begin{bmatrix} 0 & h & & & \\ h & \Delta & 2h & & \\ & 2h & 2\Delta & 3h & \\ & & 3h & 3\Delta & \ddots \\ & & & \ddots & \ddots \end{bmatrix}$$

where $\Delta = 2\left(\varepsilon_K + \frac{4\pi a\hbar^2}{m} n\right)$

$h = \frac{4\pi a\hbar^2}{m} n.$

Effect: change the gapped excitation to gap less

① HF-level. Bosonic exchange energy $\varepsilon_K + \frac{4\pi a\hbar^2}{mV}\left[\frac{(N-1)(N-2)}{2} + 2(N-1) - \frac{N(N-1)}{2}\right]$

$$= \varepsilon_K + \frac{4\pi a\hbar^2}{mV}(N-1) = \varepsilon_K + \frac{4\pi a\hbar^2}{m} n$$

② Off-diagonal transition ---- reduces the gap

$$\Delta\left(a^+a + \tfrac{1}{2}\right) + \frac{h}{2}\left(aa + a^+a^+\right) \quad \longleftarrow \quad \text{Bose gas 写复杂一些.}$$

$$\boxed{\begin{array}{l} \Delta a_K^+ a_K + a_{-K}^+ a_{-K} \\ + h(a_K a_{-K} + a_K^+ a_{-K}^+) \end{array}}$$

$$\doteq \frac{\Delta}{2}\left[\left(\frac{X}{l}\right)^2 + \left(\frac{Pl}{\hbar}\right)^2\right] + \frac{h}{2}\left(\left(\frac{X}{l}\right)^2 - \left(\frac{Pl}{\hbar}\right)^2\right)$$

$$= \left(\frac{\Delta}{2} + \frac{h}{2}\right)\left(\frac{X}{l}\right)^2 + \left(\frac{\Delta}{2} - \frac{h}{2}\right)\left(\frac{Pl}{\hbar}\right)^2 \Rightarrow E = \sqrt{(\Delta^2 - h^2)}$$

rescale $X \to KX$

$P \to \frac{1}{K} X$

$$\frac{p^2}{2m} + \frac{1}{2} m\omega^2 X^2 \to \left(\frac{1}{m} \cdot m\omega^2\right)^{1/2} \to \omega$$

Since as $k \to 0$, $\Delta \to h$, $E(k) = \left[\left(\varepsilon_K + \frac{4\pi\hbar^2}{m} an\right)\varepsilon_K\right]^{1/2}$

$\longrightarrow \hbar^2 \ldots k$

§ Many-body theory of hard core bose gas

$$H = \sum_{i=1}^{N} -\frac{\hbar^2}{2m}\nabla_i^2 + \frac{4\pi\hbar^2 a}{m}\sum_{i<j}\delta^{(3)}(r_i - \vec{r}_j)$$

reduced mass $\mu = m/2$

2nd quantization

$$H_0 = \sum_k (\varepsilon_k - \mu)\, a_k^\dagger a_k$$

$$H_{int} = \frac{1}{2}\int d^3r_1\, d^3r_2\, \psi^*(r_1)\,\psi^*(r_2)\left(\frac{4\pi\hbar^2 a}{m}\delta^{(3)}(r_1 - r_2)\right)\psi(r_2)\,\psi(r_1)$$

$$= \frac{2\pi\hbar^2 a}{m}\int d^3r\, \psi^*(r_1)\psi^*(r_2)\,\psi(r_2)\,\psi(r_1)$$

$$= \frac{2\pi\hbar^2 a}{mV}\sum_{k_1 k_2 q} a_{k_1+q}^\dagger\, a_{k_2-q}^\dagger\, a_{k_2}\, a_{k_1}$$

