Lect 6: Parity ~~for~~ non conservation

1. Symmetry and its importance in physics ~~and do~~

2. symmetry properties of Dirac Eq, P, T, C

3. symmetry analysis in high energy experiment

4: $\theta - \tau$ puzzle, CP ~~con~~ conservation and violation
   ^{60}Co β-decay!

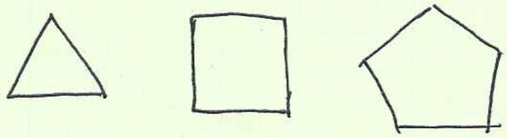
References

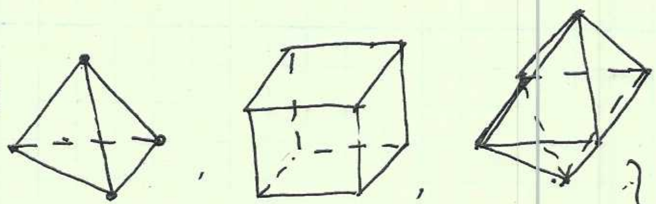
1. Lee and Yang. Physical Review 104, 254 (1956)

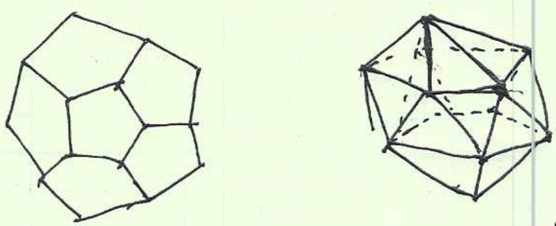
2. C. S. Wu. Nishina commemorative lecture, (1983).
   (仁科)

3. T. D. Lee, Field theory and particle physics.

§: Examples in daily life — Weyl: "Symmetry"

-  $\longrightarrow$ 

 $\xrightarrow{\text{discrete symmetry}}$  , , } Platonic solids



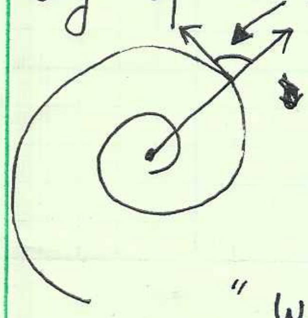
$\boxed{\text{point group}}$

dodecahedron icosahedron ↳ C_{60}

- crystal: 7 - crystalline systems
  14 - Bravis lattice
  32 - crystalline point group
  230 3 dimensional crystals — $\boxed{\text{space group}}$

discrete translational symmetry

- log - spiral fixed angle



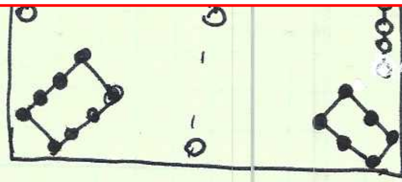
$$\rho = c\, e^{b\theta}$$

"Eadem mutata resurgo" – J. Bernou[lli]

$\boxed{\text{rotation + scaling} \longrightarrow \text{invariance}}$

"Why does a moth fly into a flame?"

⑧ Viéte theorem — $\boxed{\text{permutation group}}$



symmetric polynomials

$$x^2 + a_1 x + a_2 = 0 \quad \Rightarrow \quad \begin{cases} x_1 + x_2 = -a_1 \\ x_1 x_2 = a_2 \end{cases}$$ invariant under permutation $(1\,2)$

how about $x_1 - x_2$? $\rightarrow$ odd under $(1\,2)$

hence $(x_1 - x_2)^2$ is invaiant $\Rightarrow (x_1 - x_2)^2 = (x_1 + x_2)^2 - 4x_1 x_2 = a_1^2 - 4a_2$

$$\Rightarrow \begin{cases} x_1 + x_2 = -a_1 \\ x_1 - x_2 = \sqrt{\Delta}, \text{ with } \Delta = a_1^2 - 4a_2 \end{cases} \Rightarrow x_{1,2} = \frac{-a_1 \pm \sqrt{\Delta}}{2}$$

$x_1 \pm x_2$: reprensentations of the permutation group S_2, or Z_2

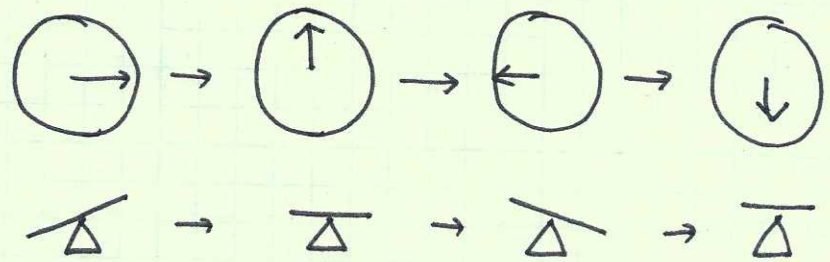
even: A_1

odd: A_2

⊛ time-involved symmetries (S. Xu, C. Wu, PRL 120, 096401 (2018))

Clock

see-saw



$\boxed{\text{space-time group}}$

§ Examples : space-time symmetry

- time translation symmetry : $\frac{dH}{dt} = -\frac{\partial L}{\partial t} = 0 \rightarrow$ energy conservation
- space translation symmetry
  $\rightarrow$ momentum conservation $\quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{\partial L}{\partial q} = 0$
  if L doesn't depend on q
- space isotropy $\longrightarrow$ angular momentum conservation

$$SO(3) \xrightarrow{c=1} \text{Lorentz group } SO(3,1) \longrightarrow \text{Poincáre group}$$

$$SO(3) \xrightarrow{c=\infty} \text{Galilean group}$$

☆ SO(n), SU(n), Sp(2n)

- Harmonic oscillators $H = \left[\left(\frac{X}{l_0}\right)^2 + \left(\frac{P \cdot l_0}{\hbar}\right)^2\right]\frac{\hbar\omega}{2}$

$$a_x = \frac{1}{\sqrt{2}}\left[\frac{x}{l_0} + i\frac{P \cdot l_0}{\hbar}\right] \qquad a_x^\dagger = \frac{1}{\sqrt{2}}\left[\frac{x}{l_0} - i\frac{P l_0}{\hbar}\right]$$

2D harmonic oscillator $H = \frac{\hbar\omega}{2}\,(a_x^\dagger\ a_y^\dagger)\begin{pmatrix} a_x \\ a_y \end{pmatrix}$

H is invariant under SU(2) transformation $\begin{pmatrix} a_x' \\ a_y' \end{pmatrix} \to u\begin{pmatrix} a_x \\ a_y \end{pmatrix}$.

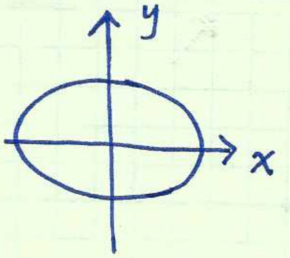
u's generator $\tau_x = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$, $\tau_y = \begin{pmatrix} & -i \\ i & \end{pmatrix}$, $\tau_z = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$

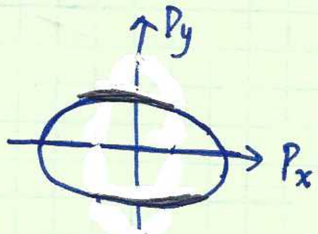
$$\Rightarrow \hbar\tau_y = -\frac{i}{\hbar}(a_x^\dagger a_y - a_y^\dagger a_x) = xP_y - yP_x \leftarrow \text{2D angular momentum}$$

$$\hbar\tau_x = \hbar[a_x^\dagger a_y + a_y^\dagger a_x] = \hbar\left[\frac{xy}{l_0^2} + \frac{P_x P_y l_0^2}{\hbar^2}\right]$$

$$\hbar\tau_z = \quad = \frac{\hbar}{2}\left[\frac{x^2 - y^2}{l_0^2} + \frac{(P_x^2 - P_y^2)l_0^2}{\hbar}\right]$$

Quadrapole





3D → SU(3) symmetry
nD → SU(N) symmetry

symmetry group widely used in high energy physics.

- Kepler problem — hydrogen problem

angular momentum conservation $\longrightarrow$ planar orbit / $2l+1$ fold degeneracy

Runge – Lentz vector $\longrightarrow$ eliptical orbit

$$\vec{L}\,, \quad \vec{A} = \frac{1}{m\gamma}\,\vec{p}\times\vec{L} - \hat{r}$$

$SO(4)$ symmetry $\rightarrow$ n^2 degeneracy

- Symplectic transformation (canonical transformation)

$$q = q(Q, \mathbb{P}) \qquad p = p(Q, \mathbb{P}) \qquad \text{define } M = \begin{pmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial p} \\ \frac{\partial \mathbb{P}}{\partial q} & \frac{\partial \mathbb{P}}{\partial p} \end{pmatrix}$$

if M satisfies $\boxed{M \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} M^T = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}}$ ← symplectic transform

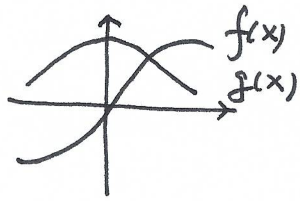
then the form of the Hamilton Eq is invariant, ie

$$\boxed{\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p} \\ \dot{p} &= -\frac{\partial H}{\partial \dot{q}} \end{aligned}} \xrightarrow{H(q,p) \to H(Q,\mathbb{P})} \boxed{\begin{aligned} \dot{Q} &= \frac{\partial H}{\partial \mathbb{P}} \\ \dot{\mathbb{P}} &= -\frac{\partial H}{\partial \dot{Q}} \end{aligned}}$$

§: Symmetry

handness.

