Lect 7 Magnetic monopole

§ Classic magnetic monopole

§ Quantum mechanical description

— Dirac string

— monopole without string (Yang and Wu's formulation, Berry phase

Quantization of ~~m~~ monopole charge.

§ Monopole harmonics

relation to D-matrix

Fractional QHE state on a sphere.

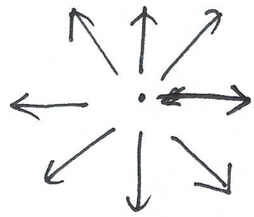
Ref: 1) P.A.M. Dirac, Proc. R. Soc. London, Ser. A 133 60 (1931)

2) C. N. Yang, Annals of New York Academy of Sciences 294, 86 (1977)

3) T.T. Wu and C. N. Yang, Nucl. Phys. B 107, 365 (1976)

§ Classic monopole

$$\vec{B} = \frac{g}{r^2}\,\hat{r}$$



$$\nabla\cdot\vec{B} = 4\pi g\,\delta^{(3)}(\vec{r}),$$

how to modify maxwell equation?

$$\nabla\cdot\vec{B} = 4\pi\rho_m(\vec{r})$$

$$\nabla\cdot\vec{E} = 4\pi\rho_e(\vec{r})$$

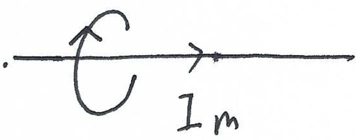
$$\nabla\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t} \quad ? - 4\pi\vec{j}_m$$

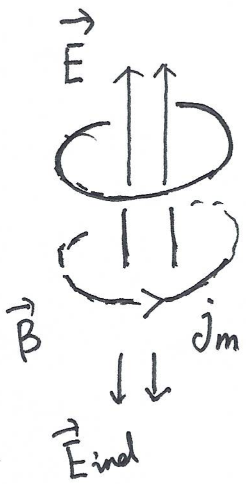
$$\nabla\times\vec{B} = \frac{1}{c}\frac{\partial\vec{E}}{\partial t} + 4\pi\vec{j}_e$$

symmetry $\vec{E}\to\vec{B}$ $\rho_e\to\rho_m$
$\vec{B}\to-\vec{E}$ $\rho_m\to-\rho_e$

imagine a world with only magnetic monopole but not electric charge. The electric field generated by a monopole current should follow the left-hand law.

The reason is as follows.





Suppose $\vec{E}$ is increasing

$$\nabla\times\vec{B} = \frac{1}{c}\frac{\partial\vec{E}}{\partial t} > 0$$

Then the induced $\vec{B}$ field is plotted as in the left.

The monopole current (if there exists a monopole wire.) should produce a counter electric field.

$$\nabla\times\vec{B} = 4\pi\,\vec{j}_e + \frac{1}{c}\frac{\partial\vec{E}}{\partial t}$$

fixed by electric charge "conservation-law". displacement current

experiment fact of Ampere's law

$$\nabla\times\vec{E} = -4\pi\,\vec{j}_m \quad -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}$$

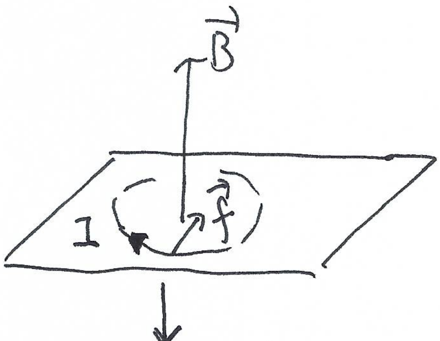
Lentz's law

counter - emf

magnetic version of Lentz law

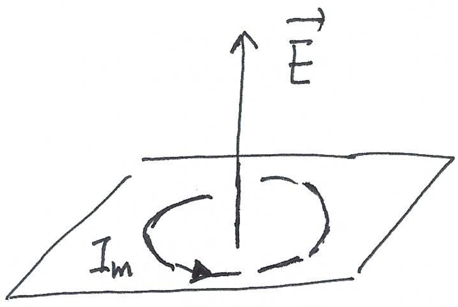
It's also consistent with monopole charge conservation.