Assume $n_0 = a_0^\dagger a_0 \sim N$,

$$\boxed{\begin{array}{ll} a\cdot n^{1/3} << 1 & n: \text{particle density} \\ ka << 1 & \end{array}}$$

$$\sum_{0000} \rightarrow a_0^\dagger a_0^\dagger a_0 a_0 = (a_0^\dagger a_0)^2 - a_0^\dagger a_0 = n_0^2 - n_0$$

$$\sum_{\substack{k_1 k_2 0 0\\ \neq 0}} a_0^\dagger a_0^\dagger a_k a_{-k} + a_k^\dagger a_{-k}^\dagger a_0 a_0 + 4a_0^\dagger a_0 (a_k^\dagger a_k) = 4n_0 \sum_k a_k^\dagger a_k + n_0 \sum_k (a_k a_{-k} + a_k^\dagger a_{-k}^\dagger)$$

but $n_0^2 - n_0 = (N - \sum_k a_k^\dagger a_k)^2 - (N - \sum_k a_k^\dagger a_k)$

$$= N^2 - N - 2N\sum_k a_k^\dagger a_k + \sum_k a_k^\dagger a_k$$

$$\sim 4N\sum_k a_k^\dagger a_k + N\sum_k (a_k^\dagger a_{-k}^\dagger + a_k a_{-k})$$

$$H_{int} = \frac{4\pi\hbar^2 a}{m}\frac{N(N-1)}{2V} + \frac{2\pi\hbar^2 a}{m}\frac{N}{V}\sum_{k\neq 0}(2a_k^\dagger a_k + a_k^\dagger a_{-k}^\dagger + a_{-k}a_k)$$

$$H = \frac{2\pi\hbar^2 a}{m} n\cdot N + \sum_{k\neq 0} a_k^\dagger a_k\left(\varepsilon_k + \frac{4\pi\hbar^2 a}{m}n\right) + \left(a_k^\dagger a_{-k}^\dagger + a_{-k}a_k\right)\frac{2\pi\hbar^2 a}{m}n$$

$$H - \frac{2\pi\hbar^2 a}{m} n N = \sum_k{}' (a_k^\dagger \; a_{-k})\begin{pmatrix} A & B \\ B & A\end{pmatrix}\begin{pmatrix} a_k \\ a_{-k}^\dagger\end{pmatrix}$$

$$- \sum_k{}' \left(\varepsilon_k + \frac{4\pi\hbar^2 a}{m}n\right) \quad \leftarrow \sum_k{}' \text{ mean's sum over half space}$$

$$\begin{pmatrix} A & B \\ B & A\end{pmatrix} = \begin{pmatrix} \varepsilon_k + \frac{4\pi\hbar^2 a}{m}n & \frac{4\pi\hbar^2 a}{m}n \\ \frac{4\pi\hbar^2 a}{m}n & \varepsilon_{-k} + \frac{4\pi\hbar^2 a}{m}n\end{pmatrix}$$

$$\begin{pmatrix} a_k \\ a_{-k}^\dagger\end{pmatrix} = \begin{pmatrix} ch\theta & -sh\theta \\ -sh\theta & ch\theta\end{pmatrix}\begin{pmatrix} \alpha_k \\ \alpha_{-k}^\dagger\end{pmatrix}$$

$$\begin{pmatrix} ch\theta & -sh\theta \\ -sh\theta & ch\theta\end{pmatrix}\left(A + B\sigma_1\right)\begin{pmatrix} ch\theta & -sh\theta \\ -sh\theta & ch\theta\end{pmatrix} = (ch\theta - sh\theta\sigma_1)(A + B\sigma_1)(ch\theta - sh\theta\sigma_1)$$

$$= A(ch2\theta - sh2\theta\sigma_1) + B(-sh\theta + ch\theta\sigma_1)(ch\theta - sh\theta\sigma_1)$$

$$= (A\,ch2\theta - B\,sh2\theta) + (-A\,sh2\theta + B\,ch2\theta)\sigma_1$$

$$\Rightarrow \frac{A}{B} = \frac{ch2\theta}{sh2\theta} \Rightarrow \qquad ch2\theta = \frac{A}{\sqrt{A^2-B^2}}$$

$$sh2\theta = \frac{B}{\sqrt{A^2-B^2}}$$

$$\begin{cases} ch^2\theta = \frac{1}{2}\left(1 + \frac{A}{\sqrt{A^2-B^2}}\right) \\ sh^2\theta = \frac{1}{2}\left(1 - \frac{A}{\sqrt{A^2-B^2}}\right)\end{cases}$$

$$H - \frac{2\pi\hbar^2 a}{m} n \cdot N = \sum_k{}' E_k (\alpha_k^\dagger \alpha_k + \alpha_{-k} \alpha_{-k}^\dagger) - \sum_k{}' \left(\varepsilon_{-k} + \frac{4\pi\hbar^2 a}{m} n\right)$$

