Lect 8 Quantum top and monopole harmonics

§ Definition of monopole harmonics

§ Quantum top's wavefunction

§ Relation to Wigner D-matrix

§ FQHE state on a sphere

Ref:

T. T. Wu and C. N. Yang. Nucl. Phys. B 107, 365 (1976)

Lect 2 Monopole harmonics

$$H = \frac{(P - \frac{e}{c}A)^2}{2m} \quad \text{where } \vec{A} = \frac{g}{r}\frac{\vec{n}\times\vec{r}}{r + (\vec{r}\cdot\hat{n})} = \frac{g}{r}\frac{1-\cos\theta}{\sin\theta}\hat{e}_\varphi$$

check that $\nabla\times\vec{A} = \frac{g}{r^2}\hat{e}_r$.

Use the formula $\nabla\times\vec{A} = \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(\sin\theta A_\phi) - \frac{\partial A_\theta}{\partial\varphi}\right)\hat{e}_r$

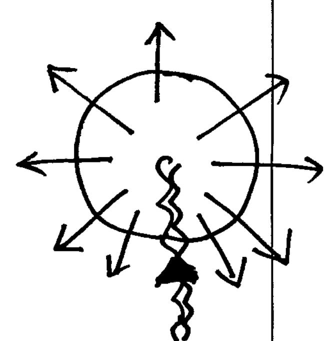
$$+ \left[\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\varphi} - \frac{1}{r}\frac{\partial}{\partial r}(rA_\varphi)\right]\hat{e}_\theta + \left[\frac{1}{r}\frac{\partial}{\partial r}[rA_\theta] - \frac{1}{r}\frac{\partial}{\partial\theta}A_r\right]\hat{e}_\varphi$$

$$\Rightarrow \quad \nabla\times\vec{A} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{g}{r}(1-\cos\theta)\right) = \frac{g}{r^2}\hat{e}_r.$$

Dirac string: singularity at the south pole.

$\oint\vec{A}\cdot d\vec{l} = -4\pi g$

for infinite-simal loop around south pole.



Electron goes around the Dirac string and then picks up a phase $\frac{eg}{\hbar c}\cdot 4\pi$. If such a phase is $2n\pi$, then this string is invisible $\Rightarrow$ $\frac{eg}{\hbar c}4\pi = 2n\pi \Rightarrow \boxed{\frac{eg}{c} = \frac{n}{2}\hbar}$

Charge quantization

From classic. elern - manopole system, we learned that $\frac{eg}{c}$ is the $\overbrace{\text{angular}}^{}$ momentum of such a system, minimum its minimum value is $\hbar/2$ according to quantum mechanics.

Define mechanical angular momentum $\vec{\Lambda} = \vec{r} \times (\vec{P} - \frac{e}{c}\vec{A})$.

$\vec{\Lambda}$ does not obey the commutation relation of angular mometum.

Please explicitly check that $H = \frac{(P - \frac{e}{c}A)^2}{2m}$ can be expressed as

$$H = \frac{-\hbar^2}{2mr^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\cdots\right) + \frac{\hbar^2\vec{\Lambda}^2}{2mr^2}\right].$$

(I leave it as a homework problem).

However, the spectra of the angular $\underbrace{\text{part are}}_{\hbar^2\vec{\Lambda}^2}$ no-longer $l(l+1)\hbar^2$.

we define $\vec{L} = \vec{r} \times (\vec{P} - \frac{e}{c}\vec{A}) - \frac{eg}{c}\hat{r}$ $\vec{L}$ satisfies the commutation relation of angular momentum, i.e.

($\hat{r}$ is the unit vector of $\vec{r}/r$)

$$\boxed{[L_i, L_j] = i\varepsilon_{ijk}L_k}$$

(I leave it as another home work problem).

we also have the following identities $\boxed{\vec{\Lambda}\cdot\hat{r} = \hat{r}\cdot\vec{\Lambda} = 0.}$

(Please check as an exercise!)

Then we have

$$\Lambda^2 = \left[\vec{L} + \frac{eg}{c}\hat{r}\right]^2 = L^2 + \left(\frac{eg}{c}\right)^2 + \frac{eg}{c}(\vec{L}\cdot\hat{r} + \hat{r}\cdot\vec{L})$$

$$\vec{L}\cdot\hat{r} = \left[\vec{\Lambda} - \frac{eg}{c}\hat{r}\right]\cdot\hat{r} = -\frac{eg}{c}, \quad \hat{r}\cdot\vec{L} = -\frac{eg}{c}$$

$$\Rightarrow \vec{\Lambda}^2 = \vec{L}^2 - \left(\frac{eg}{c}\right)^2, \quad \text{set } \frac{eg}{c} = \hbar q, \text{ we have}$$

(q can be half or integers)

$$\boxed{H = -\frac{\hbar^2}{2mr^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\cdots\right)\right] + \frac{\hbar^2}{2mr^2}\left[\vec{L}^2 - \hbar^2 q^2\right]}$$

By a litte algebra, and use the expressin in the spherical coordinate

$$\vec{p} = -i\hbar\left[\hat{e}_r\frac{\partial}{\partial r} + \frac{\hat{e}_\theta}{r}\frac{\partial}{\partial\theta} + \frac{\hat{e}_\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}\right]$$

$$\Rightarrow \boxed{\vec{L} = \vec{r}\times(\vec{p} - \frac{e}{c}\vec{A}) - \hbar q\hat{r} = \frac{\hbar}{\sin\theta}\left[i\frac{\partial}{\partial\varphi} + q(1-\cos\theta)\right]\hat{e}_\theta - i\hbar\frac{\partial}{\partial\theta}\hat{e}_\varphi - \hbar q\,\hat{e}_r}$$

Change to Cartisan coordinates, by using

$$\hat{e}_r = \sin\theta\cos\varphi\,\hat{e}_x + \sin\theta\sin\varphi\,\hat{e}_y + \cos\theta\,\hat{e}_z$$

$$\hat{e}_\theta = \cos\theta\cos\varphi\,\hat{e}_x + \cos\theta\sin\varphi\,\hat{e}_y - \sin\theta\,\hat{e}_z$$

$$\hat{e}_\varphi = -\sin\varphi\,\hat{e}_x + \cos\varphi\,\hat{e}_y$$

we have

$$\boxed{\begin{aligned} L_+ &= L_x + iL_y = \hbar e^{i\varphi}\left[\frac{\partial}{\partial\theta} + i\,\mathrm{ctg}\,\theta\frac{\partial}{\partial\varphi} - q\frac{1-\cos\theta}{\sin\theta}\right] \\ L_- &= L_x - iL_y = \hbar e^{-i\varphi}\left[-\frac{\partial}{\partial\theta} + i\,\mathrm{ctg}\,\theta\frac{\partial}{\partial\theta} - q\frac{1-\cos\theta}{\sin\theta}\right] \\ L_z &= -i\hbar\frac{\partial}{\partial\varphi} - \hbar q \end{aligned}}$$

Hw problem: please derive these formulas in the boxes.

Also by a little algebra, we have

$$\boxed{\frac{L^2}{\hbar^2} = \frac{-1}{\sin\theta}\left[\sin\theta\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta}\quad\right] + \frac{1}{\sin^2\theta}\left[i\frac{\partial}{\partial\varphi} + q(1-\cos\theta)\right]^2 + q^2}$$

~~Seek eigenstates $Y_{q,jm}(\theta,\varphi)$ satisfying~~

$$\boxed{\begin{aligned} L^2\, Y_{q,jm}(\theta,\varphi) &= j(j+1)\hbar^2\, Y_{q,jm}(\theta,\varphi) \\ L_z\, Y_{q,jm}(\theta,\varphi) &= m\hbar\, Y_{q,jm}(\theta,\varphi) \end{aligned}}$$

monopole harmonics

where $j = |q|, |q|+1, |q|+2, \cdots\cdots$

☆ Now we need to use our knowledge of D-matrix, which is also the wave-functions of rotating tops. We will build up the connetion between monopole harmonics and D-matrics.

For lecture 1, we know that the top wavefuction $\psi^{top}_{J;mk}(\alpha\beta\gamma) = \sqrt{\frac{2j+1}{8\pi^2}}\, D^{*j}_{mk}(\alpha\beta\gamma)$ which is the eigenstates for the angular momentum operators $L^2_{top}(\alpha\beta\gamma)$ and $L_{z,top}(\alpha\beta\gamma)$. We will see how to identify L^2_{top}, $L_{z,top}$ and ψ^{top} with the monopole harmonics $Y_{q,jm}(\theta,\varphi)$. Apparently, a major difference is that top has three Eulerian angles, while monopole harmonics has two angular variables.