- etymology : sym + metry



- reflection sym; Z_2 — left and right

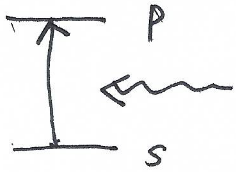
☆ $\int f(x)\, g(x)\, dx = 0$

☆ radical formula: $ax^2 + bx + c = 0$ even: $x_1 + x_2 = -\frac{b}{a}$

odd : $x_1 - x_2 \Rightarrow (x_1 - x_2)^2 = (x_1 + x_2)^2 - 4x_1 x_2 = \frac{b^2 - 4ac}{a^2}$

$\Rightarrow \quad x_1 - x_2 = \frac{1}{a}\sqrt{\Delta}$

☆ Selection rule. — diapole transition



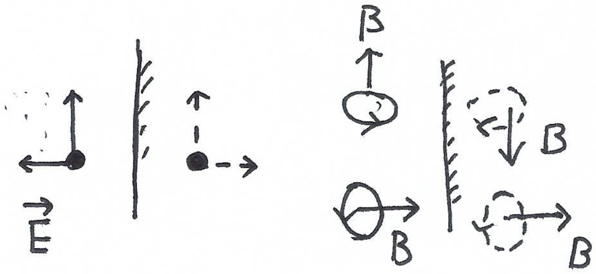
$\langle \phi_{l'} | e\vec{E}\cdot\vec{r} | \phi_l \rangle \neq 0 \quad l' = l \pm 1$

parity $\quad Y_{lm}(\theta, \varphi) \to (-)^l\, Y_{lm}(\pi - \theta, \varphi + \pi)$

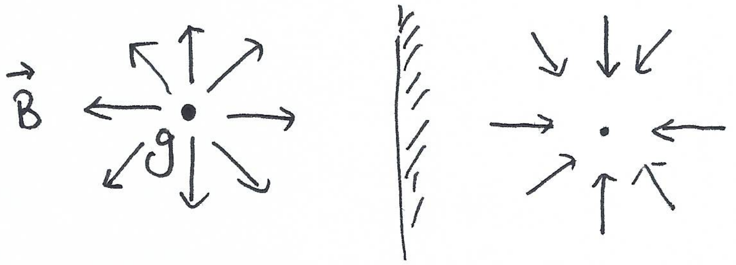
☆ polar and axial vector

$\vec{\tau} = \vec{r} \times \vec{F}$, $\quad \vec{B}$

$\vec{L} = \vec{r} \times \vec{p}$,



. pseudo-scalar (monopole)



$g' = -g$

- mixed product

$(\vec{r}_1 \times \vec{r}_2)\cdot\vec{r}_3$

invariant under rotation, Change sign under inversion or reflection

Discrete symmetry: time-reversal (TR) and parity

§1 Wigner theorem:

Generally speaking, for a transformation R (not necessarily linear), if it does not change the magnitude of the inner product between two arbitrary state vectors $|\psi\rangle$ and $|\phi\rangle$, i.e., $|\langle\psi|\phi\rangle| = |\langle R\psi|R\phi\rangle|$, then R is either a unitary transformation, or, an anti-unitary transformation. (We will omit the proof). For continuous transformation R is unitary (why?).

Anti-unitary transformation R means that: for a super-position between $|\phi_1\rangle, |\phi_2\rangle$,

$$\boxed{R\left(c_1|\phi_1\rangle + c_2|\phi_2\rangle\right) = c_1^* R|\phi_1\rangle + c_2^* R|\phi_2\rangle,} \quad \text{or } Rc = c^* R.$$

usually anti-unitary transformation can be expressed as $R = \mathcal{U} K$, where $\mathcal{U}$ is an usual unitary transformation, and K is anti-unitary satisfying $KK = 1$. In the coordinate representation, we choose K as complex conjugation.

$$\boxed{\langle \vec{r}\,|\,K|\psi\rangle = \langle \vec{r}|\psi\rangle^*}$$

Ex: please check that $R^{-1} = K\mathcal{U}^\dagger = K\mathcal{U}^{-1}$, and we can evaluate

$$\langle R\psi|R\phi\rangle = \langle \mathcal{U}K\psi|\mathcal{U}K\phi\rangle = \langle K\psi|K\phi\rangle = \int dr\, \langle K\psi|r\rangle\langle r|K\phi\rangle$$

$$= \int dr\, \langle r|K\psi\rangle^* \langle r|K\phi\rangle = \int dr\, \langle r|\psi\rangle\langle r|\Phi\rangle^* = \int dr\, \langle\Phi|r\rangle\langle r|\psi\rangle$$

$$\Rightarrow \boxed{\langle R\psi|R\phi\rangle = \langle\Phi|\psi\rangle}$$

Ex: prove that $\langle R^{-1}\psi\,|\,R^{-1}\Phi\rangle = \langle\Phi|\psi\rangle = \langle\psi|\Phi\rangle^*$.

For any states $|\psi\rangle$ and $|\psi'\rangle$, and operater O

$$\langle R\psi|O|R\psi'\rangle = \langle R\psi|R\underbrace{R^{-1}OR|\psi'\rangle} = \langle\psi|R^{-1}OR|\psi'\rangle^*$$

If $|\psi\rangle=|\psi'\rangle$, and O is an Hermitian operator $\Rightarrow$ $\langle R\psi|O|R\psi\rangle \geq 0$

$$\Rightarrow \quad \langle R\psi|O|R\psi\rangle = \langle\psi|R^{-1}OR|\psi\rangle.$$

§2. TR transformation

Consider a state vector $|\psi\rangle$, and its TR counter part $|\psi^T\rangle = T|\psi\rangle$, or, equivalently $|\psi\rangle = T^{-1}|\psi^T\rangle$, We assume T and T^{-1} satisfy Wigner theorem. Now we need to determine T is unitary or anti-unitary. We need correspondence principle.

In order to agree with classic mechanics, we need maintain

$$\begin{cases} \langle\psi^T|\vec{r}|\psi^T\rangle = \langle\psi|\vec{r}|\psi\rangle, \quad \langle\psi^T|\vec{P}|\psi^T\rangle = -\langle\psi|\vec{P}|\psi\rangle \\ \langle\psi^T|\vec{L}|\psi^T\rangle = -\langle\psi|\vec{L}|\psi\rangle \end{cases}$$

then

$$\boxed{\langle\psi^T|\vec{r}|\psi^T\rangle = \langle T\psi|\vec{r}|T\psi\rangle = \langle\psi|T^{-1}\vec{r}\,T|\psi\rangle}$$

$T^{-1}\vec{r}\,T$ is a lineary operater since the product of two anti-linear operators is a linear operator. The above relation is valid for any state vector $|\psi\rangle$. It's easy to show for two arbitary state vectors

$$\boxed{\langle\psi_1|T^{-1}\vec{r}\,T|\psi_2\rangle = \langle\psi_1|\vec{r}|\psi_2\rangle, \quad \text{Such that} \quad T^{-1}\vec{r}\,T = \vec{r}}$$

Proof: take $|\psi\rangle = |\psi_1\rangle + |\psi_2\rangle$,

$$\left(\langle\psi_1| + \langle\psi_2|\right)(T^{-1} r T)\left(|\psi_2\rangle + |\psi_1\rangle\right) = \left(\langle\psi_1| + \langle\psi_2|\right) r \left(|\psi_2\rangle + |\psi_1\rangle\right)$$

- $\Rightarrow \langle\psi_1| T^{-1}\vec{r}T|\psi_2\rangle + \langle\psi_2| T^{-1}\vec{r}T|\psi_1\rangle = \langle\psi_1|\vec{r}|\psi_2\rangle + \langle\psi_2|\vec{r}|\psi_1\rangle$

if we take $|\psi\rangle = |\psi_1\rangle + i|\psi_2\rangle \Rightarrow$

$$\langle\psi_1| T^{-1}\vec{r}T|\psi_2\rangle - \langle\psi_2| T^{-1}\vec{r}T|\psi_1\rangle = \langle\psi_1|\vec{r}|\psi_2\rangle - \langle\psi_2|\vec{r}|\psi_1\rangle$$

$$\Rightarrow \boxed{\langle\psi_1|T^{-1}\vec{r}T|\psi_2\rangle = \langle\psi_1|\vec{r}|\psi_2\rangle}.$$

Similarly, we should have

$$\boxed{\begin{array}{l} \begin{cases} T^{-1}\vec{r}\,T = \vec{r} \\ T^{-1}\vec{p}\,T = -\vec{p} \\ T^{-1}\vec{L}\,T = -\vec{L} \end{cases} \\ \text{also } T^{-1}\vec{S}\,T = -\vec{S} \end{array}} \leftarrow \text{at operator level}$$

In order to be ~~ansistent~~ consistent with these relation, T has to be anti-unitary.

- Check the commutation relation $[x, p] = i\hbar$, how does it change under T?

$$T[x,p]T^{-1} = T\, i\hbar\, T^{-1}$$

$$(TxT^{-1})(TpT^{-1}) - (TpT^{-1})(TxT^{-1}) = -(xp - px) = -i\hbar$$

$$\Rightarrow \boxed{T i T^{-1} = -i}$$

Ex: From $[L_i, L_j] = i\epsilon_{ijk}L_k$, derive that $TiT^{-1} = -i$.

§3. $T^2 = ?$

- Naively, we would expect that after TR transformation twice, the system comes back to itself, thus $T^2=1$. But we will see two possibilities.

First, T^2 is a constant.

Proof: we have $T\vec{r}T^{-1} = \vec{r} \Rightarrow T^2\vec{r}T^{-1^2} = \vec{r} \Rightarrow T^2\vec{r} = \vec{r}T^2$

$T\vec{p}T^{-1} = -\vec{p} \Rightarrow T^2\vec{p}T^{-1^2} = \vec{p} \Rightarrow T^2\vec{p} = \vec{p}T^2$

and similary $T^2\vec{L} = \vec{L}T^2$, $T^2\vec{S} = \vec{S}T^2$, $T^2 i = iT^2$.

For any operater $F(r, p, S, i)$, we have $T^2 F(r, p, S, i) = F(r, p, S, i)T^2$

$\Rightarrow$ $\boxed{T^2 \text{ is a constant.}}$

Then what's its value? $\boxed{\text{Answer: } T^4 = 1, \text{ and thus } T^2 = \pm 1.}$

Proof. $T^4 = T(T^2)T = (T^2)^* T^2 = T^2 (T^2)^*$.

For any two state vecters $|\psi\rangle$ and $|\phi\rangle$, rember T is anti-unitary, and T^2 is a complex constant $\Rightarrow$

$$\langle T\psi | T\phi\rangle = \langle\psi|\phi\rangle^*, \qquad \langle T^2\psi | T^2\phi\rangle = \langle T\psi|T\phi\rangle^* = \langle\psi|\phi\rangle$$

$$\langle T^2\psi | T^2\phi\rangle = (T^2)^* T^2 \langle\psi|\phi\rangle = \langle\psi|\phi\rangle \Rightarrow (T^2)^* T^2 = T^4 = 1$$

§4 The case of $T^2 = 1$.