$$= \sum_k E_k \alpha_k^\dagger \alpha_k + \sum_k{}' E_k - \left(\varepsilon_k + \frac{4\pi\hbar^2 a}{m} n\right)$$

where $E_k = \left[\left(\varepsilon_k + \frac{4\pi\hbar^2 a}{m} n\right)^2 - \left(\frac{4\pi\hbar^2 a}{m} n\right)^2\right]^{1/2}$

$$= \left[\varepsilon_k \left(\varepsilon_k + \frac{8\pi\hbar^2 a}{m} n\right)\right]^{1/2}$$

Neverther less, the correction diverges .

$$\frac{NV}{N} \frac{1}{2V} \sum_k \left[\frac{\hbar^2 k^2}{2m} \left(\frac{\hbar^2 k^2}{2m} + \frac{8\pi\hbar^2 a}{m} n\right)\right]^{1/2} - \left(\frac{\hbar^2 k^2}{2m} + \frac{4\pi\hbar^2 a}{m} n\right)$$

$$= N \frac{1}{2n} \int \frac{d^3 k}{(2\pi)^3} \left\{ \left(\frac{\hbar^2 k^2}{2m} \left(\frac{\hbar^2 k^2}{2m} + \frac{8\pi\hbar^2 a}{m} n\right)\right)^{1/2} - \left(\frac{\hbar^2 k^2}{2m} + \frac{4\pi\hbar^2 a}{m} n\right)\right\}$$

As $k \to +\infty$ $\quad \frac{\hbar^2 k^2}{2m} \left(1 + \frac{16\pi a n}{k^2}\right)^{1/2} - \left(\frac{\hbar^2 k^2}{2m} + \frac{4\pi\hbar^2 a}{m} n\right)$

$$= \frac{\hbar^2 k^2}{2m} \left(-\frac{1}{8}\right) \left(\frac{16\pi a n}{k^2}\right)^2 = -\frac{16\pi^2 \hbar^2 (an)^2}{m k^2}$$

$$\int \frac{d^3 k}{(2\pi)^3} \frac{1}{k^2} \sim \int dk \longrightarrow$$ ultraviolet divergence !

How to resolve this! $\longrightarrow$ Renormalization

Now consider use the regularized $\delta^{(3)}_{reg}(\vec{r})$

$$H_{int} = \frac{2\pi\hbar^2 a}{m} \int d^3r_1 d^3r_2 \, \psi^*(\vec{r}_1)\psi^*(\vec{r}_2)\, \delta^{(3)}_{reg}(\vec{r}_1-\vec{r}_2)\, \psi(\vec{r}_2)\psi(\vec{r}_1)$$

$\delta^{(3)}$ means remove the relative wavefunction $\phi(\vec{r}_2-\vec{r}_1)$ singular term $\propto \frac{1}{|\vec{r}_2-\vec{r}_1|}$, which turns out yield $\frac{1}{k^2}$ contributition's in the mumentum space, where k is the relative mumentum of two particles.

$$H_{int} = \frac{2\pi\hbar^2 a}{m} \frac{1}{V} \frac{\partial}{\partial r}\Big(r \sum_{\substack{k_1 k_2 \\ k_3 k_4}} e^{\frac{i}{2}(\vec{k}_3-\vec{k}_4)\cdot\vec{r}}\, a^+_{k_1} a^+_{k_2} a_{k_3} a_{k_4}\Big)$$

$\rightarrow$

$$H - \frac{2\pi\hbar^2 a}{m} n N = \sum_k \Big(\varepsilon_k + \frac{4\pi\hbar^2 a n}{m}\Big) a^+_k a_k + \frac{\partial}{\partial r}\Big(\underbrace{\sum_k{}'}_{r e^{i\vec{k}\cdot\vec{r}}} (a^+_k a^+_{-k} + a_{-k} a_k)\frac{4\pi\hbar^2 a}{m} n\Big)\Big|_{r\to 0}$$