Let us start with $\left[D^j_{m-q}(\alpha\beta\gamma)\right]^* = e^{im\alpha - iq\gamma}\, d^j_{m-q}(\beta)$

and we know that it satisfies

$$L_{z,top}\left[D^j_{m-q}(\alpha\beta\gamma)\right]^* = m\hbar\left[D^j_{m-q}(\alpha\beta\gamma)\right]^*$$

or $$-i\hbar\frac{\partial}{\partial\alpha}\left[e^{im\alpha-iq\gamma}\, d^j_{m-q}(\beta)\right] = m\hbar\left[e^{im\alpha-iq\gamma}\, d^j_{m-q}(\beta)\right]$$

but if we at the beginning set $\gamma = -\alpha$ before taking $\frac{\partial}{\partial \alpha}$, we have

$$-i\hbar \frac{\partial}{\partial \alpha} \left[e^{i(m+q)\alpha} d^{j}_{m-q}(\beta) \right] = (m+q)\hbar \left[e^{i(m+q)\alpha} d^{j}_{m-q}(\beta) \right]$$

$$\Rightarrow \boxed{\left(-i\hbar \frac{\partial}{\partial \alpha} - q\hbar\right) \left[e^{i(m+q)\alpha} d^{j}_{m-q}(\beta) \right] = m\hbar \left[e^{i(m+q)\alpha} d^{j}_{m-q}(\beta) \right]}$$

Again for top's $L_{top,+} = L_{top,x} + i L_{top,y}$

$$= i\hbar \left[+e^{i\alpha} \cot\beta \frac{\partial}{\partial \alpha} - i e^{+i\alpha} \frac{\partial}{\partial \beta} - \frac{e^{i\alpha}}{\sin\beta} \frac{\partial}{\partial \gamma} \right]$$

$$L_{top,+} \left[D^{j}_{m-q}(\alpha\beta\gamma) \right]^* = \sqrt{(j-m)(j+m+1)} \left[D^{j}_{m+1,-q}(\alpha\beta,\gamma) \right]^*$$

$$i\hbar \{ e^{i\alpha} [\cot\beta \frac{\partial}{\partial \alpha} \quad i\frac{\partial}{\partial \beta} - \frac{1}{\sin\beta} \frac{\partial}{\partial \gamma}] \left[e^{i(m\alpha - q\gamma)} d^{j}_{m-q}(\beta) \right]$$

$$= \sqrt{(j-m)(j+m+1)} \left[e^{i[(m+1)\alpha - q\gamma]} d^{j}_{m+1,-q}(\beta) \right]$$

$$\Rightarrow \hbar \left[-\cot\beta \; m + \frac{\partial}{\partial \beta} - \frac{q}{\sin\beta} \right] d^{j}_{m-q}(\beta) = \sqrt{(j-m)(j+m+1)} \; d^{j}_{m-q}(\beta)$$

$$\hbar \left[-\cot\beta \,(m+q) + \frac{\partial}{\partial \beta} - \frac{q}{\sin\beta}(1-\cos\beta) \right] d^{j}_{m-q}(\beta) = \sqrt{(j-m)(j+m+1)} \; d^{j}_{m-q}(\beta)$$

$$\Rightarrow \boxed{\begin{aligned} &\hbar e^{i\alpha} \left[i\cot\beta \frac{\partial}{\partial \alpha} + \frac{\partial}{\partial \beta} - q\frac{1-\cos\beta}{\sin\beta} \right] \left[e^{i(m+q)\alpha} d^{j}_{m-q}(\beta) \right] \\ &= \sqrt{(j-m)(j+m+1)} \left[d^{j}_{m+1,-q}(\beta) \, e^{i(m+q+1)\alpha} \right] \end{aligned}}$$

thus by seting $\gamma = -\alpha$, the D-matrix $\left[D^j_{m,-q}(\alpha\,\beta\,\gamma)\right]^*$

and identify $\theta = \beta$, $\varphi = \alpha$ $\quad \rightarrow \left[D^j_{m,-q}(\varphi, \theta, -\varphi)\right]^*$

$$\Rightarrow \quad \hbar e^{i\varphi}\left[\frac{\partial}{\partial\theta} + i\,\mathrm{ctg}\,\theta\frac{\partial}{\partial\varphi} - q\frac{1-\cos\theta}{\sin\theta}\right]\left[D^j_{m,-q}(\varphi,\theta,-\varphi)\right]^* = \sqrt{(j-m)(j+m+1)}\left[D^j_{m+1,-q}(\varphi,\theta,-\varphi)\right]^*$$

$$\left[-i\hbar\frac{\partial}{\partial\varphi} \mp \hbar q\right]\left[D^j_{m,-q}(\varphi,\theta,-\varphi)\right]^* = m\hbar\,\bar{D}^j_{m-q}(\varphi,\theta,-\varphi)$$

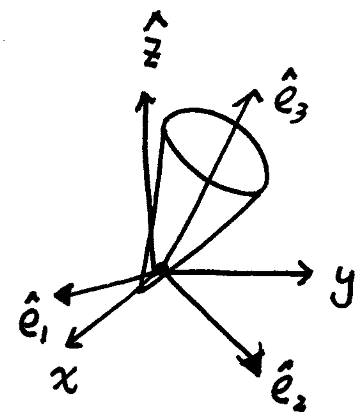
we anclude

$$\boxed{Y_{q,jm} = \sqrt{\frac{2j+1}{4\pi}}\left[D^j_{m-q}(\varphi,\theta,-\varphi)\right]^*}$$

please pay attention to the normalization factor, prove it!

Lect 1 D- matrix as rotation wavefunctions — spinning top

Let us consider a rigid rotor, how to describe its rotation wavefunction in a quantum mechanical way? Physically, this rotor can be a molecule. Now, we are quantizing the motion of a top.



The configuration space of a top can be denoted by the Eulerian angles (α, β, γ).

The relation between the body frame $(e_1 e_2 e_3)$ and the lab frame $(x\, y\, z)$ is

$$(\hat{e}_1, \hat{e}_2, \hat{e}_3) = (x, y, z) \begin{bmatrix} \cos\alpha\cos\beta\cos\gamma - \sin\alpha\sin\gamma, & -\cos\alpha\cos\beta\sin\gamma - \sin\alpha\cos\gamma, & \cos\alpha\sin\beta \\ \sin\alpha\cos\beta\cos\gamma + \cos\alpha\sin\gamma, & -\sin\alpha\cos\beta\sin\gamma + \cos\alpha\cos\gamma, & \sin\alpha\sin\beta \\ -\sin\beta\cos\gamma, & \sin\beta\sin\gamma, & \cos\beta \end{bmatrix}$$

$T(\alpha\,\beta\,\gamma) =$

SO(3) matrix. — special orthogonal matrix.

Ex: prove the above relation following the definition of Eulerian angles (α, β, γ). Hint: Check the rotation matrice for x, y, z-axes, and perform the matrice product.

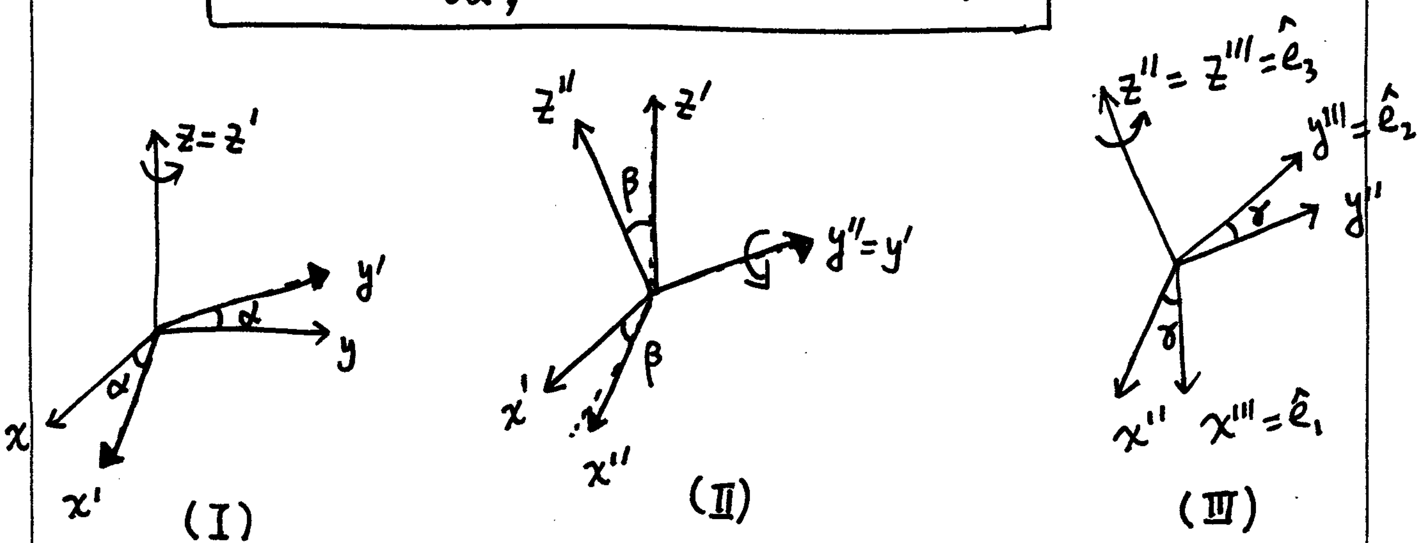
~~The angular momenta in the Lab frame is defined as~~

L_x, L_y, L_z, which are conserved quantities, and we will prove it later. Q_1, Q_2, Q_3 are angular momentum components projections on e_1, e_2 and e_3 axes. We know that from classic mechanics

$$H = \frac{Q_1^2}{2I_1} + \frac{Q_2^2}{2I_2} + \frac{Q_3^2}{2I_3}.$$ (a free top, no external torque).