For single component system, we can simply define $\psi^T(r) = \psi^*(r)$, or $\langle r|T|\psi\rangle = \langle r|\psi\rangle^*$. Please check that it satisfies:

$$\int d^3r \left(\psi^T(r)\right)^* \begin{pmatrix} \vec{r} \\ \vec{p} \\ \vec{L} \end{pmatrix} \psi^T(r) = \int d^3r\, \psi^*(r) \begin{pmatrix} \vec{r} \\ -\vec{p} \\ -\vec{L} \end{pmatrix} \psi(r).$$

Is there an example of H, that violates TR symmetry?

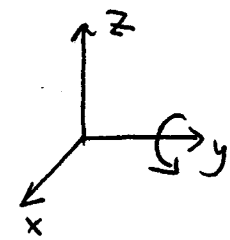
Ex: check: $H = \dfrac{(P - \frac{e}{c}A)^2}{2m}$, what's $H^T = T H T^{-1} = ?$

§5 The case of $T^2 = -1$. and Kramer degeneracy.

Let's consider a system with spin, The rotation matrix

$$D(g) = e^{-i \vec{J} \cdot \hat{n} \theta}$$

Let's consider the rotation operation and TR



$$\begin{cases} D^{\dagger}(g(\hat{y},\pi)) J_z D(g(\hat{y},\pi)) = -J_z \\ T^{-1} J_z T = -J_z \end{cases}$$

$$T^{-1} D^{-1}(g(\hat{y},\pi)) J_z D(g(y,\pi)) T = J_z$$

or $\boxed{J_z [D(g(y,\pi)) T] = [D(g(y,\pi)) T] J_z}$

Consider a J_z eigenstate, $|jm\rangle$, then $D(g(y,\pi)) T |jm\rangle$ must be the same as $|m\rangle$ up to a complex constant, because

$$J_z \left(D(g(y,\pi)) T |jm\rangle \right) = D(g(y,\pi)) T J_z |jm\rangle = m \left(D(g(y,\pi)) T |jm\rangle \right)$$

$$\Rightarrow D(g(y,\pi)) T |jm\rangle = c |jm\rangle .$$

Then $(D(g(y,\pi)) T)^2 |jm\rangle = (D(g(y,\pi)) T) c |jm\rangle = c^* c |jm\rangle$.

$$\langle D(g(y,\pi)) T jm | D(g(y,\pi)) T jm \rangle = \langle T jm | T jm \rangle = \langle jm | jm \rangle \Rightarrow c^* c = 1$$

Thus $\left(D(g(y,\pi))\, T\right)^2 = 1$ or $T\, D(g(y,\pi))\, T\, D(g(y,\pi)) = 1$

For $T e^{-iJ\cdot\hat{n}\theta} T^{-1} = e^{-(-i)(-\vec{J})\cdot\hat{n}\theta} = e^{-iJ\cdot\hat{n}\theta}$

$\Rightarrow$ $T D(g) = D(g) T$

$\Rightarrow$ $T^2 D^2(g(y,\pi)) = 1$ or $\boxed{T^2 D(g(y,2\pi)) = 1}$

But rotation around y-axis at 2π-angle, should it just an identity transformation? Not quite

$$D(g(y,2\pi)) = \begin{cases} I & j \text{ integer} \\ -I & j \text{ half-integer.} \end{cases}$$

Proof: $D(g(y,2\pi)) = D(g(x,\frac{\pi}{2}))\, D(g(z,2\pi))\, D(g^{-1}(x,\frac{\pi}{2}))$

$g(x,\frac{\pi}{2})$ rotation rotates y-axis into z-axis

$D(g(z,2\pi)) = e^{-iJ_z 2\pi} = \begin{cases} I & \text{if } J_z \text{ integer} \\ -I & \text{if } J_z \text{ half integer} \end{cases}$

$$\Rightarrow \boxed{D(g(y,2\pi)) = D(g(z,2\pi)) = \begin{cases} I & j \text{ integer} \\ -I & j \text{ half integer.} \end{cases}}$$

$$\Rightarrow \boxed{T^2 = \begin{cases} 1 & \text{for } j \text{ integer} \\ -1 & \text{for } j \text{ half integer} \end{cases}}$$

$\rightarrow$ orthogonal class,

$\rightarrow$ symplectic class.

① For spin-1/2 case, a convenient choice is $T = -i\sigma_y K = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} K$

$$\begin{cases} T\,|\uparrow\rangle = |\downarrow\rangle \\ T\,|\downarrow\rangle = -|\uparrow\rangle \end{cases} \quad \text{and} \quad T(c_1|\uparrow\rangle + c_2|\downarrow\rangle) = c_1^*|\downarrow\rangle - c_2^*|\uparrow\rangle.$$

K is the complex conjugate on complex coefficent

or $T\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -c_2^* \\ c_1^* \end{pmatrix}$, $T^2\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = -\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$.

② For a general case, we can define $T = RK$.

① if j half integer, $2j+1$ is even. we may choose $R = \begin{pmatrix} & & & -1 \\ & & 1 & \\ & -1 & & \\ 1 & & & \end{pmatrix}$,

and $R^2 = -1$, and $T^2 = -1$.

② if j is integer, $2j+1$ is odd. we choose
$$R = \begin{array}{c} \begin{array}{c} m=0 \end{array} \\ \begin{pmatrix} & & & -1 \\ & & 1 & \\ & -1 & & \\ 1 & & & \end{pmatrix} \; m=0 \end{array},$$
such that

$R^2 = T^2 = 1$.

in this case, $T|lm\rangle = (-)^m |l\,-m\rangle$, and

$$\langle\hat{n}|T|lm\rangle = Y_{lm}^*(\theta,\varphi) = (-)^m\langle\hat{n}|l\,-m\rangle = (-)^m Y_{l-m}(\theta,\varphi).$$

Consistent with $\boxed{Y_{lm}^*(\theta,\varphi) = (-)^m Y_{l-m}(\theta,\varphi)}$

§ Kramer degeneracy.

if $T^2 = -1$, then for any eigen state $|\psi\rangle$, with $H|\psi\rangle = E|\psi\rangle$,

then $H(T|\psi\rangle) = TH|\psi\rangle = E(T|\psi\rangle)$, thus $T|\psi\rangle$ is also an eigenstate with the same energy.

On the other hand

$\langle\psi|T\psi\rangle = \langle T\psi|T^2\psi\rangle^* = -\langle T\psi|\psi\rangle^* = -\langle\psi|T\psi\rangle$

$\Rightarrow \langle\psi|T\psi\rangle = 0.$ thus $T|\psi\rangle$ is an other state, and there's at least 2-fold degeneracy.

Ex: if $T^2=1$, is there always an energy level degeracy?

§ Parity transformation

Consider a state vector $|\psi\rangle$, after parity transformation P, we have $|\psi^P\rangle = \mathbb{P}|\psi\rangle$, or $|\psi\rangle = \mathbb{P}^{-1}|\psi^P\rangle$. Again we assume P satisfies Wigner theorem. Again we use correspondence principle, and arrive at

$\langle\psi^P|\vec{r}|\psi^P\rangle = -\langle\psi|\vec{r}|\psi\rangle$, $\langle\psi^P|\vec{p}|\psi^P\rangle = -\langle\psi|\vec{p}|\psi\rangle$

$\langle\psi^P|\vec{L}|\psi^P\rangle = \langle\psi|\vec{L}|\psi\rangle$, and also $\langle\psi^P|\vec{S}|\psi^P\rangle = -\langle\psi|\vec{S}|\psi\rangle$

$\Rightarrow \boxed{\mathbb{P}^{-1}\vec{r}\,\mathbb{P} = -\vec{r},\ \mathbb{P}^{-1}\vec{p}\,\mathbb{P} = -\vec{p},\ \text{and}\ \mathbb{P}^{-1}\vec{L}\,\mathbb{P} = \vec{L},\ \mathbb{P}^{-1}\vec{S}\,\mathbb{P} = \vec{S}}$

check $[x,p] = i\hbar \Rightarrow \mathbb{P}^{-1}[x,p]\mathbb{P} = [-x,-p] = i\hbar = \mathbb{P}^{-1}(i\hbar)\mathbb{P}$

$\Rightarrow \mathbb{P}i = i\mathbb{P} \Rightarrow$ $\boxed{\mathbb{P}\text{ is an unitary transformation.}}$

Similarly, we can also prove that $\mathbb{P}^2$ is a constant, and $\mathbb{P}^2(\mathbb{P}^2)^* = 1$.

Ex: without loss of generality, we choose

$\Rightarrow \mathbb{P}^2 = e^{i\delta}$ up to phase factor. $\mathbb{P}^2 = 1$.

For single component system, we simply set $\psi^P(\vec{r}) = \psi(-\vec{r})$.

We can easily check this definition satisfy the above requirement!

For the time-dependent case, we can define

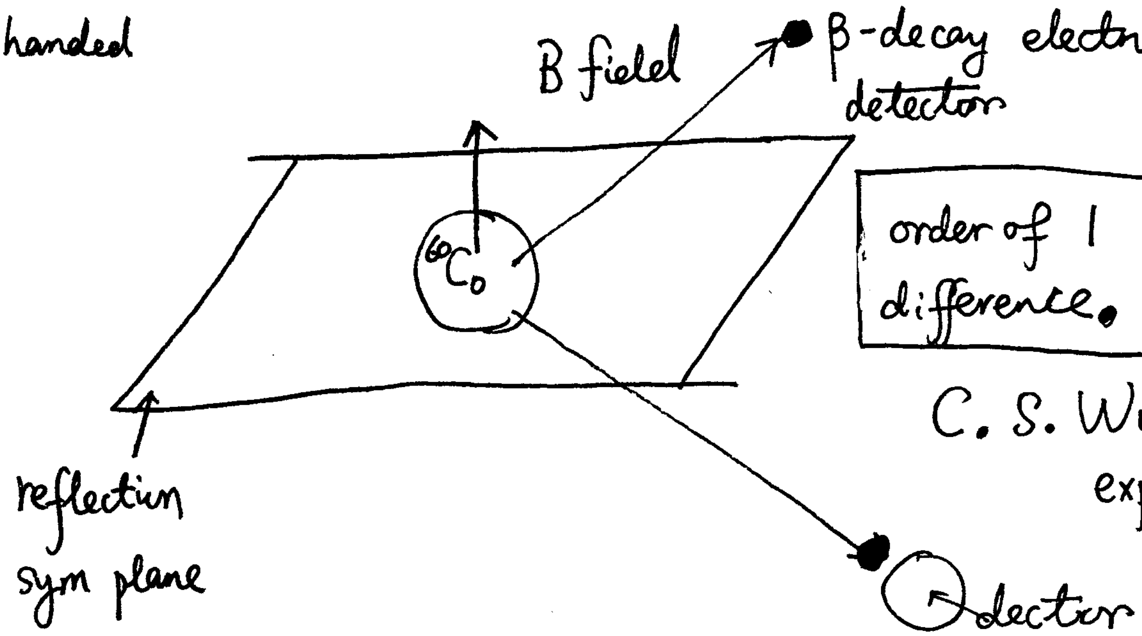
$$\boxed{\begin{aligned} \psi^T(x,t) &= \psi^*(x,-t) \\ \psi^P(x,t) &= \psi(-x,t). \end{aligned}}$$

Ex: verify for momentum eigenstate $\psi_p(x,t) = e^{ipx-i\omega t}$, what are $\psi^T_p(x,t)$, and $\psi^P_p(x,t)$? How about angular momentum eigenstates $\psi_m(x,t) = e^{im\varphi - i\omega t}$?