$$\rightarrow E - \frac{2\pi\hbar^2 a}{m} n N = \frac{N}{2n}\int \frac{d^3k}{(2\pi)^3} \frac{\partial}{\partial r}\Big[r e^{i\vec{k}\cdot\vec{r}} \Big(\frac{\hbar^2k^2}{2m}\Big(\frac{\hbar^2k^2}{2m} + \frac{8\pi\hbar^2 a}{m} n\Big)\Big)^{1/2} - \Big(\frac{\hbar^2k^2}{2m} + \frac{4\pi\hbar^2 a}{m} n\Big)\Big]$$

$$\Big[\frac{\hbar^2k^2}{2m}\Big(\frac{\hbar^2k^2}{2m} + \frac{8\pi\hbar^2 a}{m} n\Big)\Big]^{1/2} = \frac{\hbar^2k^2}{2m}\Big[\Big(1 + \frac{16\pi a n}{k^2}\Big)^{1/2} - \Big(1 + \frac{8\pi a n}{k^2}\Big)\Big]$$

at $k \to \infty$

$$\left(1+\frac{16\pi an}{k^2}\right)^{1/2} - \left(1+\frac{8\pi an}{k^2}\right) = 1+\frac{1}{2}\cdot\frac{16\pi an}{k^2} \mp \frac{1}{8}\left(\frac{16\pi an}{k^2}\right)^2 - (1+\cdots$$

$$= -\frac{32(\pi an)^2}{k^4}$$

Combain $\frac{\hbar^2 k^2}{2m}\cdot\frac{32(\pi an)^2}{k^4} = \frac{16\,\hbar^2(\pi an)^2}{mk^2} \xrightarrow{e^{i\vec{k}\cdot\vec{r}}} \frac{1}{r}$.

Hence.

$$\frac{E}{N} - \frac{2\pi\hbar^2 an}{m} = \frac{1}{2n}\int\frac{d^3k}{(2\pi)^3}\frac{\hbar^2k^2}{2m}\left[\left(1+\frac{16\pi na}{k^2}\right)^{1/2} - \left(1+\frac{8\pi na}{k^2}\right) + \frac{32(\pi an)^2}{k^4}\right]$$

define $k_0^2 = 8\pi na$

$$\to \frac{k_0^5}{2n}\frac{\hbar^2}{2m}\frac{4\pi}{(2\pi)^3}\int\left(\frac{k}{k_0}\right)^2 d\left(\frac{k}{k_0}\right)\left[\left(1+2(k_0/k)^2\right)^{1/2} - \left(1+\left(\frac{k_0}{k}\right)^2\right] + \frac{k_0^4}{2k^4}\right]$$

$$= \frac{\hbar^2}{2m}\frac{k_0^5}{n}\frac{1}{(2\pi)^2}\int_0^\infty dy\cdot y^2\left[-(1+y^{-2}) + \frac{1}{2y^4}\right) + (1+2y^{-2})^{1/2}\right]$$

$$= \frac{\hbar^2}{2m}\frac{k_0^5}{4\pi^2}\frac{1}{n}\underbrace{\int_0^\infty dy\left[-(y^2+1) + \frac{1}{2y^2} + y(y+2)^{1/2}\right]}_{A = \frac{128}{15}}$$

$$\frac{k_0^5}{4\pi^2 n} = \frac{8\pi na}{4\pi^2 n}(8\pi na)^{\frac{3}{2}} = \frac{2}{\pi}a(8\pi na)^{3/2}$$

$$\Rightarrow \boxed{\frac{E}{N} - \frac{2\pi\hbar^2 an}{m} = +\frac{2\pi\hbar^2 an}{m}\left(\frac{n^{1/2}a^{3/2}}{\pi}\right)A = \frac{2\pi\hbar^2 an}{m}\left[1+\left(\frac{a^3n}{\pi}\right)^{1/2}A\right]}$$