Next we derive the expressions of $L_{x,y,z}$ and $Q_{1,2,3}$ in terms of α, β, γ. According to the definition,

$$\boxed{\hat{L}_z = -i\hbar\frac{\partial}{\partial\alpha}, \qquad \hat{Q}_3 = -i\hbar\frac{\partial}{\partial\gamma}}$$



According to (II), $\hat{L}_{y'} = -i\hbar\frac{\partial}{\partial\beta}$, and $\hat{y}' = \cos\alpha\,\hat{y} - \sin\alpha\,\hat{x}$

$$\Rightarrow \quad \cos\alpha\,\hat{L}_y - \sin\alpha\,\hat{L}_x = \hat{L}_{y'} = -i\hbar\frac{\partial}{\partial\beta} \qquad (*)$$

$$\hat{z}'' = \cos\beta\,\hat{z} + \sin\beta\,\hat{x}' = \cos\beta\,\hat{z} + \sin\beta(\cos\alpha\,\hat{x} + \sin\alpha\,\hat{y})$$

$$\Rightarrow \hat{L}_{z''} = \hat{Q}_3 = -i\hbar\frac{\partial}{\partial\gamma} = \cos\beta\,\hat{L}_z + \sin\beta\cos\alpha\,\hat{L}_x + \sin\beta\sin\alpha\,\hat{L}_y$$

$$\Rightarrow \cos\alpha\,\hat{L}_x + \sin\alpha\,\hat{L}_y = -\frac{i\hbar}{\sin\beta}\frac{\partial}{\partial\gamma} + \cot\beta\; i\hbar\frac{\partial}{\partial\alpha} \qquad (**)$$

from (*) and (**), we arrive at

$$\boxed{\begin{aligned}\hat{L}_x &= -i\hbar\left[-\cos\alpha\cot\beta\frac{\partial}{\partial\alpha} - \sin\alpha\frac{\partial}{\partial\beta} + \frac{\cos\alpha}{\sin\beta}\frac{\partial}{\partial\gamma}\right]\\ \hat{L}_y &= -i\hbar\left[-\sin\alpha\cot\beta\frac{\partial}{\partial\alpha} + \cos\alpha\frac{\partial}{\partial\beta} + \frac{\sin\alpha}{\sin\beta}\frac{\partial}{\partial\gamma}\right]\end{aligned}}$$

Similarly $\hat{e}_1 = \hat{x}''' = \hat{x}''\cos\gamma + \hat{y}''\sin\gamma = (\cos\beta\, x' - \sin\beta\, z')\cos\gamma + \hat{y}'\sin\gamma$

$$\Rightarrow \quad = \cos\beta\cos\gamma[\cos\alpha\,\hat{x} + \sin\alpha\,\hat{y}] - \sin\beta\cos\gamma\,\hat{z}' + \hat{y}'\sin\gamma$$

$$\hat{Q}_1 = \cos\beta\cos\gamma[\cos\alpha\,\hat{L}_x + \sin\alpha\,\hat{L}_y] - \sin\beta\cos\gamma\,\hat{L}_z - i\hbar\frac{\partial}{\partial\beta}\sin\gamma$$

$$\hat{Q}_1 = -i\hbar\left[\sin\gamma\frac{\partial}{\partial\beta} - \frac{\cos\gamma}{\sin\beta}\frac{\partial}{\partial\alpha} + \cot\beta\cos\gamma\frac{\partial}{\partial\gamma}\right]$$

$$\& \quad \hat{e}_2 = \hat{y}''' = -\sin\gamma\,\hat{x}'' + \cos\gamma\,\hat{y}'$$

$$= -\sin\gamma\,[\cos\beta\, x' - \sin\beta\, z'] + \cos\gamma\,\hat{y}'$$

$$= -\sin\gamma\cos\beta\,[\cos\alpha\,\hat{x} + \sin\alpha\,\hat{y}] + \sin\gamma\sin\beta\, z' + \cos\gamma\,\hat{y}'$$

$$\Rightarrow \hat{Q}_2 = -\sin\gamma\cos\beta\left[\cos\alpha\,\hat{L}_x + \sin\alpha\,\hat{L}_y\right] + \sin\gamma\sin\beta\,\hat{L}_z - \cos\gamma\; i\hbar\frac{\partial}{\partial\beta}$$

$$\Rightarrow \boxed{\begin{aligned} Q_1 &= -i\hbar\left[-\frac{\cos\gamma}{\sin\beta}\frac{\partial}{\partial\alpha} + \sin\gamma\frac{\partial}{\partial\beta} + \cot\beta\cos\gamma\frac{\partial}{\partial\gamma}\right] \\ Q_2 &= -i\hbar\left[\frac{\sin\gamma}{\sin\beta}\frac{\partial}{\partial\alpha} + \cos\gamma\frac{\partial}{\partial\beta} - \cot\beta\sin\gamma\frac{\partial}{\partial\gamma}\right] \end{aligned}}$$

Ex: check that $Q_i = (\hat{e}_i \cdot \vec{L}) = \hat{x}_j T_{ji}(\alpha\beta\gamma) \cdot \vec{L}$

$$\Rightarrow \boxed{Q_i = T_{ij}(\alpha\beta\gamma)\hat{L}_j}$$

Since $Q_i = (\hat{e}_i \cdot \vec{L})$, under rotations, both $\hat{e}_i$ and $\vec{L}$ transform in the same way and thus keep the inner product invariant. In other words, Q_i is a scalar under rotation. $\Rightarrow$

$$D^\dagger(g)\, Q_i\, D(g) = Q_i, \quad \text{or} \quad e^{i\vec{L}\cdot\hat{n}\theta}\, Q_i\, e^{-i\vec{L}\cdot\hat{n}\theta} = Q_i$$

$\Rightarrow \boxed{[L_j, Q_i] = 0.}$ ← The Lab frame angular momentum and body frame ones commute!

Ex: please check $[L_j, Q_i] = 0$ from their expressions using (α, β, γ) by brutal force calculation.

For a free top, we have $[L^2, H] = [L_z, H] = [L^2, L_z] = 0$, thus we ~~d~~ can chose $(L^2\ L_z)$ to characterize the rotation eigenstate of a top. However, we know that (L^2, L_z) is complete for a point particle confinied on a sphere, but a top is more than a single particle. Later, we will see that we need an extra quantum number.

Now let use denote the wavefunction $\psi_{IM}(g)$, where IM are the quantum numbers for L^2, L_z, respectively. The configuration space of a top is characterized by (α, β, γ), which determines an SO(3) rotation, i.e. the rotation from the Lab frame (xyz) to the body frame (e_1, e_2, e_3). $g(\alpha\beta\gamma)$ Thus the configuration space of a top is the same of the group space of SO(3). We will use "g" as the coordinate of rotating top.

Let us apply the rotation g_0 on the wavefunction $\psi_{Im}(g)$.

$$\underset{\substack{\uparrow \\ \text{rotation}}}{R(g_0)}\ \psi_{Im}(\underset{\substack{\nwarrow \\ \text{coordinate}}}{g}) = \psi_{IM}(g_0^{-1} g)$$

product according to group operation

coordinate of the, configuration space of top

on the other hand. $R(g_0)\,\psi_{Im}(g) = \sum_{M'} \psi_{IM'}(g)\, D^I_{M'M}(g_0)$.

Let set $g_0 = g \Rightarrow \psi_{IM}(e) = \sum_{M'} \psi_{IM'}(g)\, D^I_{M'M}(g) = \sum_{M'} (D^I_{MM'})^T\, \psi_{IM'}(g)$

$$\boxed{\psi_{IM}(g) = \sum_{M'} \left(D^{I,T}\right)^{\dagger}_{MM'}(g)\, \psi_{IM'}(e) = \sum_{M'} D^{*,I}_{M,M'}(g)\, \psi_{IM'}(e)}$$

Before we move on, let's define the ~~orthogonal~~ orthagnoal conditions of $D^{*I}_{MM'}(g)$.

① measure $\int dg = \int_0^{2\pi} d\alpha \underbrace{\int_0^{\pi} d\beta}_{\sin\beta} \int_0^{2\pi} d\gamma = 8\pi^2$

$D^{I_1}_{m_1' m_1}(g) = \langle I_1 m_1' | D(g) | I_1 m_1 \rangle$, $D^{I_2}_{m_2' m_2}(g) = \langle I_2 m_2' | D(g) | I_2 m_2 \rangle$

$$\boxed{\text{Theorem: } \int dg\, D^{I_1 *}_{m_1' m_1}(g)\, D^{I_2}_{m_2' m_2}(g) = C(I_1)\, \delta_{I_1 I_2}\, \delta_{m_1' m_2'}\, \delta_{m_1 m_2}, \text{ and } C(I_1) = \frac{8\pi^2}{2I_1+1}.}$$

Proof: define an operator $P = \int dg\, D^{\dagger}(g)\, |I_1 m_1'\rangle\langle I_2 m_2'|\, D(g)$

where $D(g) = e^{-i\vec{J}\cdot\hat{n}\theta}$.