diamagnetism



$\vec{M}\,//\,-\vec{B}$

$$\vec{f} = q\vec{E} + q\frac{\vec{v}}{c}\times\vec{B}$$

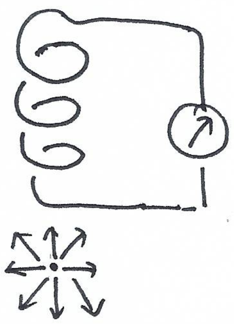


$\vec{P}\,//\,-\vec{E}$

$$\vec{f} = g\vec{B} - g\frac{\vec{v}}{c}\times\vec{E}$$

monopole current generate electric ~~mo~~ diapole moment

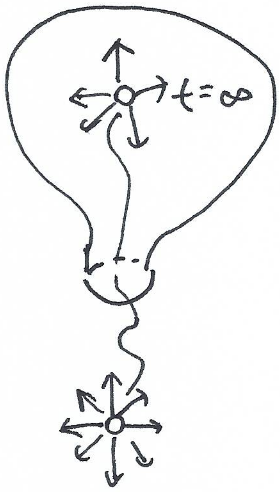
Stanford - monopole experiment



$$-L\frac{dI}{dt} = \frac{d\Phi}{dt} \Rightarrow \Delta I = -\Delta\Phi/L$$

$t \to -\infty$, $B \to 0$ set $\Phi(t=-\infty) = 0$

as $t \to \infty$, $B \to 0$, but actually $\Phi \neq 0$



$$\Phi = 4\pi g$$

$$\left.\begin{aligned} \Delta I &= \frac{4\pi g}{L} = \frac{4\pi}{eL}\, eg \\ \frac{eg}{\hbar c} &= \frac{n}{2} \end{aligned}\right\} \Rightarrow \begin{aligned} \Delta I &= \frac{2n\pi}{L}\frac{hc}{e} \\ &= \frac{n}{L}\Phi_0 \end{aligned}$$

where $\Phi_0 = hc/e$ is the fundemental flux.

西湖大學
WESTLAKE UNIVERSITY
电话 TEL | 0571-86886859
传真 FAX | 0571-85271986
地址：中国浙江省杭州市西湖区云栖小镇石龙山街18号 邮编：310024
Address: No.18,Shilongshan Road, Cloud Town, Xihu District, Hangzhou, Zhejiang Province, China. Postcode: 310024

Prob ① Angular momentum of a charge-monople system

① $\vec{B} = g\,\vec{r}/r^3$, and $\vec{F} = \frac{q}{c}\,\vec{v}\times\vec{B} = \frac{qg}{c}\,\frac{\vec{v}\times\vec{r}}{r^3}$

kinetic energy is conserved since Lorentz force does not do work

$$\frac{dT}{dt} = \frac{d}{dt}\left(\tfrac{1}{2}mv^2\right) = m\vec{v}\cdot\frac{d\vec{v}}{dt} = \vec{v}\cdot\left(\frac{\vec{v}\times\vec{r}}{r^3}\right)\frac{qg}{c} = 0$$

or v – the speed is conserved

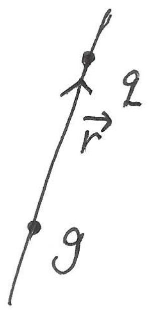
② If we use $\vec{L}' = m\vec{r}\times\vec{v}$, then

$$\frac{d}{dt}\vec{L}' = m\vec{r}\times\frac{d\vec{v}}{dt} + m\vec{v}\times\vec{v} = \frac{qg}{cr^3}\,\vec{r}\times(\vec{v}\times\vec{r})$$

$$= -\frac{qg}{mcr^3}\,\vec{r}\times\vec{L}' \neq 0$$

$$\vec{L}'\cdot\frac{d}{dt}\vec{L}' = -\frac{qg}{mcr^3}\,\vec{L}'\cdot(\vec{r}\times\vec{L}') = 0 \Rightarrow \frac{d}{dt}\vec{L}'^2 = 0$$

③ The extra contribution to the angular momentum should come from the E-M field. $\vec{L}_{em}$ should only depend on the relative displacement $\vec{r}$. There exits a rotation symmetry around the axis from "q" and "g". Then $\vec{L}_{em}$ must obey such a symmetry, i.e. $\vec{L}_{em}$ is invariant under a rotation around the line connecting "q" and "g". Hence, we denote



$\vec{L}_{em} = k\,\hat{r} = k\vec{r}/r$, where k is to be determined.