§ Parity broken in weak-interactions. — C.N Yang and T. D. Lee (Theory proposal)

$\begin{pmatrix} e_L \\ \nu_L \end{pmatrix}$, e_R, there's no ν_R.

left handed



order of 1 difference.

C. S. Wu's experiments.

§ Parity eigenstates

If $[H, \mathbb{P}] = 0$, then we can find common eigenstates of H, and $\mathbb{P}$.

For example: ① 1D harmonic oscillator $\mathbb{P}^{-1} H \mathbb{P} = H$. It's energy Wavefun

$$\psi_n(-x) = (-)^n \psi_n(x), \qquad \begin{array}{l} \text{even for } n=0,2,4\cdots \\ \text{odd for } n=1,3,5\cdots \end{array}$$

② orbital angular momentum eigenstates $Y_{lm}(\hat{n})$

$$Y_{lm}(-\hat{n}) = (-)^l Y_{lm}(\hat{n})$$

③ Selection rule $\Delta H = -e\vec{r}\cdot\vec{E}$

$$\langle nlm | \Delta H | n'l'm' \rangle \neq 0, \quad \text{only for } l' = l \pm 1.$$

§ The relation between degeneracy and symmetry.

For a Hamiltonian, all it's symmetry operations together form a group.
R

If a state $|\psi\rangle$ is an eigenstate, then all the states $R|\psi\rangle$

$$H\Big(R|\psi\rangle\Big) = R\big(H|\psi\rangle\big) = E|\psi\rangle$$

form a subspace. This subspace support a representation for the symmetry group.
degenerate

For example: ① for 3D rotation symmetry, all the states $\psi_{nlm} = R_{nl} Y_{lm}(\theta,\varphi)$ with $m=-l,\cdots l$, form a l-fold degeneracy.

② But for 1d harmonic oscillator, the parity symmetry does not bring degeneracy.

§ Dirac Equation

$$H = \vec{\alpha}\cdot\vec{P} + \beta m \qquad \vec{\alpha} = \begin{pmatrix} -\vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} \quad \beta = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

$$\gamma^0 = \beta . \quad \gamma^i = \beta\alpha^i = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}, \quad \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}$$

$$H\psi = E\psi \ , \quad \psi = \begin{pmatrix} \chi_L \\ \eta_R \end{pmatrix} \leftarrow (i\gamma^\mu\partial_\mu - m)\psi = 0$$

$$\begin{cases} (E - \vec{P}\cdot\vec{\sigma})\eta_R = m\chi_L \\ (E + \vec{P}\cdot\vec{\sigma})\chi_L = m\eta_R \end{cases} \qquad \begin{cases} \frac{\vec{\sigma}\cdot\vec{P}}{|P|}\, \xi_\pm(\vec{P}) = \pm\,\xi_\pm(\vec{P}) \\ P_\pm(\hat{P}) = \frac{1}{2}(1 \pm \vec{\sigma}\cdot\hat{P}) \end{cases}$$

① positive energy solution

$$\psi^+_{\vec{P}}(x) = e^{-iP\cdot x}\, u(\vec{P}) \qquad P\cdot x = (\omega t - \vec{P}\cdot\vec{x})$$

$$u_1(P) = \begin{pmatrix} \sqrt{E-P}\ \xi_+(P) \\ \sqrt{E+P}\ \xi_+(P) \end{pmatrix} \frac{1}{\sqrt{2(E^2+P^2)}}$$

$$u_2(P) = \begin{pmatrix} \sqrt{E+P}\ \xi_-(P) \\ \sqrt{E-P}\ \xi_-(P) \end{pmatrix} \frac{1}{\sqrt{2(E^2+P^2)}}$$

③ consider negative energy state $(-E, -\vec{p})$

$$\psi^{(-)}_{-\vec{p}}(x) = v(\vec{p})\, e^{-i\vec{p}\cdot\vec{x} + iEt}$$

$$v_1(\vec{p}) = \frac{1}{\sqrt{2(E^2+p^2)}} \begin{pmatrix} \sqrt{E+p}\ \xi_+(-\vec{p}) \\ -\sqrt{E-p}\ \xi_+(-\vec{p}) \end{pmatrix}$$

$$v_2(\vec{p}) = \frac{1}{\sqrt{2(E^2+p^2)}} \begin{pmatrix} \sqrt{E-p}\ \xi_-(-\vec{p}) \\ -\sqrt{E+p}\ \xi_-(-\vec{p}) \end{pmatrix}$$

Solution

$$\psi_p(x) = \left(a_1(\vec{p})\, u_1(\vec{p}) + a_2(\vec{p})\, u_2(\vec{p})\right) e^{i(\vec{p}\cdot\vec{x} - Et)}$$
$$+ \left(b_1^\dagger(\vec{p})\, v_1(\vec{p}) + b_2^\dagger(\vec{p})\, v_2(\vec{p})\right) e^{-i\vec{p}\vec{x} + iEt}$$

	$\|1\rangle$	$\|2\rangle$	$\|3\rangle$	$\|4\rangle$
	$a_1^\dagger(\vec{p})\|0\rangle$	$a_2^\dagger(\vec{p})\|0\rangle$	$b_1^\dagger(\vec{p})\|0\rangle$	$b_2^\dagger(\vec{p})\|0\rangle$
momentum	$\vec{p}$		$\vec{p}$	
particle #	1		-1	
helicity $\vec{\sigma}\cdot\vec{p}/\|p\|$	$1/2$	$-1/2$	$+1/2$	$-1/2$

$v(\vec{p})$ 动量 $-\vec{p}$，极化 $1/2$，负能量电子

$\longrightarrow$ 动量 $\vec{p}$ 极化 $+1/2$ 正能量正电子

(3) consider negative energy state $(-E, -\vec{p})$

$$\psi^{(-)}_{-\vec{p}}(x) = v(\vec{p})\, e^{-i\vec{p}\cdot\vec{x} + iEt}$$

$$v_1(\vec{p}) = \frac{1}{\sqrt{2(E^2+p^2)}} \begin{pmatrix} \sqrt{E+p}\; \xi_+(-\vec{p}) \\ -\sqrt{E-p}\; \xi_+(-\vec{p}) \end{pmatrix}$$

$$v_2(\vec{p}) = \frac{1}{\sqrt{2(E^2+p^2)}} \begin{pmatrix} \sqrt{E-p}\; \xi_-(-\vec{p}) \\ -\sqrt{E+p}\; \xi_-(-\vec{p}) \end{pmatrix}$$

Solution

$$\psi_p(x) = \left(a_1(\vec{p})\, u_1(\vec{p}) + a_2(\vec{p})\, u_2(\vec{p})\right) e^{i(\vec{p}\cdot\vec{x} - Et)}$$
$$+ \left(b_1^\dagger(\vec{p})\, v_1(\vec{p}) + b_2^\dagger(\vec{p})\, v_2(\vec{p})\right) e^{-i\vec{p}\vec{x} + iEt}$$

	$\vert 1\rangle$	$\vert 2\rangle$	$\vert 3\rangle$	$\vert 4\rangle$
	$a_1^\dagger(\vec{p})\vert 0\rangle$	$a_2^\dagger(\vec{p})\vert 0\rangle$	$b_1^\dagger(\vec{p})\vert 0\rangle$	$b_2^\dagger(\vec{p})\vert 0\rangle$
momentum	$\vec{p}$		$\vec{p}$	
particle #	1		-1	
helicity $\vec{\sigma}\cdot\vec{p}/\vert p\vert$	$1/2$	$-1/2$	$+1/2$	$-1/2$

$v(\vec{p})$ 动量 $-\vec{p}$，极化 $1/2$，负能量电子

$\longrightarrow$ 动量 $\vec{p}$ 极化 $+1/2$ 正能量正电子

Lorentz transformation of Dirac Eq.