§ Understanding $\sqrt{an^{1/3}}$'s dependence

Consider 2nd order perturbation theory

$$|\psi_g^{(1)}\rangle = \sum_k \cdot \frac{2\pi a\hbar^2}{mV} \frac{a_k^\dagger a_{-k}^\dagger a_0 a_0 |\psi^0\rangle}{-2\varepsilon_k}$$

$$= -\frac{\pi\hbar^2}{m}\frac{a}{V}\overbrace{\sum_k}^{\sqrt{N(N-1)}} \frac{a_k^\dagger a_{-k}^\dagger}{\varepsilon_k}|\psi^0\rangle \propto \frac{\pi a}{V} N(N-1) \sum_k \frac{1}{k^2} a_k^\dagger a_{-k}^\dagger |\psi^0\rangle$$

$$E^{(2)} = \langle \psi_g^{(1)} | H_{int} | \psi_g^{(0)} \rangle$$

$$\propto -\frac{a\hbar^2}{mV} \cdot \frac{a}{V} N(N-1) \ldots \sum_k \frac{\langle \psi^0 | a_k a_k | \psi_k' \rangle \langle \psi_k' | a_k^\dagger a_{-k}^\dagger | \psi^0 \rangle}{k^2}$$

$$\rightarrow -\frac{a^2 N^2}{L^3} \boxed{\frac{1}{V}\sum_k \frac{\partial}{\partial r}\left[e^{i\vec{k}\cdot\vec{r}} \cdot \frac{1}{k^2}\right]} \rightarrow \int d^3k \, e^{i\vec{k}\cdot\vec{r}} \frac{1}{k^2}$$

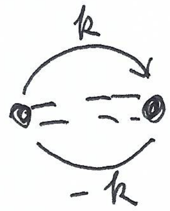
$$\lim_{r\to 0} \qquad = \frac{1}{r} - \frac{1}{L}$$

$$\underset{A_2}{=} \frac{a^2N^2}{L^4} \propto \frac{aN}{L^3}\left(\frac{aN}{L}\right) = Nan\left(\frac{a}{L}\right)$$

~~we may further consider to excite another pair, then it yield~~

$$\cancel{\frac{a^2N^2}{L^4} \cdot \left(\frac{aN}{L}\right)}$$

This process can be depicted as



$$\rightarrow \frac{a^2N^2}{V^2}\sum_k \frac{-1}{k^2} = \frac{a^2N^2}{V}\boxed{\frac{1}{V}\sum \frac{-1}{k^2}} = \frac{a^2N^2}{L^4}$$

For an extra order. $H'_{int} = \frac{2\pi a\hbar^2}{mV} a^+_{k_1} a^+_{k_2} a^+_{k_3} a^+_{k_4}$

$$\longrightarrow \frac{\hbar^2 a}{mV} N \sum \frac{1}{\frac{\hbar^2 k^2}{2m}} \sim \frac{aN}{V} \sum \frac{1}{k^2} \sim \frac{Na}{L}$$

Hence we have

$$E \sim \frac{1}{L^2}\left[A_2\left(\frac{aN}{L}\right)^2 + A_3\left(\frac{aN}{L}\right)^3 + \cdots\right]$$

$$\Rightarrow \frac{E}{N} = \frac{1}{NL^2} f\left(\frac{aN}{L}\right) = \frac{1}{NL^2}\left(\frac{aN}{L}\right)^x$$

$$= a^x N^{x-1} / L^{x+2}$$

in order to arrive at an intensive quantity $3(x-1) = x+2$

$$\Rightarrow 2x = 5$$

$$\Rightarrow \frac{E}{N} = a^{\frac{5}{2}} N^{\frac{3}{2}} / V^{\frac{3}{2}} = an(a^3 n)^{1/2}$$

§ Phonon — low energy excitation

analogy : harmonic oscillator ground state $\psi_0 \sim e^{-\alpha x^2}$ nodeless.

first excitation $\psi_1(x) \sim x e^{-\alpha x^2}$ — creating node

For a ^4He-type condensate, $|\psi_k\rangle = \frac{1}{\sqrt{N}} \rho_{\vec{k}} |\Phi_0\rangle$
$\underbrace{\text{type}}_{\text{bose}}$

{ assume — $H \simeq \frac{P_K P_{-K}}{2m} + \frac{1}{2} m \omega_K^2 \rho_K \rho_{-K}$

$$\rho(r) = \sum_{i=1}^{N} \delta(r - r_i) \longrightarrow \psi^\dagger(r)\psi(r) = \frac{1}{V} \sum_k e^{i\vec{k}\cdot\vec{r}} \sum_l a_l^\dagger a_{l+k}$$

$$\rho(r) = \frac{1}{V} \sum_{\vec{k}} \rho(k) e^{i\vec{k}\cdot\vec{r}} \Rightarrow \rho(k) = \sum_l a_l^\dagger a_{l+k}$$

$$\rho(k) = \int e^{-i\vec{k}\cdot\vec{r}} \rho(r) dr = \sum_{i=1}^{N} e^{-i\vec{k}\cdot\vec{r}_i}$$