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传真 FAX | 0571-85271986
地址：中国浙江省杭州市西湖区云栖小镇石龙山街18号 邮编：310024
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Then define $\vec{L} = \vec{L}' + k\hat{r}$

$$\frac{d}{dt}\vec{L} = \frac{d}{dt}\vec{L}' + k\frac{d}{dt}(\hat{r})$$

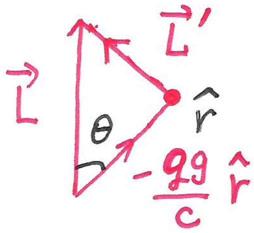
$$\frac{d\vec{L}'}{dt} = \frac{qg}{cr^3}\,\vec{r}\times(\dot{\vec{r}}\times\vec{r}) = \frac{qg}{cr^3}\left(r^2\dot{\vec{r}} - (\vec{r}\cdot\dot{\vec{r}})\,\vec{r}\right)$$

$$\vec{r}\cdot\dot{\vec{r}} = \frac{1}{2}\frac{d}{dt}(\vec{r}\cdot\vec{r}) = \frac{1}{2}\frac{d}{dt}r^2 = r\frac{d}{dt}r$$

$$\frac{d\vec{L}'}{dt} = \frac{qg}{cr^3}\left(r^2\dot{\vec{r}} - r\vec{r}\frac{d}{dt}r\right) = \frac{qg}{c}\left[\frac{1}{r}\frac{d}{dt}\vec{r} - \frac{\vec{r}}{r^2}\frac{d}{dt}r\right]$$

$$= \frac{qg}{c}\frac{d}{dt}\left(\frac{\vec{r}}{r}\right) = \frac{qg}{c}\frac{d}{dt}\hat{r} \quad \Rightarrow \text{ we take } k = -qg/c$$

$$\Rightarrow \boxed{\vec{L} = m\vec{r}\times\vec{v} - \frac{qg}{c}\hat{r}}$$



$$\vec{L}\cdot\hat{r} = -qg/c$$

$|\vec{L}'| = mvb$, where b is the closest distance between q and g

then the charge q moves on a surface of a cone, the angle of the cone $\boxed{\cot\theta = \frac{qg/c}{L'} = \frac{qg}{mvbc}}$.

i.e. $\hat{L}$ precesses around $\vec{L}$ and $L = \dfrac{qg}{c\cos\theta}$

5) when $\vec{v} = 0$, $\vec{L} = -\frac{qg}{c}\hat{r} \Rightarrow L_z = -\frac{qg}{c}$, its minimal $\frac{\hbar}{2}$

we have $\frac{qg}{c} = \frac{n}{2}\hbar \Rightarrow \boxed{\frac{qg}{\hbar c} = \frac{n}{2}}$, since $\frac{e^2}{\hbar c} = \alpha \sim \frac{1}{137}$

n is an integer

$$\Rightarrow \boxed{g/e = \frac{n}{2\alpha} \simeq 70n}$$

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电话 TEL | 0571-86886859
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地址：中国浙江省杭州市西湖区云栖小镇石龙山街18号 邮编：310024
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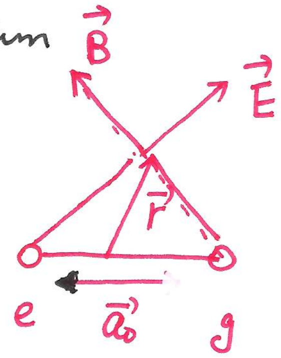
(*) Explicit check the charge-monople angular momentum



$$\vec{L}_{em} = \frac{1}{4\pi c}\int d^3\vec{r}\ \vec{r}\times(\vec{E}\times\vec{B})$$

$$\vec{E} = -\nabla\phi, \text{ where } \phi = \frac{e}{|\vec{r}-\vec{a}_0/2|}$$

$$\vec{E}\times\vec{B} = -\nabla\phi\times\vec{B} = -\nabla\times(\phi\vec{B}) \leftarrow \underline{\nabla\times(u\vec{A}) = u\nabla\times\vec{A} - \vec{A}\times\nabla u}.$$

since $\nabla\times\vec{B} = 0$ for a monople field.