$x^\mu = (t, -\vec{x}), \; x_\mu = (t, \vec{x}$

$\partial^\mu = (\partial_t, \partial_x) \; \partial_\mu = (\partial_t, -\partial$

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu, \quad \psi'(x') = S\psi(x)$$

$$\bar\psi(x') = \bar\psi(x) S^{-1} \Leftarrow \quad S^\dagger = \gamma^0 S^{-1} \gamma^0$$

plug in $S^{-1}\psi'(x')$

$$\left[i\hbar \gamma^\mu \left(\partial_\mu + \frac{ie}{\hbar c} A_\mu\right) - \frac{mc}{\hbar} \right] \psi(x) = 0$$

$$\left[i\, \underbrace{S\, \gamma^\nu}_{S^{-1}} \left(\partial_\nu + \frac{ie}{\hbar c} A_\nu\right) - \frac{mc}{\hbar} \right] \psi'(x') = 0 \qquad \text{(move } S^{-1} \text{ to left)}$$

$$\partial_\nu = \Lambda^\mu{}_\nu \partial'_\mu, \quad A_\nu(x) = \Lambda^\mu{}_\nu A'_\mu(x')$$

$$\Rightarrow \left\{ i\, S\gamma^\nu S^{-1} \Lambda^\mu{}_\nu \left(\partial'_\mu + \frac{ie}{\hbar c} A'_\mu(x')\right) - \frac{mc}{\hbar} \right\} \psi'(x') = 0$$

$$\Rightarrow \quad \Lambda^\mu{}_\nu S \gamma^\nu S^{-1} = \gamma^\mu \longrightarrow \boxed{S^{-1} \gamma^\mu S = \Lambda^\mu{}_\nu \gamma^\nu}$$

$\Rightarrow$ $\boxed{\text{Lorentz scalar } \bar\psi\psi, \; i\bar\psi\gamma^5\psi}$

$$\bar\psi'(x')\psi'(x') = \bar\psi(x) S^{-1} S \psi(x) = \bar\psi(x)\psi(x)$$

$$i\bar\psi'(x')\gamma^5\psi'(x') = i^2 \bar\psi(x) (S^{-1}\gamma^0 S)(S^{-1}\gamma^1 S) \cdots (S^{-1}\gamma^3 S)\psi(x)$$

$$= i^2 \bar\psi(x)\, \Lambda^0{}_{\nu_1} \Lambda^1{}_{\nu_2} \Lambda^2{}_{\nu_3} \Lambda^3{}_{\nu_4} \gamma^{\nu_1}\gamma^{\nu_2}\gamma^{\nu_3}\gamma^{\nu_4} \psi(x)$$

$$= \det\Lambda \; \bar\psi(x)\psi(x)$$

$$\boxed{\text{Lorentz vector:}\quad \bar\psi\gamma^\mu\psi \qquad \bar\psi\gamma^\mu\gamma^5\psi}$$

$$\rightarrow \bar\psi'(x')\gamma^\mu\psi'(x') = \Lambda^\mu{}_\nu\,\bar\psi(x)\gamma^\nu\psi(x) \quad \text{— current}$$

$$\bar\psi'(x')\gamma^\mu\gamma^5\psi'(x') = \det(\Lambda)\,\Lambda^\mu{}_\nu\,\bar\psi\gamma^\nu\gamma^5\psi \quad \text{— axial current}$$

$$\boxed{\text{Lorentz tensor}\quad \bar\psi i\gamma^\mu\gamma^\nu\psi}$$

$$\rightarrow \bar\psi'(x')i\gamma^\mu\gamma^\nu\psi'(x') = \Lambda^\mu{}_{\mu'}\Lambda^\nu{}_{\nu'}\,\bar\psi(x)\gamma^{\mu'}\gamma^{\nu'}\psi(x)$$

§ Space reflection (P)

$$\Lambda = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \qquad \psi'(x') = S_p \psi(x)$$

Similar to the regular Lorentz transform $\quad S_p^{-1} \gamma^\mu S_p = \Lambda^\mu{}_\nu \gamma^\nu$

i.e. $\quad S_p^{-1} \gamma^0 S_p = \gamma^0$

$\quad S_p^{-1} \gamma^i S_p = \gamma^i$ $\quad \Rightarrow \boxed{\text{we can choose } S_p = \gamma^0}$

again $\quad \bar\psi(x') = \bar\psi(x) S_p^{-1}$

§ time-reversal transformation (T)

$$\Lambda = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \qquad \psi'(x') = T\psi(x) \quad \text{i.e } \psi'(x_i, t) = T\psi(x, -t)$$

T is antiunitary transformation: $\quad T = \theta\, \mathcal{U} \leftarrow$ unitary transf

$\uparrow$ complex-conjugate

$$\Rightarrow \psi(x) = T^{-1} \psi'(x') = \mathcal{U}^{-1} \theta\, \psi'(x')$$

$$\left\{ i\,\gamma^\mu \left(\partial_\mu + \frac{ie}{\hbar c} A_\mu(x)\right) - \frac{mc}{\hbar} \right\} u^{-1}\theta \;\; \psi'(x') = 0$$

$$\left\{ \theta u \left[i\,\gamma^\mu \left(\partial_\mu + \frac{ie}{\hbar c} A_\mu\right) u^{-1}\theta \right] - \frac{mc}{\hbar} \right\} \psi'(x') = 0$$

we want

$$-i\,\theta u\,\gamma^\mu \left(\partial_\mu + \frac{ie}{\hbar c} A_\mu\right) u^{-1}\theta = i\,\gamma^\mu \left(\partial_\mu' + \frac{ie}{\hbar c} A_\mu'(x')\right)$$

$$\partial_0 = -\partial_{0'} \qquad \partial_i = \partial_{i'}$$

$$A_0(x) = A_0'(x') \qquad A_i(x) = -A_i'(x')$$

$$\Rightarrow \quad \theta u\,\gamma^0 u^{-1}\theta = \gamma^0 \quad \Rightarrow \quad u\gamma^0 u^{-1} = (\gamma^0)^*$$

$$\theta u\,\gamma^i u^{-1}\theta = -\gamma^i \qquad\qquad u\gamma^i u^{-1} = -(\gamma^i)^*$$

$$T = \theta u \quad \Rightarrow \quad T^2 = \theta u [\theta u] = u^* u$$

in our representation $\quad u\,\gamma^{1,3} u^{-1} = -\gamma^{1,3}$, $\quad u\,\gamma^{0,2} u^{-1} = +\gamma^{0,2}$

$$\Rightarrow \quad u = \gamma_1\gamma_3 \quad \Rightarrow \quad T^2 = (\gamma_1\gamma_3)^*(\gamma_1\gamma_3) = \gamma_1\gamma_3\gamma_1\gamma_3 = -1.$$

$$\boxed{T = \theta\,\gamma_1\gamma_3 = \gamma_1\gamma_3\,\theta}$$

$$\bar\psi'(x') = \psi^{\dagger\prime}(x')\,\gamma^0 = \left[\gamma_1\gamma_3\,\psi^*(x)\right]^\dagger \gamma^0 = \psi^T(x)\,\gamma_3^\dagger\gamma_1^\dagger\,\gamma^0$$

$$= \psi^T(x)\,(\gamma_1\gamma_3)^T\,\gamma^0$$

§ Charge conjugation (C)

$$\left\{ i\gamma^\mu\left(\partial_\mu + \frac{ie}{\hbar c}A_\mu(x)\right) - \frac{mc}{\hbar}\right\}\psi(x) = 0$$

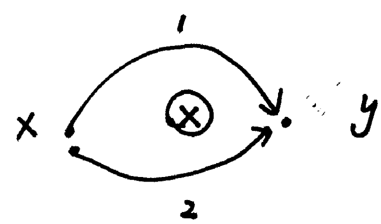
under charge conjugation, $\psi(x) \to \varphi(x)$, $\quad e \to -e$

$$\left\{ i\gamma^\mu\left(\partial_\mu - \frac{ie}{\hbar c}A_\mu(x)\right) - \frac{mc}{\hbar}\right\}\psi'(x) = 0$$

$\uparrow$ anti-particle wavefunction

This has to be an anti-unitary transformation because if

$$\psi(y) = \psi(x)\left[e^{i\int_1 d\vec{x}\, e\vec{A}} + e^{ie\int_2 d\vec{x}\, \vec{A}}\right]$$



$$\to \psi'(y) = \psi'(x)\left(e^{-i\int_1 d\vec{x}\, e\vec{A}} + e^{-i\int d\vec{x}\, e\vec{A}}\right)$$

define $\quad \psi'(x) = \mathcal{U}\psi^*(x) \quad$ or $\quad \psi(x) = (\mathcal{U}^{-1})^* \psi'^*(x)$

$$\Rightarrow \left[i\gamma^\mu\left(\partial_\mu + \frac{ie}{\hbar}A_\mu\right) - \frac{mc}{\hbar}\right](\mathcal{U}^{-1})^*\psi'^*(x) = 0 \qquad \boxed{C^2 = 1}$$

$$\mathcal{U}\left[-i\gamma^{*\mu}\left(\partial_\mu - \frac{ie}{\hbar}A_\mu\right) - \frac{mc}{\hbar}\right]\mathcal{U}^{-1}\psi'(x) = 0$$

$$\Rightarrow \left[-i\,\mathcal{U}\gamma^{*\mu}\mathcal{U}^{-1}\left(\partial_\mu - \frac{ie}{\hbar}A_\mu\right) - \frac{mc}{\hbar}\right]\psi'(x) = 0$$

$$\mathcal{U}\gamma^{*\mu}\mathcal{U}^{-1} = -\gamma^\mu \quad \text{or} \quad \boxed{\mathcal{U}^{-1}\gamma^\mu\mathcal{U} = -\gamma^{\mu *}}$$

i.e. $\quad \mathcal{U}^{-1}\gamma^{0,1,3}\mathcal{U} = -\gamma^{0,1,3}, \qquad \mathcal{U}^{-1}\gamma^2\mathcal{U} = +\gamma^2$

$\Rightarrow \mathcal{U} = i\gamma^2$, $\quad$ it is often write $\quad \boxed{\psi'(x) = i\gamma^2\gamma^0\gamma^0\psi^*(x) = \underbrace{i\gamma^2\gamma^0}_{C}\bar{\psi}^T}$

Test symmetry: ① $P = \gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ $\boxed{\psi'(x,t) = \gamma^0 \psi(-x,t)}$