Let us define a rotation g_0, and $P \rightarrow P' = D^{\dagger}(g_0)\, P\, D(g_0)$

$$P' = \int dg\, D^{\dagger}(g_0) D^{\dagger}(g)\, |I_1 m_1'\rangle\langle I_2 m_2'|\, D(g) D(g_0)$$

$$= \int dg\, D^{\dagger}(g g_0)\, |I_1 m_1'\rangle\langle I_2 m_2'|\, D(g g_0)$$

$$= \int d(g' g_0)\, D^{\dagger}(g')\, |I_1 m_1'\rangle\langle I_2 m_2'|\, D(g') = \int dg\, D^{\dagger}(g)\, |I_1 m'\rangle\langle I_2 m_2'|\, D(g)$$

↑ measure is invariant

$= P$

then $\int dg\; D^{I_1^*}_{m_1' m_1}(g)\, D^{I_2}_{m_2' m_2}(g) = \langle I_1 m_1 | P | I_2 m_2 \rangle$

since P is rotationally invariant $\Rightarrow \int dg\; D^{I_1^*}_{m_1' m_1}(g)\, D^{I_2}_{m_2' m_2}(g) \propto \delta_{I_1 I_2}\, \delta_{m_1 m_2}$

Similarly $\Rightarrow \propto \delta_{m_1' m_2'}$

The result is also only dependent on I_1, but not m's. ~~[illegible]~~

i.e. $C(I_1)$

$$\Rightarrow C(I_1) = \frac{1}{2I_1+1} \sum_{m_1} \int D^{*\, I_1}_{m_1' m_1}(g)\, D^{I_1}_{m_1' m_1}(g)\, dg$$

$$= \frac{1}{2I_1+1} \sum_{m_1} \int \left[D^{I_1}_{m_1 m_1'}(g) \right]^{\dagger} D^{I_1}_{m_1' m_1}(g)\, dg = \frac{1}{2I_1+1} \int dg = \frac{8\pi^2}{2I_1+1}$$

$$\Rightarrow \boxed{\int D^{I_1^*}_{m_1' m_1}(g)\, D^{I_2}_{m_2' m_2}(g)\, dg = \frac{8\pi^2}{2I_1+1}\, \delta_{I_1 I_2}\, \delta_{m_1' m_1}\, \delta_{m_2' m_2}}$$

Now we can interpret $D^{*,I}_{MM'}(g)$ as the basis of rotation functions of a top. We also know that IM are not complete to describe top. We can assign M' as another quantum number to classify bases.

orth-normal basis (IMM')

$$\boxed{\psi_{IMM'}(g) = \sqrt{\frac{2I+1}{8\pi^2}}\; D^{*,I}_{MM'}(g)}$$

Next, we need to figure out what is the physical meaning of the quantum number of M'.

~~by definition, we apply rotation on top wavefunction~~

$$e^{-i\theta\hat{n}\cdot\vec{J}}\,\psi(g) = \psi(g^{-1}(\hat{n},\theta)\cdot g).$$

Replace $\psi(g) = D^{I*}_{MM'}(\alpha\beta,\gamma)$, where $g = g(\alpha\beta\gamma) \Rightarrow$

$$e^{-i\theta\hat{n}\cdot\vec{J}}\,D^{I*}_{MM'}(\alpha\beta\gamma) = \left[D^{I}_{MM'}(g_0^{-1}(\hat{n},\theta)\,g)\right]^*$$

$$= \sum_{M''} D^{I,*}_{MM''}(g_0^{-1}(\hat{n},\theta))\,D^{I,*}_{M''M'}(g) = \sum_{M''} D^{I,*}_{M''M'}(g)\,D^{I}_{M'',M}(g_0(\hat{n},\theta))$$

→ take infinitesimal rotation, and remember $g = g(\alpha\beta\gamma)$

$$\boxed{\hat{n}\cdot\vec{J}\,D^{I,*}_{MM'}(\alpha\beta\gamma) = \sum_{M''} D^{I,*}_{M'',M'}(\alpha\beta\gamma)\,\langle IM''|\,\hat{n}\cdot\vec{J}\,|IM\rangle}$$

operation on the first m-index.

now take $\hat{n}$ to be one of the top body axes, $\hat{e}_{k=1,2,3}$

$\hat{e}_K = g(\alpha\beta\gamma)\,\hat{i}_k$. $\hat{i}_k = \hat{x}, \hat{y}, \hat{z}$; the fixed frame axis.

$$Q_K = \hat{e}_K\cdot\vec{J} \Rightarrow Q_K D^{I,*}_{MM'}(\alpha\beta\gamma) = \sum_{M''} D^{I,*}_{M''M'}(\alpha\beta\gamma)\langle IM''|\,\hat{e}_K\cdot\vec{J}\,|IM\rangle$$

$\Rightarrow$

$$\hat{e}_K\cdot\vec{J} = \vec{J}\cdot g(\alpha\beta\gamma)\,\hat{i}_K = (g^{-1}\vec{J})\cdot\hat{i}_K = D(g)\,(\vec{J}\cdot\hat{i}_K)\,D^{-1}(g)$$

$$\Rightarrow \langle IM''|\,Q_K\,|IM\rangle = \sum_{K'K''}\langle IM''|\,D(g)\,|IK'\rangle\langle IK'|\,\vec{J}\cdot\hat{i}_K\,|IK''\rangle\langle IK''|\,D^{-1}(g)\,|IM\rangle$$

$$= \sum_{K',K''} D^{I}_{M''K'}(g)\,D^{I}_{K''M}(g^{-1})\,\langle IK'|\,J_K\,|IK''\rangle$$

$\Rightarrow$

$$Q_K\, D^{I,*}_{MM'}(\alpha\beta\gamma) = \sum_{\substack{M''\\K'K''}} D^{I,*}_{M''M'}(\alpha\beta\gamma)\, D^{I}_{M'',K'}(g)\, D^{I}_{K''M}(g^{-1})\, \langle IK' | J_k | IK''\rangle$$

Sum over M''

$$\sum_{M''} D^{I,*}_{M''M'}(\alpha\beta\gamma)\, D^{I}_{M''K'}(\alpha\beta\gamma) = \delta_{K'M'}$$

$$\Rightarrow Q_K\, D^{I,*}_{MM'}(\alpha\beta\gamma) = \sum_{K''} D^{I}_{K''M}(g^{-1})\, \langle I M' | J_k | I K''\rangle$$

$$= \sum_{K''} D^{I,*}_{MK''}(g)\, \langle I M' | J_k | I K''\rangle$$

set $k=3 \Rightarrow \langle I M' | J_z | I K''\rangle = \delta_{M'K''}\, M'$.

$$\Rightarrow \boxed{Q_3\, D^{I,*}_{MM'} = M'\, D^{I,*}_{MM'}}$$ M' is the eigen-number of Q_3.

more generally $\Rightarrow$

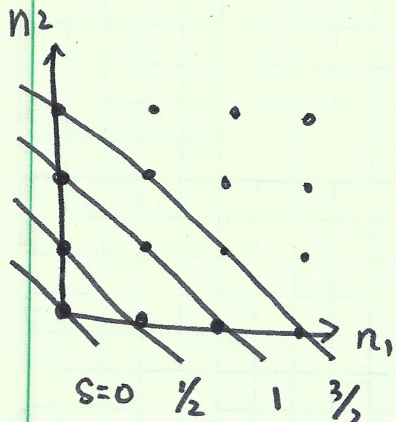
$$\boxed{(\vec{J}\cdot\hat{e}_k)\, D^{I,*}_{MM'}(\alpha\beta\gamma) = \sum_{M''} D^{I,*}_{MM''}(\alpha\beta\gamma)\, \langle I M' | \vec{J}\cdot\hat{e}_k | I M''\rangle}$$

operations on ~~the second index in the lower line~~ the second m-index.

Lect 4 Representations of SU(2) — Wigner D-matrix

The 2D harmonic oscillator has an SU(2) symmetry

$$H = \hbar\omega\,[a_1^\dagger a_1 + a_2^\dagger a_2] \quad \text{where} \quad a_i = \frac{1}{\sqrt{2}}\left(\frac{x_i}{l} + i\,\frac{P_i l}{\hbar}\right)$$



① define $J_z = \frac{1}{2}(a_1^\dagger a_1 - a_2^\dagger a_2)$

$J_+ = a_1^\dagger a_2, \quad J_- = a_2^\dagger a_1$

then $[J_i, J_j] = i\epsilon_{ijk} J_k$, — Schwinger boson Representation of SU(2) algebra.

② $a_1^\dagger a_1 + a_2^\dagger a_2 = 2S$ is conserved, which specify each representation.

Let us check each energy level

1) $S=0$: the ground state $|\Omega\rangle$, $a_i|\Omega\rangle = 0$. — trivial Rep.

2) $S = \frac{1}{2}$ $\quad |i\rangle = a_i^\dagger |\Omega\rangle$ — fundamental Rep

$$\hat{\mathcal{U}}(R) = e^{-iJ_z\alpha}\, e^{-iJ_y\beta}\, e^{-iJ_z\gamma}$$

For $S=\frac{1}{2}$ $\quad \mathcal{U}(R) = e^{-i\frac{\sigma_z}{2}\alpha}\, e^{-i\frac{\sigma_y}{2}\beta}\, e^{-i\frac{\sigma_z}{2}\gamma}$

and $\boxed{\langle i|\hat{\mathcal{U}}(R)|j\rangle = \mathcal{U}_{ij}(R)}$.

How $a_i^\dagger$ transform under $\hat{\mathcal{U}}(R)$?

$$\mathcal{U}(R)|i\rangle = \sum_j |j\rangle\, \mathcal{U}_{ji}(R) \Rightarrow \mathcal{U}(R)\, a_i^\dagger\, \mathcal{U}^{-1}(R)\,|\Omega\rangle$$

$$= \sum_j a_j^\dagger\, \mathcal{U}_{ji}(R)\, |\Omega\rangle$$

$$\Rightarrow \quad \hat{u}(R)\, a_i^\dagger\, \hat{u}^\dagger(R) = \sum_j a_j^\dagger\, u_{ji} \rightarrow \text{Representation } 2\times2 \text{ matrix}$$

Rotation operator in terms of $a_i^\dagger, a_i$,

For example, for $\hat{u}(R) = e^{-iJ_y\beta}$, we have

$$e^{-iJ_y\beta}[a_1^\dagger \;\; a_2^\dagger]\, e^{iJ_y\beta} = [a_1^\dagger \;\; a_2^\dagger]\, e^{-i\frac{\sigma_y}{2}\beta} = [a_1^\dagger \; a_2^\dagger]\begin{bmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\beta/2 & \cos\frac{\beta}{2} \end{bmatrix}$$

i.e.
$$\begin{cases} e^{-iJ_y\beta}\, a_1^\dagger\, e^{iJ_y\beta} = a_1^\dagger \cos\frac{\beta}{2} + a_2^\dagger \sin\frac{\beta}{2} \\ e^{-iJ_y\beta}\, a_2^\dagger\, e^{iJ_y\beta} = -a_1^\dagger \sin\frac{\beta}{2} + a_2^\dagger \cos\frac{\beta}{2} \end{cases}$$