Denote $\vec{W} = \phi\vec{B}$, we have

$$\vec{r}\times(\vec{E}\times\vec{B}) = -\vec{r}\times\nabla\times\vec{W} = -\nabla(\vec{r}\cdot\vec{\omega}) + (\vec{r}\cdot\vec{\nabla})\vec{W} + (\vec{\omega}\cdot\vec{\nabla})\vec{r}$$

where we have used $\underline{\nabla(\vec{A}\cdot\vec{B}) = \vec{A}\times(\nabla\times\vec{B}) + \vec{B}\times(\nabla\times\vec{A}) + (\vec{A}\cdot\vec{\nabla})\vec{B} + (\vec{B}\cdot\vec{\nabla})\vec{A}}$

look at the last two terms

$$r_j\partial_j\omega_i + \omega_j\partial_j r_i = r_j\partial_j\omega_i + \omega_j\delta_{ij} = \partial_j(r_j\omega_i) - (\partial_j r_j)\omega_i + \omega_i$$

$$= \partial_j(r_j\omega_i) - 2\omega_i$$

check $\partial_j(r_i\omega_j) = r_i\partial_j\omega_j + (\partial_j r_i)\omega_j = r_i\partial_j\omega_j + \omega_i$

$$\Rightarrow (r_j\partial_j)\omega_i + 2\partial_j(r_i\omega_j) = 2r_i(\partial_j\omega_j) + \partial_j(r_j\omega_i)$$
$$+(\omega_j\partial_j)r_i$$

$$\Rightarrow (\vec{r}\cdot\vec{\nabla})\vec{W} + (\vec{\omega}\cdot\vec{\nabla})\vec{r} = -2\partial_j(\vec{r}w_j) + 2\vec{r}(\nabla\cdot\vec{W}) + \partial_j(r_j\vec{W})$$

$$\Rightarrow \vec{r}\times(\vec{E}\times\vec{B}) = \underline{-\nabla(\vec{r}\cdot\vec{\omega}) - 2\partial_j(\vec{r}w_j)} + 2\vec{r}(\nabla\cdot\vec{W})$$
$$\underline{+\partial_j(r_j\vec{W})}$$

we can prove they are zero.

The first three terms are total derivatives,

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电话 TEL | 0571-86886859
传真 FAX | 0571-85271986
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$$\int dxdydz\ \nabla(\vec{r}\cdot\phi\vec{B}) = \int dxdydz\left[\partial_x(\phi\vec{r}\cdot\vec{B}) + \partial_y(\phi\vec{r}\cdot\vec{B}) + \partial_z(\phi\vec{r}\cdot\vec{B})\right]$$

$\phi\vec{r}\cdot\vec{B} \sim \frac{e}{r}\cdot\frac{g}{r}$ nearly an even function, and decays $1/r^2$

$$\Rightarrow \int d^3\vec{r}\ \nabla(\vec{r}\cdot\vec{w}) = 0.$$

$$\int d^3\vec{r}\ \partial_j(\vec{r}w_j) = \oiint (d\vec{S}\cdot\vec{w})\,\vec{r}\,, \qquad \vec{w} = \phi\vec{B} = \frac{eg}{r^3}\hat{r}$$

$$= \lim_{r\to\infty} 4\pi \oiint r^2\cdot\frac{eg\,d\Omega}{r^3}\ \vec{r} = 4\pi eg \oiint d\Omega\ \hat{r} = 0$$

$$\int d^3\vec{r}\ \partial_j(r_j\vec{w}) = \oiint (d\vec{S}\cdot\vec{r})\,\vec{w}$$

$$= \underbrace{4\pi \oiint d\Omega\ r^2\cdot r\cdot\frac{eg}{r^3}\hat{r}}_{\lim_{r\to\infty}} = 4\pi eg \oiint d\Omega\ \hat{r} = 0$$

$$\Rightarrow \int d^3\vec{r}\ \vec{r}\times(\vec{E}\times\vec{B}) = \int d^3\vec{r}\ 2\vec{r}(\nabla\cdot\vec{W}) = \int d^3\vec{r}\ 2\vec{r}\left((\nabla\phi)\cdot\vec{B} + \phi\,\nabla\cdot\vec{B}\right)$$

$$= \int d^3\vec{r}\left[-2\vec{r}(\vec{E}\cdot\vec{B}) + 2\vec{r}\phi\cdot 4\pi g\,\delta^{(3)}(\vec{r}+\vec{a}_0/2)\right]$$

$\vec{E}\cdot\vec{B}$ is an even function of $\vec{r}$, $\Rightarrow$ the contribution of the 1st term $=0$

$$\Rightarrow \boxed{\vec{L}_{em} = \frac{4\pi g}{4\pi c}\int 2\vec{r}\,\phi\,\delta^{(3)}(\vec{r}+\vec{a}_0/2)\,d^3\vec{r} = -\frac{eg}{c}\vec{a}_0/a_0 = -\frac{eg}{c}\hat{a}_0.}$$