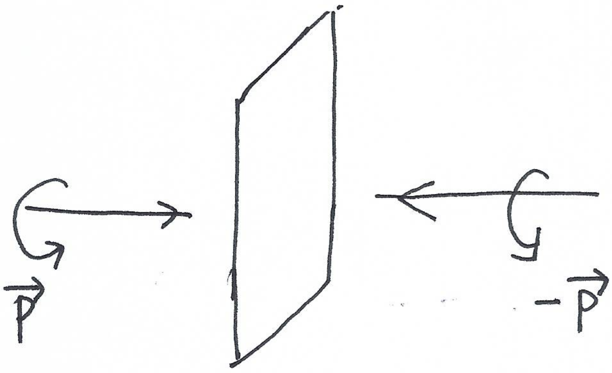
$$P u_1(p) = \begin{pmatrix} \sqrt{E+p}\ \xi_+(p) \\ \sqrt{E-p}\ \xi_+(p) \end{pmatrix} = \begin{pmatrix} \sqrt{E+p}\ \xi_-(-p) \\ \sqrt{E-p}\ \xi_-(-p) \end{pmatrix} = u_2(-p)$$

$$P u_2(p) = \begin{pmatrix} \sqrt{E-p}\ \xi_-(p) \\ \sqrt{E+p}\ \xi_{\pm}(p) \end{pmatrix} = \begin{pmatrix} \sqrt{E-p}\ \xi_+(-p) \\ \sqrt{E+p}\ \xi_+(-p) \end{pmatrix} = u_1(-p)$$

$$\boxed{\begin{aligned} \xi_+(\vec{p}) &= \xi_-(-\vec{p}) \\ \xi_-(\vec{p}) &= \xi_+(-\vec{p}) \end{aligned}}$$

$$P v_1(p) = \begin{pmatrix} \sqrt{E-p}\ \xi_+(-\vec{p}) \\ -\sqrt{E+p}\ \xi_+(-\vec{p}) \end{pmatrix} = -v_2(-p)$$

$$P v_2(p) = -\begin{pmatrix} \sqrt{E+p}\ \xi_-(-\vec{p}) \\ -\sqrt{E-p}\ \xi_-(-\vec{p}) \end{pmatrix} = -\begin{pmatrix} \sqrt{E+p}\ \xi_+(\vec{p}) \\ -\sqrt{E-p}\ \xi_-(\vec{p}) \end{pmatrix} = -v_1(-p)$$



$$\swarrow \quad P \hat{\psi}(x) P^{-1} = \gamma^0 \psi(-x,t)$$

$$\Rightarrow \quad P a_1(\vec{p}) P^{-1} = a_2(-\vec{p}) \qquad P b_1(\vec{p}) P^{-1} = -b_2(-\vec{p})$$

$$P a_2(\vec{p}) P^{-1} = a_1(-\vec{p}) \qquad P b_2(\vec{p}) P^{-1} = -b_1(-\vec{p})$$

Consider ~~$e\bar{e}$ - s-wave system~~

~~$P \int dk\ f(|k|)$~~ a

$\boxed{\text{helicity flip the sign}}$

⊛ time-reversal

$$\boxed{\psi'(x,t) = \gamma^1\gamma^3\,\psi^*(x,-t)} \qquad \gamma^1\gamma^3 = \begin{pmatrix} i\sigma_2 & 0 \\ 0 & i\sigma_2 \end{pmatrix}$$

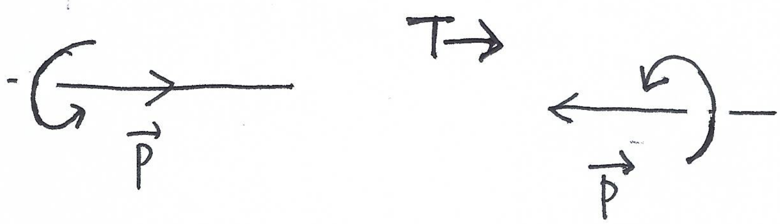
$$\gamma^1\gamma^3 u_1^*(\vec{p}) = \begin{pmatrix} \sqrt{E-p}\; i\sigma_2\,\xi_+^*(\vec{p}) \\ \sqrt{E+p}\; i\sigma_2\,\xi_+^*(\vec{p}) \end{pmatrix} = \begin{pmatrix} \sqrt{E-p}\;\tilde{\xi}_+(-\vec{p}) \\ \sqrt{E+p}\;\tilde{\xi}_+(-\vec{p}) \end{pmatrix} \sim u_1(-\vec{p})$$

$$\gamma^1\gamma^3 u_2^*(\vec{p}) = \begin{pmatrix} \sqrt{E+p}\; i\sigma_2\,\xi_-^*(\vec{p}) \\ \sqrt{E-p}\; i\sigma_2\,\xi_-^*(\vec{p}) \end{pmatrix} = -\begin{pmatrix} \sqrt{E+p}\;\tilde{\xi}_-(-\vec{p}) \\ \sqrt{E-p}\;\tilde{\xi}_-(-\vec{p}) \end{pmatrix} \sim u_2(-\vec{p})$$

$$\boxed{\text{define} \quad \cancel{\xi_+(-\vec{p}) = i\sigma_2\,\xi_+^*(\vec{p}) = \xi_-(\vec{p})} \\ \cancel{\xi_-(-\vec{p}) = -i\sigma_2\,\xi_-^*(\vec{p}) = \xi_+(\vec{p})}}$$

$$\gamma^1\gamma^3 v_1^*(\vec{p}) = \begin{pmatrix} \sqrt{E+p}\; i\sigma_2\,\xi_+^*(-\vec{p}) \\ -\sqrt{E-p}\; i\sigma_2\,\xi_+^*(-\vec{p}) \end{pmatrix} = \begin{pmatrix} \sqrt{E+p}\;\tilde{\xi}_+(\vec{p}) \\ -\sqrt{E-p}\;\tilde{\xi}_+(\vec{p}) \end{pmatrix} \sim v_1(-\vec{p})$$

$$\gamma^1\gamma^3 v_2^*(\vec{p}) = \begin{pmatrix} \sqrt{E-p}\; i\sigma_2\,\xi_-^*(-\vec{p}) \\ -\sqrt{E+p}\; i\sigma_2\,\xi_-^*(-\vec{p}) \end{pmatrix} = \begin{pmatrix} \sqrt{E-p}\;\tilde{\xi}_-(-p) \\ -\sqrt{E+p}\;\tilde{\xi}_-(-p) \end{pmatrix} \sim v_2(-\vec{p})$$



$i\sigma_2\,\xi_+^*(\vec{p})$ is + helicity state of $-\vec{p}$ but phase is complicated!

C:

$$\psi'(x) = i\gamma^2\psi^*(x) = \begin{pmatrix} 0 & i\sigma_2 \\ -i\sigma_2 & 0 \end{pmatrix}\psi^*(x) = i\gamma^2\gamma^0\gamma^0\psi^* = i\gamma^2\gamma^0\bar{\psi}^T$$ (3) (5)

for $\psi(x) = v_1(\vec{p})$ $\quad \boxed{C\hat{\psi}C^{-1} = i\gamma^2\hat{\psi}^* = i\gamma^2\gamma^0\bar{\hat{\psi}}^T}$

$$\begin{pmatrix} 0 & i\sigma_2 \\ -i\sigma_2 & 0 \end{pmatrix}\begin{pmatrix} \sqrt{E+p}\;\xi_+^*(-\vec{p}) \\ -\sqrt{E-p}\;\xi_+^*(-\vec{p}) \end{pmatrix} = \begin{pmatrix} \sqrt{E-p}\;(-i\sigma_2\,\xi_+^*(-\vec{p})) \\ \sqrt{E+p}\;(-i\sigma_2\,\xi_+^*(-\vec{p})) \end{pmatrix}$$

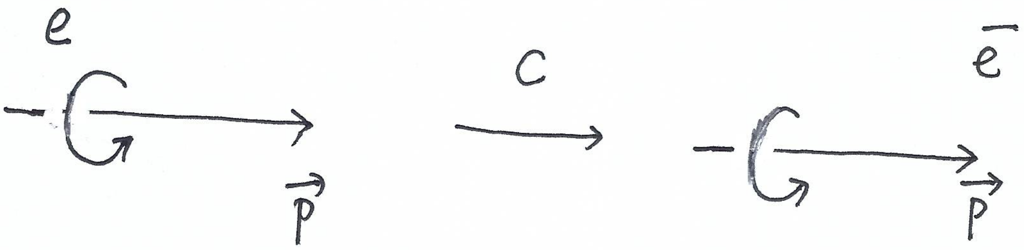
we define

$$\begin{cases} \xi_-(\vec{p}) = i\sigma_2\,\xi_+^*(\vec{p}), & \xi_+(-\vec{p}) = i\sigma_2\,\xi_-^*(-\vec{p}) \qquad \xi_+(\vec{p}) = \xi_-(-\vec{p}) \\ \xi_+(\vec{p}) = -i\sigma_2\,\xi_-^*(\vec{p}), & \xi_-(-\vec{p}) = -i\sigma_2\,\xi_+^*(-\vec{p}) \qquad \xi_-(\vec{p}) = \xi_+(-\vec{p}) \end{cases}$$

$$\Rightarrow i\gamma^2 v_1^*(\vec{p}) = \begin{pmatrix} \sqrt{E-p}\;\xi_-(-\vec{p}) \\ \sqrt{E+p}\;\xi_-(-\vec{p}) \end{pmatrix} = \begin{pmatrix} \sqrt{E-p}\;\xi_+(\vec{p}) \\ \sqrt{E+p}\;\xi_+(\vec{p}) \end{pmatrix} = u_1(\vec{p})$$

$$i\gamma^2 v_2^*(\vec{p}) = \begin{pmatrix} 0 & i\sigma_2 \\ -i\sigma_2 & 0 \end{pmatrix}\begin{pmatrix} \sqrt{E-p}\;\xi_-^*(-\vec{p}) \\ -\sqrt{E+p}\;\xi_-^*(-\vec{p}) \end{pmatrix} = \begin{pmatrix} \sqrt{E+p}\;(-i\sigma_2)\,\xi_-^*(-\vec{p}) \\ \sqrt{E-p}\;(-i\sigma_2)\,\xi_-^*(-\vec{p}) \end{pmatrix}$$

$$= -\begin{pmatrix} \sqrt{E+p}\;\xi_+(-\vec{p}) \\ \sqrt{E-p}\;\xi_+(-\vec{p}) \end{pmatrix} = -u_2(\vec{p})$$



$$P\bar{\psi}(\vec{x},t)P^{-1} = \left[P\psi P^{-1}\right]^{\dagger}\gamma^0 = \psi^{\dagger}(-x,t)\gamma^0\gamma^0 = \bar{\psi}(-x,t)\gamma^0$$

$$\Rightarrow \quad P\bar{\psi}(x,t)\psi(x,t)P^{-1} = \bar{\psi}(-x,t)\psi(-x,t)$$

$$P\bar{\psi}(x,t)i\gamma^5\psi(x,t)P^{-1} = \bar{\psi}(-x,t)\gamma^0\gamma^5\gamma^0\psi(-x,t) = -\bar{\psi}(x,t)\psi(-x,t)$$

$$P\bar{\psi}(x,t)\gamma^\mu\psi(x,t)P^{-1} = \begin{cases} \bar{\psi}(-x,t)\gamma^\mu\psi(-x,t) & \text{for } \mu=0 \\ -\bar{\psi}(-x,t)\gamma^\mu\psi(-x,t) & \text{for } \mu=1,2,3 \end{cases}$$