HW: Directly prove the above results from operator calculation by using $J_y = \frac{1}{2i}(a_1^\dagger a_2 - a_2^\dagger a_1)$, $[J_y\, a_1^\dagger] = -\frac{1}{2i} a_2^\dagger$, $[J_y\, a_2^\dagger] = \frac{1}{2i} a_1^\dagger$

③ For a general energy level $a_1^\dagger a_1 + a_2^\dagger a_2 = 2S$, we can label the state

$$|j\, m\rangle = \frac{a_1^{\dagger\, j+m}\, a_2^{\dagger\, j-m}}{\sqrt{(j+m)!(j-m)!}}\, |\Omega\rangle ,$$

$$\langle jm'|\, \hat{u}(R)\, |jm\rangle = D^j_{m'm}(R) = e^{-im'\alpha - im\gamma}\, d^j_{m'm}(\beta)$$

$$\langle jm'|\, e^{-iJ_z\alpha}\, e^{-iJ_y\beta}\, e^{-iJ_z\gamma}\, |jm\rangle = e^{-im'\alpha - im\gamma}\, \langle jm'|\, e^{-iJ_y\beta}\, |jm\rangle$$

where $d^j_{m'm}(\beta) = \langle jm'|\, e^{-iJ_y\beta}\, |jm\rangle$

$$e^{-iJ_y\beta}|jm\rangle = \frac{1}{\sqrt{(j+m)!(j-m)!}}\left(a_1^\dagger\cos\frac{\beta}{2}+a_2^\dagger\sin\frac{\beta}{2}\right)^{j+m}\left(-a_1^\dagger\sin\frac{\beta}{2}+a_2^\dagger\cos\frac{\beta}{2}\right)^{j-m}|\Omega\rangle$$

$$= \frac{1}{\sqrt{(j+m)!(j-m)!}}\sum_{m'=-j}^{j}\sum_{\sigma}\binom{j+m}{j-m'-\sigma}\binom{j-m}{\sigma}\left(a_1^\dagger\cos\frac{\beta}{2}\right)^{m+m'+\sigma}\left(a_2^\dagger\sin\frac{\beta}{2}\right)^{j-m'-\sigma}$$

$$(-)^{j-m-\sigma}\left(a_1^\dagger\sin\frac{\beta}{2}\right)^{j-m-\sigma}\left(a_2^\dagger\cos\frac{\beta}{2}\right)^{\sigma}|\Omega\rangle$$

$$= \frac{1}{\sqrt{(j+m)!(j-m)!}}\sum_{m'=-j}^{j}\sum_{\sigma}\binom{j+m}{j-m'-\sigma}\binom{j-m}{\sigma}(a_1^\dagger)^{j+m'}(a_2^\dagger)^{j-m'}$$

$$(-)^{j-m-\sigma}\cdot\left(\cos\frac{\beta}{2}\right)^{m+m'+2\sigma}\left(\sin\frac{\beta}{2}\right)^{2j-2\sigma-m'-m}|\Omega\rangle$$

$$\left.\begin{array}{l} 0\le\sigma\le j-m \\ -m-m'\le\sigma\le j-m' \end{array}\right\}\Rightarrow \max(0,-m-m')\le\sigma\le\min(j-m,j-m')$$

$$|jm'\rangle = \frac{1}{\sqrt{(j+m')!(j-m')!}}(a_1^\dagger)^{j+m'}(a_2^\dagger)^{j-m'}|\Omega\rangle$$

$$\Rightarrow d^j_{m'm} = \frac{\sqrt{(j+m')!(j-m')!}}{\sqrt{(j+m)!(j-m)!}}\sum_\sigma\binom{j+m}{j-m'-\sigma}\binom{j-m}{\sigma}(-)^{j-m-\sigma}$$

$$\left(\cos\frac{\beta}{2}\right)^{2\sigma+m+m'}\left(\sin\frac{\beta}{2}\right)^{2j-2\sigma-m-m'}$$

$$= \sqrt{\frac{(j+m)!(j-m)!}{(j+m')!(j-m')!}}\left(\cos\frac{\beta}{2}\right)^{m+m'}\left(\sin\frac{\beta}{2}\right)^{m-m'}\left\{\sum_\sigma\frac{(j+m')!(j-m')!}{(j+m)!(j-m)!}\cdot\right.$$

$$\left.(-)^{j-m-\sigma}\binom{j+m}{j-m'-\sigma}\binom{j-m}{\sigma}\left(\cos\frac{\beta}{2}\right)^{2\sigma+m+m'}\left(\sin\frac{\beta}{2}\right)^{2j-2\sigma-m-m'}\right\}$$

It can be representeel in terms of Jacobi polynomial

$$P_n^{(\alpha,\beta)}(x) = \frac{(-)^n}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^n}{dx^n}\left((1-x)^{\alpha+n}(1+x)^{\beta+n}\right)$$

HW: Prove that

① $$P_n^{(\alpha,\beta)}(x) = \sum_{l=0}^{n}(-)^{n+l}\binom{\alpha+n}{l}\binom{\beta+n}{n-l}\left(\frac{1-x}{2}\right)^{n-l}\left(\frac{1+x}{2}\right)^{n-l}$$

and then

$$d^j_{m'm}(\beta) = \sqrt{\frac{(j+m)!(j-m)!}{(j+m')!(j-m')!}}\left(\cos\frac{\beta}{2}\right)^{m+m'}\left(\sin\frac{\beta}{2}\right)^{m-m'} P_{j-m}^{m-m',\,m+m'}(\cos\beta)$$

HW 2: Prove the following properties of D-matrices

① $$d^j_{m'm}(\beta) = (-)^{m'-m} d^j_{mm'}(\beta) = (-)^{m'-m} d^j_{-m',-m}(\beta)$$

② $$d^l_{0m}(\beta) = \left[\frac{(l-m)!}{(l+m)!}\right]^{1/2} P_l^m(x) -$$

where $P_l^m(x)$ is the associated Lengdre polynomial. You can use the follwing results.

$$P_l^m(x) = \frac{1}{2^l l!}(1-x^2)^{m/2}\frac{d^{l+m}}{dx^{l+m}}(x^2-1)^l = \frac{(l+m)!}{2^m l!}(1-x^2)^{\frac{m}{2}} P^{m,m}_{l-m}(x)$$

③ $$D^l_{00}(\alpha\beta\gamma) = d^l_{00}(\beta) = P_l(\cos\beta)$$

$$D^l_{0m}(\alpha\beta\gamma) = e^{-im\gamma} d^l_{0m}(\beta) = (-)^m\left(\frac{4\pi}{2l+1}\right)^{1/2} Y^*_{lm}(\beta,\gamma)$$

for $J=1/2$, the D-matrix is relatively simple

$$e^{-i\frac{\sigma_y}{2}\beta} = \cos\frac{\beta}{2} - i\sin\frac{\beta}{2}\,\sigma_y = \begin{array}{cc} |\uparrow\rangle & |\downarrow\rangle \\ \end{array}$$
$$\begin{array}{c} \\ \end{array}\begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix}\begin{array}{c} |\uparrow\rangle \\ |\downarrow\rangle \end{array}$$

$$D^{1/2}_{m'm}(\alpha\beta\gamma) = e^{-im'\alpha}\, e^{-im\gamma}\cdot d^{1/2}_{m'm} = \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{i\frac{-\alpha+\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}$$

$$Y_{q\,lm}(\theta,\varphi) = \sqrt{\frac{2l+1}{4\pi}}\, e^{i(m+q)\varphi}\, d^{l}_{m\,-q}(\theta)$$

$$\Rightarrow\; \boxed{\begin{aligned} Y_{\frac12\frac12\frac12}(\theta,\varphi) &= \sqrt{\tfrac{1}{2\pi}}\, e^{i\varphi}\, d^{\frac12}_{\frac12\,-\frac12}(\theta) = -\sqrt{\tfrac{1}{2\pi}}\, e^{i\varphi}\sin\tfrac{\theta}{2} \\ Y_{\frac12\frac12-\frac12}(\theta,\varphi) &= \sqrt{\tfrac{1}{2\pi}}\, d^{\frac12}_{-\frac12-\frac12}(\theta) = \sqrt{\tfrac{1}{2\pi}}\cos\tfrac{\theta}{2} \end{aligned}}$$

south pole singular

$$\boxed{\begin{aligned} Y_{-\frac12\frac12\frac12}(\theta,\varphi) &= \sqrt{\tfrac{1}{2\pi}}\, d^{\frac12}_{\frac12\frac12}(\theta) = \sqrt{\tfrac{1}{2\pi}}\cos\tfrac{\theta}{2} \\ Y_{-\frac12\frac12-\frac12}(\theta,\varphi) &= \sqrt{\tfrac{1}{2\pi}}\, e^{-i\varphi}\, d^{\frac12}_{-\frac12\frac12}(\theta) = \sqrt{\tfrac{1}{2\pi}}\, e^{-i\varphi}\sin\tfrac{\theta}{2} \end{aligned}}$$

for $J=1$ $\quad (J_y)_{m'm} = \begin{pmatrix} 0 & \frac{-i}{\sqrt2} & 0 \\ \frac{i}{\sqrt2} & 0 & \frac{-i}{\sqrt2} \\ 0 & \frac{i}{\sqrt2} & 0 \end{pmatrix}$