$\hat{a}_0$ is the vector from the monopole to the charge the high energy state

- $$|\psi_H^{(1)}(\hat{n})\rangle = \frac{1}{N} P_- \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{N} \begin{pmatrix} 1-n_3 \\ n_1 - i n_2 \end{pmatrix} = \frac{1}{\sqrt{2(1-n_3)}} \begin{pmatrix} 1-n_3 \\ n_1 - i n_2 \end{pmatrix}$$

or we can use that state $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ for projection

$$|\psi_L^{(2)}(\hat{n})\rangle = \frac{1}{N} \begin{pmatrix} n_1 - i n_2 \\ 1-n_3 \end{pmatrix} = \frac{1}{\sqrt{2(1-n_3)}} \begin{pmatrix} n_1 - i n_2 \\ 1-n_3 \end{pmatrix}$$

$$|\psi_H^{(2)}(\hat{n})\rangle = \frac{1}{N} \begin{pmatrix} n_1 + i n_2 \\ 1+n_3 \end{pmatrix} = \frac{1}{\sqrt{2(1+n_3)}} \begin{pmatrix} n_1 + i n_2 \\ 1+n_3 \end{pmatrix}$$

- Next we calculate Berry connection / Berry curvature.

For low energy level and gauge 1 $\Rightarrow$

$$\frac{d}{dt} |\psi_L^{(1)}(\hat{n}(t))\rangle = \frac{1}{\sqrt{2(1+n_3)}} \begin{pmatrix} \dot{n}_3 \\ \dot{n}_1 + i\dot{n}_2 \end{pmatrix} + \frac{-\dot{n}_3}{2\sqrt{2}(1+n_3)^{3/2}} \begin{pmatrix} 1+n_3 \\ n_1 + i n_2 \end{pmatrix}$$

$$\langle \psi_L^{(1)}(\hat{n}(t)) | \frac{d}{dt} | \psi_L^{(1)}(\hat{n}(t))\rangle = \frac{1}{2(1+n_3)} \left[(1+n_3)\dot{n}_3 + (n_1 - i n_2)(\dot{n}_1 + i\dot{n}_2) \right]$$

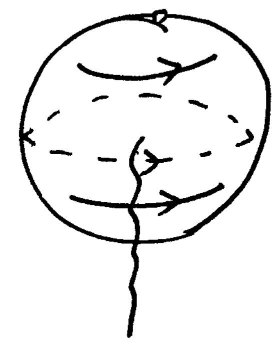
$$- \frac{1}{4(1+n_3)^2} \left[(1+n_3)^2 \dot{n}_3 + (n_1^2 + n_2^2)\dot{n}_3 \right]$$

- $$= \frac{1}{2(1+n_3)} \left(\dot{n}_3 + n_3\dot{n}_3 + n_1\dot{n}_1 + n_2\dot{n}_2 + i(n_1\dot{n}_2 - n_2\dot{n}_1) - \frac{1}{2}\left[(1+n_3)\dot{n}_3 + (1-n_3)\dot{n}_3 \right] \right.$$

$$= \frac{i}{2(1+n_3)} (n_1\dot{n}_2 - n_2\dot{n}_1)$$

$$\gamma_L = -i\int dt \; A(t) = -i\int d\vec{n} \; \langle \psi_L^{(1)} | \nabla_n | \psi_L^{(1)} \rangle$$

$$= \int dn_\alpha \; A_\alpha(\hat{n})$$



where $\boxed{A_\alpha dn_\alpha = \frac{1}{2(1+n_3)}(n_1 dn_2 - n_2 dn_1)}$

we can choose $A_1 = \frac{-n_2}{2(1+n_3)}$, $A_2 = \frac{n_1}{2(1+n_3)}$, $A_3 = 0$

$$\Rightarrow \left.\begin{aligned} A_1 &= \frac{-\sin\theta\sin\varphi}{2(1+\cos\theta)} = -\frac{\sin\varphi}{2}\, tg\frac{\theta}{2} \\ A_2 &= +\frac{\cos\varphi}{2}\, tg\frac{\theta}{2}, \quad A_3 = 0 \end{aligned}\right\} \Rightarrow \underset{(\hat{n})}{\vec{A}} = \frac{1}{2} tg\frac{\theta}{2}\, \hat{e}_\varphi$$

$\vec{A}$ has a singular point at south pole $n_3 = -1$.

using the formula

$$\nabla \times \vec{A} = \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(\sin\theta A_\phi) - \frac{\partial A_\theta}{\partial\varphi}\right)\hat{e}_r$$

$$+ \left[\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\varphi} - \frac{1}{r}\frac{\partial}{\partial r}(rA_\varphi)\right]\hat{e}_\theta$$