$$P\bar{\psi}(x,t)\gamma^5\gamma^\mu\psi(x,t)P^{-1} = \begin{cases} -\bar{\psi}(-x,t)\gamma^5\gamma^\mu\psi(-x,t) & \text{for } \mu=0 \\ \bar{\psi}(-x,t)\gamma^5\gamma^\mu\psi(-x,t) & \text{for } \mu=1,2,3 \end{cases}$$

$$P\bar{\psi}(x,t)\gamma^\mu\gamma^\nu\psi(x,t)P^{-1} = \begin{cases} \bar{\psi}(-x,t)\gamma^\mu\gamma^\nu\psi(-x,t) & \text{for } \mu,\nu=1,2,3 \\ -\bar{\psi}(-x,t)\gamma^\mu\gamma^\nu\psi(-x,t) & \text{for } \mu \ldots 0;\ \nu=1,2. \end{cases}$$

$$T\bar{\psi}\psi T^{-1} = \bar{\psi}\psi(x,-t)$$ 利用 $(\gamma^{0,1,3})^* = \gamma^{0,1,3}$, $(\gamma^2)^* = -\gamma^2$

$$T\bar{\psi}i\gamma^5\psi T^{-1} = \bar{\psi}\gamma^3\gamma^1 \underset{(-i)}{\gamma^{5*}}\gamma^1\gamma^3\psi = -i\bar{\psi}\gamma^5\psi(x,-t)$$

$$T\bar{\psi}\gamma^\mu\psi T^{-1} = \bar{\psi}\gamma^3\gamma^1\gamma^{\mu*}\gamma^1\gamma^3\psi = \begin{cases} \bar{\psi}\gamma^\mu\psi & (\mu=0) \\ -\bar{\psi}\gamma^\mu\psi & (\mu=1,2, \end{cases}$$

$$T\bar{\psi}\gamma^5\gamma^\mu\psi T^{-1} = \bar{\psi}\gamma^3\gamma^1\gamma^5\gamma^{\mu*}\gamma^1\gamma^3\psi = \begin{cases} \bar{\psi}\gamma^5\gamma^\mu\psi & (\mu=0) \\ -\bar{\psi}\gamma^5\gamma^\mu\psi & (\mu=1,2 \end{cases}$$

$$T\bar{\psi}i\gamma^\mu\gamma^\nu\psi T^{-1} = \bar{\psi}\gamma^3\gamma^1 \underset{(-i)}{(\gamma^\mu\gamma^\nu)^*}\gamma^1\gamma^3\psi = \begin{cases} \bar{\psi}\gamma^\mu\gamma^\nu\psi & (\mu=0,\nu= \\ -\bar{\psi}\gamma^\mu\gamma^\nu\psi & (\mu,\nu=1, \end{cases}$$

define $\boxed{C a_{\vec p \pm} C^{-1} = b_{\vec p \pm}(\pm e^{\mp i\varphi_p})} \to \boxed{C a^\dagger_{\vec p\pm} C^{-1} = b^\dagger_{\vec p\pm}(\pm e^{\pm i\varphi_p}}$

$\boxed{C b^\dagger_{p\pm} C^{-1} = a^\dagger_{\vec p\pm}(\pm e^{\mp i\varphi_p})}$

$$\Rightarrow C\psi(x,t)C^{-1} = \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\; b_{\vec p\pm}(-i\gamma^2 v^*_\pm(p))\, e^{i\vec p\vec x - iEt} + a^\dagger_{p\pm}(-i\gamma^2)u^*_\pm(p)\, e^{-i\vec p\vec x + iEt}$$

$$= -i\gamma^2\left[\psi^\dagger(x,t)\right]^T = -i[\psi^\dagger\gamma^2]^T = -i(\bar\psi\gamma^0\gamma^2)^T$$

$$C\psi^\dagger(x,t)C^{-1} = \left[-i\gamma^2(\psi^\dagger)^T\right]^\dagger = \psi^T(-i\gamma^2) = (-i\gamma^2\psi)^T \qquad \boxed{\gamma^2 \text{ is symmet}}$$

$$C\bar\psi C^{-1} = C\psi^\dagger C^{-1}\gamma^0 = (-i\gamma^2\psi)^T\gamma^{0T} = (-i\gamma^0\gamma^2\psi)^T$$

$$\Rightarrow C\bar\psi\psi C^{-1} = (-i\gamma^0\gamma^2\psi)^T(-i\bar\psi\gamma^0\gamma^2)^T = \underset{\text{Fermion ordering}}{(-)}(-i)^2[\bar\psi\underbrace{\gamma^0\gamma^2\gamma^0\gamma^2}_{1}\psi]^T = \bar\psi$$

$$C\bar\psi i\gamma^5\psi C^{-1} = (-i\gamma^0\gamma^2\psi)^T i\gamma^5(-i\bar\psi\gamma^0\gamma^2)^T$$

$$= (-)(-i)^2[\bar\psi\gamma^0\gamma^2 i\gamma^5\gamma^0\gamma^2\psi]^T = \bar\psi i\gamma^5\psi$$

$$C\bar\psi\gamma^\mu\psi C^{-1} = (-i\gamma^0\gamma^2\psi)^T\gamma^\mu(-i\bar\psi\gamma^0\gamma^2)^T = (-)(-i)^2[\bar\psi\gamma^0\gamma^2(\gamma^\mu)^T\gamma^0\gamma^2\psi$$

$$= -\bar\psi\gamma^\mu\psi \qquad \boxed{\gamma^5, \gamma^0, \gamma^2 \text{ sym} \quad \gamma^1, \gamma^3 \text{ anti-sy}}$$

$$C\bar\psi\gamma^\mu\gamma^5\psi C^{-1} = (-i\gamma^0\gamma^2\psi)^T\gamma^\mu\gamma^5(-i\bar\psi\gamma^0\gamma^2)^T = (-)(-i)^2(\bar\psi\gamma^0\gamma^2(\gamma^\mu)^T\gamma^5\gamma^0$$

$$= \bar\psi\gamma^\mu\gamma^5\psi$$

$$C\bar\psi i\gamma^\mu\gamma^\nu\psi C^{-1} = (-i\gamma^0\gamma^2\psi)^T i\gamma^\mu\gamma^\nu(-i\bar\psi\gamma^0\gamma^2)^T = (-)(-i)^2$$

$$(\bar\psi\gamma^0\gamma^2 i(\gamma^\mu)^T(\gamma^\nu)^T\gamma^0\gamma^2\psi]^T$$

$$= \bar\psi i\gamma^\mu\gamma^\nu\psi$$

	$\bar\psi\psi$	$i\bar\psi\gamma^5\psi$	$\bar\psi\gamma^\mu\psi$	$\bar\psi\gamma^\mu\gamma^5\psi$	$\bar\psi i\gamma^\mu\gamma^\nu\psi$	∂_μ	eA
P	1	-1	$(-)^\mu$	$-(-)^\mu$	$(-)^\mu(-)^\nu$	$(-)^\mu$	$(-)$
T	1	-1	$(-)^\mu$	$(-)^\mu$	$(-)(-)^\mu(-)^\nu$	$-(-)^\mu$	$(-)$
C	1	1	-1	1	-1	1	-1
CPT	1	1	-1	-1	1	-1	-1

$$(-)^\mu = \begin{cases} 1 & \text{for } \mu = 0 \\ -1 & \mu = 1, 2, 3 \end{cases}$$

> CPT theorem: A local Langrangian density $\mathcal{L}(\vec{x}, t)$ satisfying Lorentz invariance, then it is invariant under a combined operation of CPT.

P-breaking: $\quad \bar\psi\gamma^\mu\gamma^5\psi A_\mu$

$$\mathcal{L} = \frac{G}{\sqrt{2}} \bar\psi_1 \gamma_\mu (1-\gamma_5)\psi_2 \bar\psi_3 \gamma^\mu (1-\gamma_5)\psi_4 + h.c.$$

$$p \to n + e^- + \bar\nu_{e^-}$$

$$\to P\mathcal{L}P^{-1} = \frac{G}{2} \bar\psi_1 \gamma_\mu (1+\gamma_5)\psi_2 \bar\psi_3 \gamma^\mu (1+\gamma_5)\psi_4 \underbrace{(-)^\mu(-)^\mu}_{=1}$$

$$(CP)\mathcal{L}P^{-1}C^{-1} = \frac{G}{2} \bar\psi_1 \gamma_\mu)(-1+\gamma_5)\psi_2 \psi_3 \gamma^\mu(-1+\gamma_5)\psi_4$$

$$= \mathcal{L}$$

Example

1. Consider spin 0-particle decays $\rightarrow \gamma + \gamma$, in the center of mass frame

if $P=-1$, ~~then~~ $\Longleftrightarrow$ the polarizations are ~~is~~ perpendicular.

$P=1$, $\Longleftrightarrow$ their polarizations are parallel.

Proof: $J=0 \Rightarrow$ isotropic

$$|2\gamma\rangle = \int d^3\vec{p}\ \underbrace{\chi_{ij}(\vec{p})}_{\swarrow\ J=0\ \text{isotropic}}\ \alpha_i^\dagger(\vec{p})\ \alpha_j^\dagger(-\vec{p})\ |0\rangle$$

$\alpha_i^\dagger(\vec{p})$ represent photon polarization along $\hat{e}_i$ $(i=x,y,z)$

$= \hat{e}_i \cdot \vec{\alpha}^\dagger(\vec{p})$ $\qquad \vec{A}(p) \propto$

~~??~~ constraint $\quad P_i\,\alpha_i^\dagger(\vec{p}) = \vec{p}\cdot\vec{\alpha}^\dagger(\vec{p}) = 0$ (transvers field).

$$\chi_{ij}(\vec{p}) = A\delta_{ij} + B\epsilon_{ijk}P_k + C P_i P_j\ ,$$ A, B, C only depends on $|p|$

The C-term vanishes

$$\Rightarrow P|2\gamma\rangle = \int d^3p\ \chi_{ij}(\vec{p})\ P\alpha_i^\dagger(\vec{p})P\ P^\dagger\alpha_j^\dagger(-\vec{p})P|0\rangle$$

$$= \int d^3p\ \chi_{ij}(\vec{p})\ (-)\alpha_i^\dagger(-\vec{p})\ (-)\ \alpha_j^\dagger(-\vec{p})\ |0\rangle$$

$$= \int d^3p\ \chi_{ij}(-\vec{p})\ \alpha_i^\dagger(\vec{p})\,\alpha_j^\dagger(-\vec{p})\,|0\rangle$$

$\Rightarrow$ if $\chi_{ij} = A\delta_{ij} \Rightarrow P|2\gamma\rangle = |2\gamma\rangle \Rightarrow P=1$

$\chi_{ij} = B\epsilon_{ijk}P_k \Rightarrow \chi_{ij}(\vec{p}) = -\chi_{ij}(p) \Rightarrow P=-1$

(2) In QED, $e\bar{e}$ pair $J=0$ ~~state~~, the intrinsic parity is -1,

if $L=0$. $\Rightarrow$ $^1S_0$ (l, J) $\qquad {}^{2S+1}L_J$

$\Rightarrow$ polarization is ~~perped~~ perpendicular

If $L=1 \Rightarrow S=1,\ J=0 \quad {}^3P_0 \Rightarrow$ parity $= (-)(-)^l = 0$

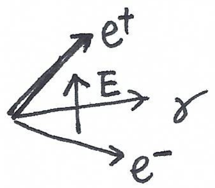
polarization is parallel.