$$e^{-iJ_y\beta}\,|m\rangle = \frac{1}{\sqrt{(m+1)!\,(1-m)!}}\left(a_1^\dagger\cos\frac{\beta}{2} + a_2^\dagger\sin\frac{\beta}{2}\right)^{1+m}\left(-a_1^\dagger\sin\frac{\beta}{2} + a_2^\dagger\cos\frac{\beta}{2}\right)^{1-m}|0\rangle$$

$$e^{-iJ_y\beta}\,|1\rangle = \frac{1}{\sqrt{2!}}\left[(a_1^\dagger)^2\cos^2\frac{\beta}{2} + (a_2^\dagger)^2\sin^2\frac{\beta}{2} + 2a_1^\dagger a_2^\dagger\cos\frac{\beta}{2}\sin\frac{\beta}{2}\right]|0\rangle$$

$$= \cos^2\frac{\beta}{2}\,|1\rangle + \frac{1}{\sqrt2}\sin\beta\,|0\rangle + \sin^2\frac{\beta}{2}\,|-1\rangle$$

$$e^{-iJ_y\beta}|0\rangle = \left(a_1^\dagger\cos\frac{\beta}{2} + a_2^\dagger\sin\frac{\beta}{2}\right)\left(-a_1^\dagger\sin\frac{\beta}{2} + a_2^\dagger\cos\frac{\beta}{2}\right)|0\rangle$$

$$= \left(-\frac{1}{2}(a_1^\dagger)^2\sin\beta + \frac{1}{2}(a_2^\dagger)^2\sin\beta + a_1^\dagger a_2^\dagger\cos\beta\right)|0\rangle$$

$$= -\frac{1}{\sqrt{2}}\sin\beta\,|1\rangle + \cos\beta\,|0\rangle + \frac{1}{\sqrt{2}}\sin\beta\,|-1\rangle$$

$$e^{-iJ_y\beta}|-1\rangle = \frac{1}{\sqrt{2!}}\left[-a_1^\dagger\sin\frac{\beta}{2} + a_2^\dagger\cos\frac{\beta}{2}\right]^2|0\rangle = \frac{1}{\sqrt{2!}}\left[(a_1^\dagger)^2\sin^2\frac{\beta}{2} - a_1^\dagger a_2^\dagger\sin\beta + (a_2^\dagger)^2\cos^2\frac{\beta}{2}\right]|0\rangle$$

$$= \sin^2\frac{\beta}{2}|1\rangle - \frac{1}{\sqrt{2}}\sin\beta|0\rangle + \cos^2\frac{\beta}{2}|-1\rangle$$

$$\Rightarrow \langle m'|e^{-iJ_y\beta}|m\rangle = \begin{pmatrix} \cos^2\frac{\beta}{2} & -\frac{1}{\sqrt{2}}\sin\beta & \sin^2\frac{\beta}{2} \\ \frac{1}{\sqrt{2}}\sin\beta & \cos\beta & \frac{-1}{\sqrt{2}}\sin\beta \\ \sin^2\frac{\beta}{2} & \frac{1}{\sqrt{2}}\sin\beta & \cos^2\beta/2 \end{pmatrix}$$

$$Y_{q\,lm}(\theta\,\varphi) = \sqrt{\frac{2l+1}{4\pi}}\,e^{i(m+q)\varphi}\,d^l_{m\,-q}(\theta)$$

$$\boxed{\begin{aligned} Y_{1;11}(\theta,\varphi) &= \sqrt{\frac{3}{4\pi}}\,e^{i2\varphi}\sin^2\frac{\theta}{2} \\ Y_{1;10}(\theta,\varphi) &= \sqrt{\frac{3}{4\pi}}\,e^{i\varphi}\frac{-1}{\sqrt{2}}\sin\theta \\ Y_{1;1-1}(\theta,\varphi) &= \sqrt{\frac{3}{4\pi}}\cos^2\frac{\theta}{2} \end{aligned}},\quad \boxed{\begin{aligned} Y_{-1;11}(\theta,\varphi) &= \sqrt{\frac{3}{4\pi}}\cos^2\frac{\theta}{2} \\ Y_{-1;10}(\theta,\varphi) &= \sqrt{\frac{3}{4\pi}}\,e^{-i\varphi}\frac{1}{\sqrt{2}}\sin\theta \\ Y_{-1;1-1}(\theta,\varphi) &= \sqrt{\frac{3}{4\pi}}\,e^{-i2\varphi}\sin^2\frac{\theta}{2} \end{aligned}}$$

$$\boxed{\begin{aligned} Y_{11}(\theta,\varphi) &= \sqrt{\frac{3}{4\pi}}\,e^{i\varphi}\left(\frac{-1}{\sqrt{2}}\sin\theta\right) \\ Y_{10}(\theta\,\varphi) &= \sqrt{\frac{3}{4\pi}}\cos\theta \\ Y_{1-1}(\theta,\varphi) &= \sqrt{\frac{3}{4\pi}}\,e^{-i\varphi}\left(\frac{1}{\sqrt{2}}\sin\theta\right) \end{aligned}}$$

for $J = 3/2$

$$e^{-iJ_y\beta}|m\rangle = \frac{1}{\sqrt{(m+3/2)!(3/2-m)!}}\left(a_1^\dagger\cos\frac{\beta}{2} + a_2^\dagger\sin\frac{\beta}{2}\right)^{\frac{3}{2}+m}\left(-a_1^\dagger\sin\frac{\beta}{2} + a_2^\dagger\cos\frac{\beta}{2}\right)^{\frac{3}{2}-m}|0\rangle$$

$$e^{-iJ_y\beta}|\tfrac{3}{2}\rangle = \frac{1}{\sqrt{3!}}\left[(a_1^\dagger)^3\cos^3\frac{\beta}{2} + 3(a_1^\dagger)^2(a_2^\dagger)\cos^2\frac{\beta}{2}\sin\frac{\beta}{2} + 3a_1^\dagger(a_2^\dagger)^2\cos\frac{\beta}{2}\sin^2\frac{\beta}{2} + (a_2^\dagger)^3\sin^3\frac{\beta}{2}\right]|0\rangle$$

$$= \cos^3\frac{\beta}{2}|\tfrac{3}{2}\rangle + \underbrace{\sqrt{\frac{2!}{3!}}\cdot 3}_{\sqrt{3}}\cos^2\frac{\beta}{2}\sin\frac{\beta}{2}|\tfrac{1}{2}\rangle + \underbrace{\sqrt{\frac{2!}{3!}}\,3}_{\sqrt{3}}\cos\frac{\beta}{2}\sin^2\frac{\beta}{2}|-\tfrac{1}{2}\rangle + \sin^3\frac{\beta}{2}|-\tfrac{3}{2}\rangle$$

$$e^{-iJ_y\beta}|\tfrac{1}{2}\rangle = \frac{1}{\sqrt{2!}}\left(a_1^\dagger\cos\frac{\beta}{2} + a_2^\dagger\sin\frac{\beta}{2}\right)^2\left(-a_1^\dagger\sin\frac{\beta}{2} + a_2^\dagger\cos\frac{\beta}{2}\right)|0\rangle$$

$$= \frac{1}{\sqrt{2!}}\left[(a_1^\dagger)^2\cos^2\frac{\beta}{2} + 2a_1^\dagger a_2^\dagger\cos\frac{\beta}{2}\sin\frac{\beta}{2} + (a_2^\dagger)^2\sin^2\frac{\beta}{2}\right]\left[-a_1^\dagger\sin\frac{\beta}{2} + a_2^\dagger\cos\frac{\beta}{2}\right]|0\rangle$$

$$= \frac{1}{\sqrt{2!}}\Big[-\cos^2\frac{\beta}{2}\sin\frac{\beta}{2}(a_1^\dagger)^3 + \left[\cos^3\frac{\beta}{2} - 2\cos\frac{\beta}{2}\sin^2\frac{\beta}{2}\right](a_1^\dagger)^2 a_2 + \left[2\cos^2\frac{\beta}{2}\sin\frac{\beta}{2} - \sin^3\frac{\beta}{2}\right]a_1^\dagger a_2^{\dagger 2}$$
$$+ (a_2^\dagger)^3\cos\frac{\beta}{2}\sin^2\frac{\beta}{2}\Big]|0\rangle$$

$$= -\sqrt{\frac{3!}{2!}}\cos^2\frac{\beta}{2}\sin\frac{\beta}{2}|\tfrac{3}{2}\rangle + \cos\frac{\beta}{2}\left[\cos^2\frac{\beta}{2} - 2\sin^2\frac{\beta}{2}\right]|\tfrac{1}{2}\rangle + \sin\frac{\beta}{2}\left[2\cos^2\frac{\beta}{2} - \sin^2\frac{\beta}{2}\right]|-\tfrac{1}{2}\rangle$$
$$+ \sqrt{\frac{3!}{2!}}\cos\frac{\beta}{2}\sin^2\frac{\beta}{2}|-\tfrac{3}{2}\rangle$$

$$e^{-iJ_y\beta}|-\tfrac{1}{2}\rangle = \frac{1}{\sqrt{2!}}\left[a_1^\dagger\cos\frac{\beta}{2} + a_2^\dagger\sin\frac{\beta}{2}\right]\left[a_1^{\dagger 2}\sin^2\frac{\beta}{2} - 2a_1^\dagger a_2^\dagger\sin\frac{\beta}{2}\cos\frac{\beta}{2} + a_2^{\dagger 2}\cos^2\frac{\beta}{2}\right]|0\rangle$$