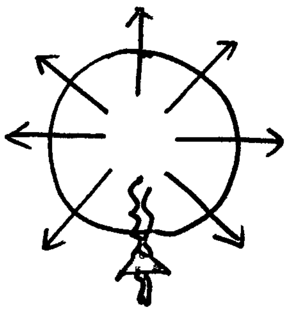
$$+ \left[\frac{1}{r}\frac{\partial}{\partial r}(rA_\theta) - \frac{1}{r}\frac{\partial A_r}{\partial\theta}\right]\hat{e}_\varphi$$

plug in $\Rightarrow$ $\nabla \times \vec{A} = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left[\frac{1}{2}\, 2\sin^2\frac{\theta}{2}\right]\hat{e}_r = \frac{1}{2}\hat{e}_r$

Magnetic Monople field !!!

How can a curl describle a monople field? $\nabla\cdot(\nabla\times A)=0$?
$\nabla\cdot(\hat{e}_r)\neq 0$

Singularity: Dirac string / Dirac monopole



$\vec{A}$ is not well-define over the entire sphere

Let us choose a small path around south pole

$\theta = \pi - 0^+$, $\varphi: 0\to 2\pi$

$$\oint A^a dn_a = -2\pi \sin\theta \ \frac{1}{2} \operatorname{tg}\frac{\theta}{2} \Big|_{\theta\to\pi}$$

$$= -\pi \ 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} \cdot \sin\frac{\theta}{2}/\cos\frac{\theta}{2} = -2\pi$$

$$\Rightarrow \nabla\times\vec{A} = \frac{1}{2}\left[1 - 4\pi\,\delta(\vec{r} = \text{south pole})\right]\hat{e}_r$$

Dirac string

however, the Dirac string should not be physical, because we have rotational symmetry. $\vec{B}$ should be uniform over the entire sphere. This is the artifact that, we insist to use vector potential to discrible a monopole field.

Let us recalculate the Berry connection and Berry curvature, but use a different gauge
$$|\psi_L^{(2)}\rangle = \frac{1}{\sqrt{2(1-n_3)}}\begin{pmatrix} n_1 - in_2 \\ 1-n_3 \end{pmatrix}$$

$$\frac{d}{dt}|\psi_L^{(2)}(\hat{n}(t))\rangle = \frac{1}{\sqrt{2(1-n_3)}}\begin{pmatrix} \dot{n}_1 - i\dot{n}_2 \\ -\dot{n}_3 \end{pmatrix} + \frac{\dot{n}_3}{2\sqrt{2}(1-n_3)^{3/2}}\begin{pmatrix} n_1 - in_2 \\ 1-n_3 \end{pmatrix}$$

Similarly we will get

$$\langle \psi_L^{(2)} | \frac{d}{dt} | \psi_L^{(2)} \rangle = \frac{1}{2(1-n_3)} \left[(n_1 + i n_2)(\dot{n}_1 - i \dot{n}_2) + (1-n_3)(-)\dot{n}_3 \right]$$

$$+ \frac{1}{4(1-n_3)^2} \left[(n_1^2 + n_2^2)\dot{n}_3 + (1-n_3)^2 \dot{n}_3 \right]$$

$$= \frac{-i}{2(1-n_3)} \left[n_1 \dot{n}_2 - n_2 \dot{n}_1 \right]$$

$$\tilde{A}_\alpha d n_\alpha = \frac{-1}{2(1-n_3)} (n_1 dn_2 - n_2 dn_1) \Rightarrow \tilde{A} = -\frac{1}{2} \operatorname{ctg} \frac{\theta}{2} \hat{e}_\varphi$$

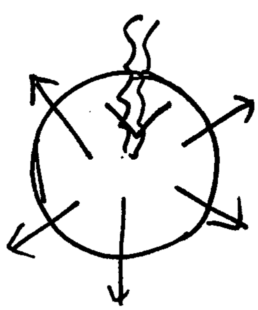
$$\tilde{A}_1 = \frac{n_2}{2(1-n_3)} = \frac{\sin\theta \sin\varphi}{2(1-\cos\theta)} = \frac{\sin\varphi}{2} \operatorname{ctg}\frac{\theta}{2}$$

$$\tilde{A}_2 = \frac{-n_1}{2(1-n_3)} = \frac{-\sin\theta\cos\varphi}{2(1-\cos\theta)} = -\frac{\cos\varphi}{2} \operatorname{ctg}\frac{\theta}{2}$$

$$\nabla \times \tilde{A} = \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(-\frac{1}{2} \, 2\cos^2\frac{\theta}{2} \right) \hat{e}_r = \frac{1}{2} \hat{e}_r$$

the singular point is at the north pole.