Let us calculate the variational energy

$$E = \frac{\langle \psi_k | H | \psi_k \rangle}{\langle \psi_k | \psi_k \rangle} \rightarrow \Delta E = E - E_0 = \frac{\langle \psi_k | H - E_0 | \psi_k \rangle}{\langle \psi_k | \psi_k \rangle}$$

numerator $f(k) = \frac{1}{N} \langle \psi_0 | \rho_k^\dagger (H - E_0) \rho_k | \psi_0 \rangle = \frac{1}{N} \langle \psi_0 | \rho_k^\dagger [H, \rho_k] | \psi_0 \rangle$

$$f(k) = \frac{1}{N} \langle \psi_0 | [H, \rho_k]^\dagger, \rho_k | \psi_0 \rangle = \frac{1}{N} \langle \psi_0 | [\rho_k^\dagger, H], \rho_k | \psi_0 \rangle$$

$$= \frac{-1}{N} \langle \psi_0 | [H, \rho_{-k}^\dagger], \rho_k | \psi_0 \rangle \Rightarrow f(-k) = -\frac{1}{N} \langle \psi_0 | [H, \rho_k], \rho_k^\dagger | \psi_0 \rangle$$

$f(k) = f(-k)$

$$f(k) = \frac{1}{2N} \langle \psi_0 | [\rho_k^+, [H, \rho_k]] | \psi_0 \rangle$$

$$H = \underbrace{\sum_k \varepsilon_k a_k^+ a_k}_{H_0} + \underbrace{v(k) \sum \rho_k^+ \rho_k}_{H_{int}} \qquad [H_{int}, \rho_k] = 0$$

$$[H_0, \rho_k] = \sum \varepsilon_{k'} [a_{k'}^+ a_{k'}\ a_q^+ a_{q+k}] = \sum_q (\varepsilon_q - \varepsilon_{q+k}) a_q^+ a_{q+k}$$

$$[\rho_k^+, [H_0, \rho_k]] = \left[\sum_{q'} a_{q'+k}^+ a_{q'}, \sum_q (\varepsilon_q - \varepsilon_{q+k}) a_q^+ a_{q+k} \right]$$

$$= \sum_q (\varepsilon_q - \varepsilon_{q+k}) (a_{q'+k}^+ a_{q+k} - a_q^+ a_q)$$

$$= \sum_q (\varepsilon_{q+k} - \varepsilon_q)(a_q^+ a_q - a_{k+q}^+ a_{k+q})$$

$$= \sum_q (\underbrace{\varepsilon_{q+k} - \varepsilon_q - \varepsilon_q + \varepsilon_{q-k}}_{\frac{\hbar^2 k^2}{m}}) a_q^+ a_q = \frac{\hbar^2 k^2}{m} N$$

$$\Rightarrow \quad f(k) = \frac{\hbar^2 k^2}{2m} = \varepsilon(k)$$

denominator $\langle \psi_k | \psi_k \rangle = \frac{1}{N} \langle \psi_0 | \rho_k^+ \rho_k | \psi_0 \rangle = S(k)$

↙

structure factor

$$S(k) = \frac{1}{N}\int dr_1 \cdots dr_N \; |\psi_0(r_1 \cdots r_N)|^2 \; \left|\sum_{i=1}^{N} e^{-i\vec{k}\vec{r}_i}\right|^2$$

$$\sum_{i=1}^{N} |e^{-i\vec{k}\cdot\vec{r}_i}|^2 = N + \sum_{i\neq j} e^{i\vec{k}\cdot(\vec{r}_i - \vec{r}_j)}$$

if $k=0 \Rightarrow S(k) = N.$

$$k \neq 0 \quad s(k) = 1 + \frac{1}{N}\cdot N(N-1)\int dr_1 dr_2 \; e^{i\vec{k}\cdot(\vec{r}_1-\vec{r}_2)} \underbrace{|\psi_0(r_1, r_2, r_3, \cdots r_N)|^2}_{\int dr_3 \cdots dr_N}$$