$$= \sqrt{\frac{3!}{2!}}\left[\frac{1}{\sqrt{3!}}(a_1^\dagger)^3\cos\frac{\beta}{2}\sin^2\frac{\beta}{2}\right] + \frac{1}{\sqrt{2}}(a_1^\dagger)^2(a_2^\dagger)\left[-2\cos^2\frac{\beta}{2}\sin\frac{\beta}{2} + \sin^3\frac{\beta}{2}\right] + \frac{1}{\sqrt{2}}a_1^\dagger(a_2^\dagger)^2\left\{\cos^3\frac{\beta}{2} - 2\cos\frac{\beta}{2}\sin^2\frac{\beta}{2}\right\}$$
$$+ \sqrt{\frac{3!}{2!}}\left(\frac{1}{\sqrt{3!}}(a_2^\dagger)^3\cos^2\frac{\beta}{2}\sin\frac{\beta}{2}\right)|0\rangle$$

$$= \sqrt{3}\cos\frac{\beta}{2}\sin^2\frac{\beta}{2}|\tfrac{3}{2}\rangle + \left[-2\cos^2\frac{\beta}{2}\sin\frac{\beta}{2} + \sin^3\frac{\beta}{2}\right]|\tfrac{1}{2}\rangle + \left(\cos^3\frac{\beta}{2} - \cos\frac{\beta}{2}\sin^2\frac{\beta}{2}\right)|-\tfrac{1}{2}\rangle + \underbrace{\cos^2\frac{\beta}{2}\sin\frac{\beta}{2}}|-\tfrac{3}{2}\rangle$$

$$e^{-iJ_y\beta}|-\tfrac{3}{2}\rangle = \frac{1}{\sqrt{3!}}\left[(-)^3\sin^3\frac{\beta}{2}(a_1^\dagger)^3 + 3\sin^2\frac{\beta}{2}\cos\frac{\beta}{2}(a_1^\dagger)^2 a_2 - 3\sin\frac{\beta}{2}\cos^2\frac{\beta}{2}a_1^\dagger a_2^{\dagger 2} + (a_2^\dagger)^3\cos^3\frac{\beta}{2}\right]$$

$$= (-)^3\sin^3\frac{\beta}{2}|\tfrac{3}{2}\rangle + \sqrt{3}\sin^2\frac{\beta}{2}\cos\frac{\beta}{2}|\tfrac{1}{2}\rangle - \sqrt{3}\sin\frac{\beta}{2}\cos^2\frac{\beta}{2}|-\tfrac{1}{2}\rangle + \cos^3\frac{\beta}{2}|-\tfrac{3}{2}\rangle$$

$$\langle m'|e^{-iJ_y\beta}|m\rangle = \begin{pmatrix} \cos^3\frac{\beta}{2} & -\sqrt{3}\cos^2\frac{\beta}{2}\sin\frac{\beta}{2} & \sqrt{3}\cos\frac{\beta}{2}\sin^2\frac{\beta}{2} & -\sin^3\frac{\beta}{2} \\ \sqrt{3}\cos^2\frac{\beta}{2}\sin\frac{\beta}{2} & \cos\frac{\beta}{2}\left[\cos^2\frac{\beta}{2}-2\sin^2\frac{\beta}{2}\right] & -\sin\frac{\beta}{2}\left[+2\cos^2\frac{\beta}{2}-\sin^2\frac{\beta}{2}\right] & \sqrt{3}\cos\frac{\beta}{2}\sin^2\frac{\beta}{2} \\ \sqrt{3}\cos\frac{\beta}{2}\sin^2\frac{\beta}{2} & \sin\frac{\beta}{2}\left[2\cos^2\frac{\beta}{2}-\sin^2\frac{\beta}{2}\right] & \cos\frac{\beta}{2}\left[\cos^2\frac{\beta}{2}-2\sin^2\frac{\beta}{2}\right] & -\sqrt{3}\cos^2\frac{\beta}{2}\sin\frac{\beta}{2} \\ \sin^3\frac{\beta}{2} & \sqrt{3}\cos\frac{\beta}{2}\sin^2\frac{\beta}{2} & \sqrt{3}\cos^2\frac{\beta}{2}\sin\frac{\beta}{2} & \cos^3\frac{\beta}{2} \end{pmatrix}$$

$$\mathcal{Y}_{q\ell m}(\theta,\varphi) = \sqrt{\frac{2\ell+1}{4\pi}}\, e^{i(m+q)\varphi}\, d^{\ell}_{m-q}(\theta)$$

$$\mathcal{Y}_{3/2;3/2,j_z}(\theta,\varphi) = \begin{array}{ll} \frac{3}{2} & \sqrt{\frac{1}{\pi}}\, e^{i3\varphi}\,(-)\sin^3\frac{\beta}{2} \\ \frac{1}{2} & \sqrt{\frac{1}{\pi}}\, e^{i2\varphi}\,\sqrt{3}\cos\frac{\beta}{2}\sin^2\frac{\beta}{2} \\ -\frac{1}{2} & \sqrt{\frac{1}{\pi}}\, e^{i\varphi}\,(-\sqrt{3})\cos^2\frac{\beta}{2}\sin\frac{\beta}{2} \\ -\frac{3}{2} & \sqrt{\frac{1}{\pi}}\,\cos^3\frac{\beta}{2} \end{array};$$

$$\mathcal{Y}_{1/2;3/2,j_z}(\theta,\varphi) = \begin{array}{l} \sqrt{\frac{1}{\pi}}\, e^{i2\varphi}\,\sqrt{3}\cos\frac{\beta}{2}\sin^2\frac{\beta}{2} \\ \sqrt{\frac{1}{\pi}}\, e^{i\varphi}\,(-)\sin\frac{\beta}{2}\left[2\cos^2\frac{\beta}{2}-\sin^2\frac{\beta}{2}\right] \\ \sqrt{\frac{1}{\pi}}\,\cos\frac{\beta}{2}\left[\cos^2\frac{\beta}{2}-2\sin^2\frac{\beta}{2}\right] \\ \sqrt{\frac{1}{\pi}}\, e^{-i\varphi}\sqrt{3}\cos^2\frac{\beta}{2}\sin\frac{\beta}{2} \end{array}$$

$$\mathcal{Y}_{-1/2;3/2\,j_z} = \begin{array}{l} \sqrt{\frac{1}{\pi}}\, e^{i\varphi}\,(-\sqrt{3})\cos^2\frac{\beta}{2}\sin\frac{\beta}{2} \\ \sqrt{\frac{1}{\pi}}\,\cos\frac{\beta}{2}\left[\cos^2\frac{\beta}{2}-2\sin^2\frac{\beta}{2}\right] \\ \sqrt{\frac{1}{\pi}}\, e^{-i\varphi}\sin\frac{\beta}{2}\left[2\cos^2\frac{\beta}{2}-\sin^2\frac{\beta}{2}\right] \\ \sqrt{\frac{1}{\pi}}\, e^{-i2\varphi}\,\sqrt{3}\cos\frac{\beta}{2}\sin^2\frac{\beta}{2} \end{array};$$

$$\mathcal{Y}_{-3/2\,3/2\,j_z} = \begin{array}{l} \sqrt{\frac{1}{\pi}}\,\cos^3\frac{\beta}{2} \\ \sqrt{\frac{1}{\pi}}\, e^{-i\varphi}\,\sqrt{3}\cos^2\frac{\beta}{2}\sin\frac{\beta}{2} \\ \sqrt{\frac{1}{\pi}}\, e^{-i2\varphi}\,\sqrt{3}\cos\frac{\beta}{2}\sin^2\frac{\beta}{2} \\ \sqrt{\frac{1}{\pi}}\, e^{-i3\varphi}\,\sin^3\frac{\beta}{2} \end{array}$$

Define $u = \cos\frac{\beta}{2}$ Fractional QHE WF

$v = \sin\frac{\beta}{2}\, e^{i\varphi}$

The LLL wavefunction on the sphere with the monopole

charge $\cancel{2g+1=}\; n-1$: $\quad u^m v^{n-m} = \vartheta$

Then the slater determinate WF

$$\begin{vmatrix} u_1^0 v_1^n & u_2^0 v_2^n & \cdots & u_n^0 v_n^n \\ u_1^1 v_1^{n-1} & u_2^1 v_2^{n-1} & \cdots & u_n^1 v_n^{n-1} \\ \vdots & & & \vdots \\ u_1^n v_1^0 & & \cdots & u_n^n v_n^0 \end{vmatrix} = \prod_{i<j} (u_i v_j - u_j v_i)$$

$\to$ Laughlin WF

$$\psi(u_1 v_1, \cdots u_n v_n) = \prod_{i<j} (u_i v_j - u_j v_i)^3$$

Haldane's paper (QHE on the sphere)

define a spiner $u = \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} \cos\frac{\theta}{2}\, e^{i\phi/2 + i\chi/2} \\ \sin\frac{\theta}{2}\, e^{-i\phi/2 - i\chi/2} \end{pmatrix} e^{i\chi/2}$

u defines a direction $\hat{n}_i$ in the sphere as

$$\hat{n}_i = u^{\dagger}\sigma_i u ,$$

Single particle ~~states~~ :

$$H = \frac{\vec{\Lambda}^2}{2mR^2} \qquad \omega_c = \frac{eB}{M}$$

$$= \frac{1}{2}\omega_c \frac{|\vec{\Lambda}|^2}{\hbar S} \qquad \boxed{B = \frac{\hbar S}{eR^2}} \Rightarrow 4\pi R^2 B = \frac{h}{e} 2S$$

where $\vec{\Lambda} = \vec{R} \times (-i\hbar\nabla + e\vec{A}(r))$

$$\nabla \times \vec{A} = B\hat{\Omega} \quad \Rightarrow \quad \vec{\Lambda}\cdot\hat{\Omega} = 0$$

now $\vec{\Lambda}$ is the mechrical memomtum, while ~~$\vec{L}$~~

~~$\vec{L} = \vec{R}\times(-i\hbar\nabla) = \vec{\Lambda} - e(\vec{R}\times\vec{A}(r))$~~