Let us choose a small loop at $\theta = 0^+$



$$\oint \tilde{A}_\alpha dn_\alpha = \sin\theta \left(-\frac{1}{2}\right) \operatorname{ctg}\frac{\theta}{2} \cdot 2\pi \Big|_{\theta \to 0^+} = -2\pi$$

$$\nabla \times \tilde{A} = \frac{1}{2} \left[1 - 2\pi \delta(\hat{n} = \text{north pole}) \right] \hat{e}_r$$

Dirac monopole defines a topologically non-trivial $U(1)$ fiber bundle over the two-sphere S^2. We are not able define well-defined $\vec{A}$ over the entire sphere. If you insist to use a single definition of $\vec{A}$, you suffer from the unphysical Dirac string. (An analogy in differential geometry is that, you cannot define a non-singular coordinate over a sphere. This is a result of the intrinsic curvature).

The best job we can do: cut the sphere into two hemisphere.

A_α is well-defined in northern hemisphere $\Big\}$ $\Leftarrow$ Locally
$\tilde{A}$ is well-defined in the south hemisphere $\Big\}$ but not globally

They overlap at the equator, in which

$$\tilde{A}_\alpha = A_\alpha - i\omega^{-1}\partial_\alpha \omega \qquad \omega = e^{-i\varphi} \leftarrow \text{gauge transform}$$

At equation $|\psi_L^{(1)}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ n_1+in_2 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ e^{i\varphi} \end{pmatrix} = e^{-i\varphi}|\psi_u^{(1)}\rangle$

$$|\psi_L^{(2)}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} n_1-in_2 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{-i\varphi} \\ 1 \end{pmatrix}$$

ω is also the group space of $U(1)$

This maps the equator $S^1 \longrightarrow U(1)$ group space.

We can define the winding number C_1

$$2\pi C_1 = i\oint_{S_1} \omega^{-1}\partial_\alpha \omega \, dn_\alpha$$

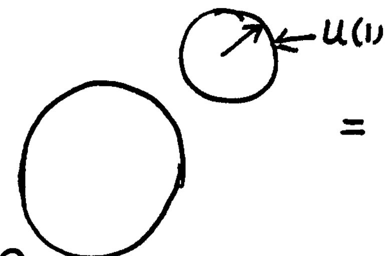
$$2\pi C_1 = \oint_{S_1} A^a dn_a - \oint_{S_1} \tilde{A}^a dn_a$$

$$= \underset{\text{north}}{\iint d\vec{S}\cdot\vec{B}} + \underset{\text{south}}{\iint d\vec{S}\cdot\vec{B}} = \oiint d\vec{S}\cdot\vec{B}$$

Winding number

C_1 has to be integer : $i \oint_{S_1} \omega^{-1} \partial_\alpha \omega \, dn_\alpha$

generally speaking $\omega = e^{-i\delta(\varphi)}$



S_1 equation

$$= i\int_0^{2\pi} d\varphi \; e^{i\delta(\varphi)} \, \partial_\varphi \, e^{-i\delta(\varphi)}$$

$$= \int_0^{2\pi} d\varphi \; \partial_\varphi \delta(\varphi) = \oint d\delta = 2\pi C_1$$

angle is multiple valued.

$\boxed{\pi_1(U(1)) = \mathbb{Z}}$

$$\Rightarrow \oiint d\vec{S}\cdot\vec{B} = 2\pi C_1$$

the first chern number!

For the Low energy level $C_1 = 1$.

Similary, we can repeat the above calculation, and arrive at the Berry connection / curvature for the high energy level, which is also a monopole with opposite charge -1.

$$\boxed{\oiint \vec{B}_H \cdot d\vec{S} = \oint_{S_1} A_H^a dn_a - \oint_{S_1} \tilde{A}_H^a dn_a = -2\pi}$$

In the subspace of $f(\Omega)\begin{pmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}e^{i\varphi}\end{pmatrix}$, we check the reduce expression of $\vec{J}=\vec{L}+\frac{\vec{\sigma}}{2}$. Denote $\chi_+(\Omega)=\begin{pmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}e^{i\varphi}\end{pmatrix}$

$\Rightarrow$ for the matrix element between two states in this subspace, we have