(3) $\pi^0 \rightarrow \gamma_a + \gamma_b$ in detectors a and b

$\gamma_a \rightarrow e_a + \bar{e}_a$

$\gamma_b \rightarrow e_b + \bar{e}_b$

γ: polarization is along the e^+e^- plane.



$\Rightarrow$ γ_a and γ_b ~~are~~ polarizations are perpendicular $\Rightarrow$ π^0 - odd parity

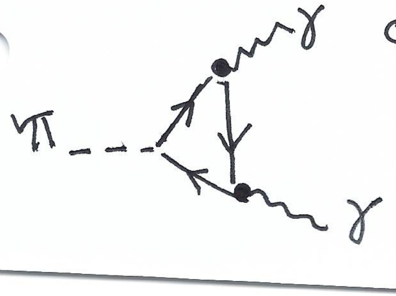
pseudo-scalar

(4) $\pi^0 \rightarrow 2\gamma$, how to construct $\mathcal{L} = ?$

$$\mathcal{L} \propto \varphi(x) F_{\mu\nu} F^{\mu\nu} = \varphi(x)(E^2 - B^2) \qquad \times$$

$$\varphi(x)\, \epsilon_{\mu\nu\lambda\delta} F^{\mu\nu} F^{\lambda\delta} = \varphi(x)\, \vec{E}\cdot\vec{B} \qquad \checkmark$$

ABJ anomaly



$\theta - \tau$' puzzle

$\theta \to \pi + \pi$

$\tau \to \pi + \pi + \pi$

Part II: Conservation of Parity Operation in Radioactive Decays

To use "Tau" and "Theta" particles themselves in these tests is impractical. However, the beta decays of radioisotopes are perfectly suited for this experimentation. To understand the meaning of the experiment on polarized nuclei, one must first examine the meaning of conservation of parity in radioactive decays.

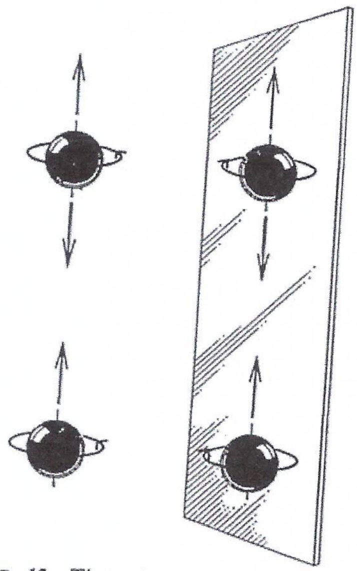
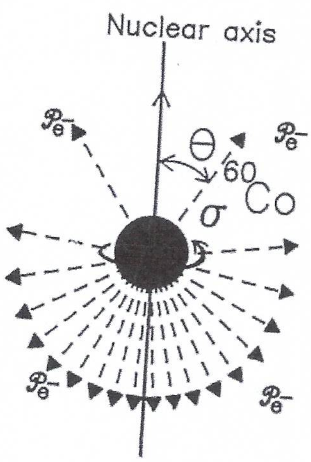


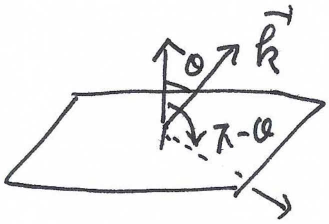
Fig. 12 The mirror reflection of a spinning ball. The image and the real object could not be distinguished because the top right one looks just like the real ball turned upside down. Reflection can be detected if there is a preferred direction.

The law of parity dictates that the physics phenomena of atomic or nuclear systems in the original and its mirror twin should be indistinguishable. Fig. 12 shows the mirror reflection of a spinning ball. If the ball ejected particles equally in both directions along its axis, the image and the real object could not be distinguished because the top right one looks just like the real ball turned upside down. However, if there is a preferred direction for the ejection of particles, then the reflection can be detected. The image at bottom cannot be mistaken for the real thing, as they have reversed handedness.

Mathematically, it states that a pseudoscalar term $\langle \sigma \cdot p \rangle$ changes sign under space inversion where p is the electron momentum and σ the spin of the nucleus. If the distribution of emitted electrons from polarized nuclei is asymmetrical (see Fig. 13) the pseudoscalar term $\langle \sigma \cdot p \rangle$ of the radioactive decay is not identically equal to zero. The pseudoscalar term $\langle \sigma \cdot p \rangle \neq 0$ will change sign under space inversion therefore the parity is not conserved.

β Particle distribution about nuclear axis





θ

Pseudoscalar Quantity $<(\sigma \cdot \mathcal{P}_{e^-})>$

$\langle \cos\theta \rangle \xrightarrow{P} \langle \cos(\pi-\theta) \rangle$

Fig. 13 σ, the spin of the nucleus; p_e the electron momentum.
If parity conservation is valid; the expectation value of $(\sigma \cdot p_e) \equiv 0$.

$$\langle\langle \sigma \cdot p_e \rangle\rangle \equiv \int dr \phi^*(r)[\sigma(r) \cdot p_e(r)]\phi(r)$$

$\Rightarrow -\langle \cos\theta \rangle$

If parity invariance is valid; then P-operation gives $P\phi(r) = \phi(-r) = \pm\phi(r)$

$= 0$

then
$$P\langle\langle \sigma \cdot p_e \rangle\rangle = \int dr \phi^*(r)[\sigma(-r) \cdot p_e(-r)]\phi(r)$$
$$= -\int dr \phi^*(r)[\sigma(r) \cdot p_e(r)]\phi(r)$$
$$= -\langle\langle \sigma \cdot p_e \rangle\rangle$$

If parity invariance is valid, $\langle\langle \sigma \cdot p_e \rangle\rangle \equiv 0$, the expectation value $\langle\langle\ \rangle\rangle$ of pseudoscalar quantity must be identically zero.

we had to prove that this asymmetry effect was not due to the strong magnetic field of the CMN crystals produced at extremely low temperatures. We also needed to show that this effect was not due to the remnant magnetization in the sample induced by the strong demagnetization field. The most clear-cut control experiment would be one in which a beta activity would be introduced into the CMN crystal, but in which the radioactive nucleus would be known not to be polarized; thus no asymmetry effect should be detected. To carry out all these experiments would take several weeks.

On Christmas Eve I returned to New York on the last train as the airport was closed because of heavy snow. I told Dr. Lee that the observed asymmetry was reproducible and huge, but we had not exhausted all experimental checks yet. When I started to make a quick rough estimate of the asymmetry parameter A, I found it was nearly -1. The asymmetry parameter A was estimated as follows:

The electron angular distribution is

$$W(\theta)=1+A\frac{\langle I_z\rangle}{I}\frac{v}{c}\cos\theta$$

$1+\cos\theta$ interference term

"θ" is the angle between the nuclear spin and electron momentum direction. The actually observed asymmetry is $\sim 25\%$

$$\frac{W(0)-W(\pi)}{W(0)+W(\pi)}=-0.25=A\frac{\langle I_z\rangle}{I}\frac{v}{c}$$

where $\frac{\langle I_z\rangle}{I}=0.65$ calculated from observed γ anisotropy,

$\frac{v}{c}\cong 0.6$ from the calibrated pulse height analysis.

The back scattering of the electrons from the CMN crystal was found in a magnetic spectrometer to be 30—35%.

Therefore $A\cong -0.25\times(0.65\times 0.60)^{-1}\times\frac{3}{2}\cong -1$.

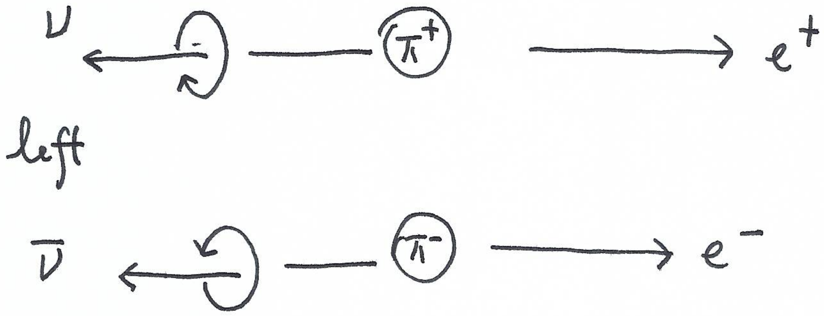
The result of $A=-1$ was the first indication that the interference between parity conserving and parity non-conserving terms in the G-T interaction Hamiltonian was close to maximum or, $C_A=C_A'$. This result is just what one should expect for a two component theory of the neutrino in a pure Gamow-Teller transition. It also implies that, in this case, *the charge conjugation is also non invariant*. Dr. Lee realized it then

① CP-conservation

$\pi^+ \longrightarrow \bar{e} + \nu_e$

$\pi^- \longrightarrow e^+ + \bar{\nu}$

breaks C and P, but not CP.



② CP-violation neutral K_L-meson

5×10^{-8} s

$$\frac{d\bar{s} - s\bar{d}}{\sqrt{2}}$$

497 M ev

$$\frac{K_L^0 \rightarrow e^+ + \pi^- + \nu_e}{K_L^0 \rightarrow e^- + \pi^+ + \bar{\nu}_e} = 1.0066.$$