Check commutation relation

$$[\Lambda_i, \Lambda_j] = \epsilon_{ilk}\,\epsilon_{jl'k'}\,[r_l(-i\hbar\partial_k) + e\,r_l A_k,\; r_{l'}(-i\hbar\partial_{k'}) + e\,r_{l'} A_{k'}]$$

$$= \epsilon_{ilk}\,\epsilon_{jl'k'}\,(-i\hbar)^2\,(\delta_{kl'}\,r_l\,\partial_{k'} - \delta_{k'l}\,r_{l'}\,\partial_k)$$

$$+ e\,\epsilon_{ilk}\,\epsilon_{jl'k'}\,(-i\hbar)\,[\delta_{kl'}\,r_l A_{k'} + r_l r_{l'}\,\partial_k A_{k'} - \delta_{k'l}\,r_{l'} A_k - r_{l'} r_l\,\partial_{k'} A_k]$$

$$\epsilon_{ilk}\,\epsilon_{jkk'}\,(r_l\,\partial_{k'}) - \epsilon_{ilk}\,\epsilon_{jl'l}\,r_{l'}\,\partial_k$$

$$= \epsilon_{ilk}\,\epsilon_{k'jk}\,r_l\,\partial_{k'} - \epsilon_{kil}\,\epsilon_{jl'l}\,r_{l'}\,\partial_k$$

$$= [\delta_{ik'}\delta_{lj} - \delta_{ij}\delta_{lk'}]\,r_l\,\partial_{k'} - [\delta_{kj}\delta_{il'} - \delta_{kl'}\delta_{ij}]\,r_{l'}\,\partial_k$$

$$= r_j\,\partial_i - \delta_{ij}\,\vec{r}\cdot\vec{\partial} - r_i\,\partial_j + \delta_{ij}\,\vec{r}\cdot\vec{\partial} = -(r_i\,\partial_j - r_j\,\partial_i)$$

$\Rightarrow$ First term is just $i\hbar\,\epsilon_{ijk}\,\epsilon_{klm}\,(-i\hbar)\,[r_l\,\partial_m]$

$$= i\hbar\,\epsilon_{ijk}\,(\vec{R}\times(-i\hbar\vec{\nabla}))_k$$

$$e\,\epsilon_{ilk}\,\epsilon_{jl'k'}\,(-i\hbar)\,[\delta_{kl'}\,r_l A_{k'} - \delta_{k'l}\,r_{l'} A_k]$$

$$= (-i\hbar)\,e\,[r_j A_i - r_i A_j] = i\hbar\,\epsilon_{ijK}\,\epsilon_{Klm}\,e\,r_l A_m$$

First term + second term $\Rightarrow i\hbar\,\epsilon_{ijk}\,\Lambda_k$

the 3-rd term

$$-i\hbar e \, \epsilon_{ilk} \epsilon_{jl'k'} r_l r_{l'} [\underbrace{\partial_k A_{k'} - \partial_{k'} A_k}_{\neq \epsilon_{kk'm} B_m}]$$

$$\epsilon_{ilk} \epsilon_{jl'k'} \epsilon_{kk'm} = \epsilon_{ilk} \epsilon_{jl'k'} \epsilon_{mkk'}$$

$$= \epsilon_{ilk} [\delta_{jm} \delta_{l'k} - \delta_{jk} \delta_{l'm}] = \epsilon_{ill'} \delta_{jm} - \epsilon_{ilj} \delta_{l'm}$$

$$\Rightarrow \quad -i\hbar e \, r_l r_{l'} B_m \, [\epsilon_{ill'} \delta_{jm} - \epsilon_{ilj} \delta_{l'm}]$$

$$= -i\hbar e (-) \epsilon_{ilj} r_l r_{l'} B_{l'} = -i\hbar e \, \epsilon_{ijl} r_l (\vec{r} \cdot \vec{B})$$

$$= -i\hbar e \, \epsilon_{ijk} \frac{r_k}{R} \frac{\hbar S}{e} = -i\hbar (\hbar S) \epsilon_{ijk} \Omega^k$$

$$\Rightarrow \boxed{[\Lambda_i, \Lambda_j] = i\hbar \epsilon^{ijk} (\Lambda_k - \hbar S \Omega_k)}$$

Generator of rotations:

$$\vec{L} = \vec{\Lambda} + \hbar S \hat{\Omega},$$ Question: Why use this defination?

Check $[L_i, L_j] = [\Lambda_i, \Lambda_j] + [\Lambda_i, \hat{\Omega}_j] \hbar S + \hbar S [\hat{\Omega}_i, \Lambda_j]$

$$[\Lambda_i, \Lambda_j] = i\hbar \epsilon_{ijk} (\Lambda_k - \hbar S \Omega_k)$$

$$[\Lambda_i, \hat{\Omega}_j] = \epsilon_{ilm} [r_l (-i\hbar\partial_m), r_j/r] = \varepsilon_{ilm} r_l (-i\hbar) \left[\frac{\delta_{mj} r_j}{r^3} + \frac{r_j r_m}{-r^{3/2}}\right]$$

$$= \varepsilon_{ilj} (-i\hbar) \frac{r_l}{r} = +i\hbar\, \epsilon_{ijk} \hat{\Omega}_k$$

$$\Rightarrow \boxed{\begin{aligned} [L_i, L_j] &= i\hbar\, \epsilon_{ijk} (\Lambda_k - \hbar s \hat{\Omega}_k) + 2i\hbar\, \epsilon_{ijk} \hat{\Omega}_k = i\epsilon_{ijk} L_k \\ [L_i, \Omega_j] &= i\hbar\, \epsilon_{ijk} \hat{\Omega}_k \end{aligned}}$$

$$L^2 = \hbar^2 l(l+1)$$

$$\underline{\vec{L}\cdot\hat{\Omega} = \hat{\Omega}\cdot\vec{L} = \hbar s} \qquad \Rightarrow l = s, s+1, s+2, \cdots = s+n$$

Lower bound!

$$\Lambda^2 = L^2 - \hbar^2 s^2 = \hbar^2 \left[(s+n)(s+n+1) - s^2\right]$$

$$= \hbar^2 [n(n+1) + (2n+1)s]$$

where "n" is the Landau level index.

set $u = \cos\frac{\theta}{2} e^{\frac{i}{2}\varphi}$

$v = \sin\frac{\theta}{2} e^{-\frac{i}{2}\varphi}$

$$\longrightarrow \boxed{\hat{\Omega}(u,v) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)}$$

use (u, v) as coordinate, $(u,v) \nearrow CP(1)$ Rep

- define coherent states. which is $\{\hat{\Omega}(\alpha\beta)\cdot\hat{L}\}\psi_{\alpha\beta}$

$$= \hbar S\, \psi_{\alpha\beta}$$ ← spin coherent state

highest weight state

$$\boxed{\psi_{\alpha\beta}(u,v) = (\alpha^* u + \beta^* v)^{2S}}$$

$\alpha\beta$ is the label parameter, u, v contains coordinates.

the operator

$$\begin{cases} L^+ = \hbar u \partial/\partial v, \quad L^- = \hbar v \dfrac{\partial}{\partial u}, \quad L^z = \dfrac{\hbar}{2}\left[u\dfrac{\partial}{\partial u} - v\dfrac{\partial}{\partial v}\right] \\ S = \dfrac{1}{2}\left[u\dfrac{\partial}{\partial u} + v\dfrac{\partial}{\partial v}\right] \end{cases}$$

- Schwinger boson operator

Eigenstate ~~...~~ of LLL

$$|Sm\rangle = \sqrt{\frac{(2S)!}{(S+m)!\,(S-m)!}}\; u^{S+m}\, v^{S-m}$$

Check normalization

$$N^2 \int \frac{d\theta}{4\pi} \int d\varphi \; \cos^{S+m}\frac{\theta}{2}\, \sin^{S-m}\frac{\theta}{2} = 1$$ → I don't how to do this integral.

- ⇒ $\hbar u \dfrac{\partial}{\partial v} |Sm\rangle = \sqrt{\dfrac{(2S)!}{(S+m+1)!\,(S-m-1)!}}\; \dfrac{u^{S+m+1}\, v^{S-m}}{\sqrt{(S-m)(S+m+1)}} = \sqrt{(S-m)(S+m+1)}\; |S, m+1\rangle$

and so on ......

the LLL has degeneracy $2S+1$

$\nu = 1$ $\quad \psi(1\,2\cdots N) = \prod_{1\le i<j\le N} (u_i v_j - v_i u_j)$

$N = 2S+1$

$\nu = 1/3$ $\quad \psi = \prod_{1\le i<j\le N} (u_i v_j - v_i u_j)^3$

$S' = \underset{\substack{\uparrow \\ \text{power of } u_1}}{\underline{3(N-1)}} \left(\frac{1}{2}\right) \Rightarrow$ degeneracy $\sim 3(N-1)+1$

$\Rightarrow \nu \simeq \frac{N}{3N} \to \frac{1}{3}$

quasi-hole

$$A^{+}(\alpha,\beta) = \prod_{i=1}^{N} (\beta u_i - \alpha v_i)$$

particle

$$A(\alpha,\beta) = \prod_{i=1}^{N} \left(\beta^* \frac{\partial}{\partial u_i} - \alpha^* \frac{\partial}{\partial v_i}\right)$$