$\psi_1(\Omega)=f(\Omega)\chi_+(\Omega)$ and $\psi_2(\Omega)=g(\Omega)\chi_+(\Omega)$

$$\Rightarrow \langle\psi_1(\Omega)|\vec{J}|\psi_2(\Omega)\rangle=\int d\Omega\; f^*(\Omega)\underbrace{\chi_+^\dagger(\Omega)\;\vec{L}\;\chi_+(\Omega)g(\Omega)}+f^*(\Omega)\frac{\hat{\Omega}}{2}g(\Omega)$$

$$\vec{L}=\vec{r}\times(-i\hbar\vec{\nabla}),\qquad -i\hbar\vec{\nabla}\,(\chi_+(\Omega)g(\Omega))$$
$$=-i\hbar(\vec{\nabla}\chi_+(\Omega))g(\Omega)-i\hbar\chi_+(\Omega)\vec{\nabla}g(\Omega)$$

$$\Rightarrow f^*(\Omega)\chi_+^\dagger(\Omega)[\vec{r}\times(-i\hbar\vec{\nabla})]\chi_+(\Omega)g(\Omega)$$
$$=f^*(\Omega)\,\vec{r}\times\langle\chi_+^\dagger(\Omega)|-i\hbar\nabla|\chi_+(\Omega)\rangle g(\Omega)+f^*(\Omega)(-i\hbar)\vec{r}\times\vec{\nabla}g(\Omega)$$
$$=f^*(\Omega)\vec{r}\times[-i\hbar\vec{\nabla}-i\hbar\langle\chi_+^\dagger(\Omega)|\nabla|\chi_+(\Omega)\rangle]g(\Omega)$$

$\Rightarrow$ the projected $\vec{J}=\vec{r}\times(\vec{p}-\vec{A})+\frac{\hbar}{2}\hat{e}_r \quad\Rightarrow\quad g=\frac{-1}{2}$

$$\vec{A}=i\hbar\langle\chi_+^\dagger(\Omega)|\nabla|\chi_+(\Omega)\rangle$$

$$\vec{\nabla}=\hat{e}_r\frac{\partial}{\partial r}+\hat{e}_\theta\frac{1}{r}\partial_\theta f+\frac{1}{r\sin\theta}\hat{e}_\varphi\partial_\varphi f$$

$$\nabla|\chi_+(\Omega)\rangle=\begin{pmatrix}\hat{e}_\theta\frac{1}{r}(-)\sin\frac{\theta}{2}\cdot\frac{1}{2}\\ \hat{e}_\theta\frac{1}{r}\cos\frac{\theta}{2}\cdot\frac{e^{i\varphi}}{2}+\sin\frac{\theta}{2}\frac{i}{r\sin\theta}e^{i\varphi}\,\hat{e}_\varphi\end{pmatrix}$$

$$\langle\chi_+(\Omega)|\nabla|\chi_+(\Omega)\rangle=\frac{\hat{e}_\theta}{r}\cancel{\left[(-)\cos\frac{\theta}{2}\sin\frac{\theta}{2}\frac{1}{2}+\sin\frac{\theta}{2}\cos\frac{\theta}{2}\frac{1}{2}\right]}+i\frac{\sin^2\frac{\theta}{2}}{r\sin\theta}\hat{e}_\varphi=\frac{i}{2r}tg\frac{\theta}{2}\hat{e}_\varphi$$

$$\Rightarrow \vec{A} = -\frac{\hbar}{2r}\frac{1-\cos\theta}{\sin\theta}\hat{e}_\varphi$$

Compare the usual monopole problem. $\vec{P} - \frac{e}{c}\vec{A} \leftarrow \vec{A} = \frac{g}{r}\frac{1-\cos\theta}{\sin\theta} e_\varphi$

$$\Rightarrow \frac{eg}{c} = -\frac{\hbar}{2} \Rightarrow g = -1/2 .$$

Similarly for the subspace of $f(\nu_2)\begin{pmatrix}-\sin\frac{\theta}{2}e^{-i\varphi}\\ \cos\frac{\theta}{2}\end{pmatrix}$, the projected

$$L = \vec{r}\times(\vec{p}-\vec{A}) - \frac{\hbar}{2}\hat{e}_r \quad \text{wher} \quad \vec{A} = \frac{\hbar}{2r}\frac{1-\cos\theta}{\sin\theta}\hat{e}_\varphi .$$

Positive monopole charge.

So a little counter-intuitive, that the monopole charge and hedgoe hog config,  $\to g = -1/2$.