Define pair-distribution function

$$n \; g(r_1 - r_2) = \sum_i \sum_j \int \delta(r_1 - r_i')\,\delta(r_2 - r_j') \; |\psi_0(r_1' r_2' \cdots r_N')|^2 \, dr_1' dr_2' \cdots dr_N$$

$$= \sum_{i=j} \int \cancel{dr_1 \cdots dr_{j-1}\, dr_{j+1} \cdots dr_N} \; \cancel{\delta(r_1 - r_2)} \; |\psi_0(r_1' \cdots r_i', \cdots r_i')|^2$$

$$N \int \cancel{dr_2 \cdots dr_N} \; |\cancel{\psi}(r_1$$

$$= N \int \delta(r_1 - r_1')\,\delta(r_2 - r_1') \; |\psi_0(r_1', r_2', \cdots r_N')|^2 \, dr_1' \, dr_2' \cdots dr_N'$$

$$+ N(N-1) \int \delta(r_1 - r_1')\,\delta(r_2 - r_2') \; |\psi_0(r_1, r_2, r_3', \cdots r_N')|^2 \, dr_3' \cdots dr_N' \, dr_1' dr_2'$$

$$= N\,\delta(r_1 - r_2) \int |\psi_0(r_1, r_2', r_3', \cdots r_N')|^2 \, dr_2' \cdots dr_N'$$

$$+ N(N-1) \int |\psi_0(r_1, r_2, r_3' \cdots r_N')|^2 \, dr_3' \cdots dr_N'$$

$$= \frac{N}{V}\,\delta(r_1 - r_2) + N(N-1) \int |\psi_0(r_1 r_2, r_3' \cdots r_N')|^2 \, dr_3' \cdots dr_N'$$

$$\Rightarrow \; N(N-1) \int |\psi_0(r_1 r_2, r_3' \cdots r_N')|^2 \, dr_3' \cdots dr_N' = n\,(g(r_1 - r_2) - \delta(r_1 - r_2))$$

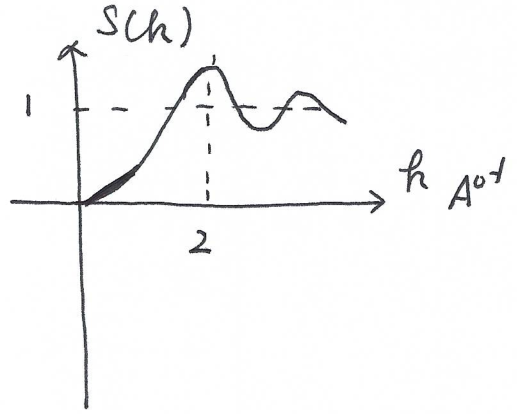
$$\Rightarrow k \neq 0, \quad S(k) = 1 + \frac{1}{N}\int dr_1 dr_2\, e^{i\vec{k}(\vec{r}_1-\vec{r}_2)}\, n\,(g(r_1-r_2) - \delta(r_1-r_2))$$

$$= 1 + n\frac{V}{N}\int dr\, e^{i\vec{k}\cdot\vec{r}}\,[g(r) - \delta(r)]$$

$$= 1 + \int dr\, e^{i\vec{k}\cdot\vec{r}}\,[g(r) - \delta(r)]$$

$$= \int dr\, e^{i\vec{k}\cdot\vec{r}}\, g(r) \qquad (k \neq 0)$$

where $g(r)$ is the pair-distribution function.



① as $k \to 0$, $S(k) \to k$.

this means that as $r \to \infty$,

$$g(r) \sim n_0 + \frac{1}{r^4}$$

$$\int dr \frac{1}{r^4} e^{i\vec{k}\cdot\vec{r}} \sim k.$$

$$\Big\} \Rightarrow \Delta E(k) \simeq \frac{\frac{\hbar^2 k^2}{2m}}{ck}$$

$$\approx \frac{\hbar^2}{2m\xi} k$$

$$\xi^{-1} = \sqrt{na}$$

② since $g(r) = \begin{cases} \delta(r) & r \to 0 \\ n_0 & r \to +\infty \end{cases}$

hence there's a background of $S(k)$ of 1.

③ The hump at $k \simeq 2\,\mathring{A}$, $\Rightarrow$ a dip of $\Delta E(k)$

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