Lect 9: Symmetry dictates interactions

— Yang-Mills theory

1. Weyl's idea

2. $U(1)$ gauge field, — quantization,

   Anderson-Higgs mechanism $U(1)$

3. Yang-Mills, non-Abelian Berry phase

   Fadder-popov quantization

4: Glashow-Weinberg-Salam model

   (Higgs mechanism). $SU_L(2) \otimes U(1)$

Weyl's gauge theory 1918~1919

Quantity	value at point 1	value at neighbourhood
coordinate	x^μ	$x^\mu + dx^\mu$
field	f	$f + (\partial_\mu f)\, dx^\mu$
scale	1	$1 + S_\mu dx^\mu$
scaled field	f	$\underline{f + (\partial_\mu + S_\mu) f\, dx^\mu}$

$$f(x+\Delta x) - e^{S_\mu \cdot \Delta x^\mu} f(x)$$

Fock 1927: $\pi_\mu = P_\mu - \frac{e}{c} A_\mu = -i\hbar \left[\partial_\mu - \frac{ie}{\hbar c} A_\mu \right]$

$$\psi(x+\Delta x) - e^{-i\frac{e}{\hbar c} A_\mu \cdot \Delta x^\mu} \psi(x)$$

phase change — non-integrable phase factor.

$\mathcal{U}(1)$ gauge field

$$\psi(x) \to e^{i\alpha(x)}\psi(x) \qquad \text{local } \mathcal{U}(1) \text{ phase}$$

$$\psi(x) - \psi(x-a) \ ? \qquad \psi(x) - \mathcal{U}(x, x-a)\,\psi(x-a)$$

$$\mathcal{U}(x, x-a) = e^{-i\frac{e}{\hbar c}\vec{A}\cdot\vec{a}} \simeq 1 - \frac{ie}{\hbar c}\vec{A}\cdot\vec{a}$$

$$e^{i\alpha(x)}\psi(x) - \mathcal{U}'(x, x-a)\, e^{i\alpha(x-a)}\psi(x-a)$$

$$= e^{i\alpha(x)}\left[\psi(x) - e^{-i\alpha(x)}\mathcal{U}'(x, x-a)\, e^{i\alpha(x-a)}\psi(x-a)\right)$$

Hence we require $\qquad = e^{i\alpha(x)}\left(\psi(x) - \mathcal{U}(x, x-a)\,\psi(x-a)\right)$

$$e^{-i\alpha(x)}\mathcal{U}'(x, x-a)\, e^{i\alpha(x-a)} = \mathcal{U}(x, x-a)$$

$$\exp\left[\left(-i\frac{e}{\hbar c}A_x' - i\partial_x\alpha\right)\cdot a\right] = \exp\left[-\frac{ie}{\hbar c}A_x\cdot a\right]$$

$$\Rightarrow -i\frac{e}{\hbar c}A_x' - i\partial_x\alpha = -\frac{ie}{\hbar c}A$$

$$\Rightarrow \vec{A}_x' = \vec{A}_x - \frac{\hbar c}{e}\partial_x\alpha, \quad \text{or} \quad \boxed{A_\mu' = A_\mu - \frac{\hbar c}{e}\partial_\mu\alpha}$$

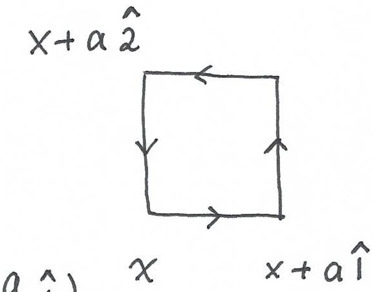
Connection:

$$\lim_{a\to 0}\frac{1}{a}\left[\psi(x) - \mathcal{U}(x, x-a)\,\psi(x-a)\right] = \frac{1}{a}\left[\psi(x) - \psi(x-a) + \frac{ie}{\hbar c}A_x\cdot a\,\psi(x)\right]$$

$$= \left(\partial_x + \frac{ie}{\hbar c}A_x\right)\psi(x) = D_x\psi(x)$$

$$\boxed{D_\mu\psi(x) \longrightarrow e^{i\alpha(x)}D_\mu\psi(x)}$$

$$\mathcal{U}(x, x+a\hat{2})\,\mathcal{U}(x+a\hat{2}, x+a\hat{1}+a\hat{2})\,\mathcal{U}(x+a\hat{1}+a\hat{2}, x+a\hat{1})\,\mathcal{U}(x+a\hat{1}, x)$$



$$= \exp\Big[-\frac{iea}{\hbar c}\Big[-A_2(x+\tfrac{a}{2}\hat{2}) - A_1(x+a\hat{2}+\tfrac{a}{2}\hat{1})$$
$$+ A_2(x+a\hat{1}+\tfrac{a}{2}\hat{2}) + A_1(x+\tfrac{a}{2}\hat{1})\Big]$$

$$= \exp\Big[-\frac{ie}{\hbar c}\,a^2\Big[\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}\Big]\Big]\Big\}$$

Define $\quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

$$[D_\mu, D_\nu] = [\partial_\mu + \frac{ie}{\hbar c}A_\mu, \partial_\nu + \frac{ie}{\hbar c}A_\nu] = \frac{ie}{\hbar c}F_{\mu\nu}$$

$$\to \quad \underbrace{\mathcal{L} = \bar{\psi}(i\not{D})\psi - \frac{1}{4}(F_{\mu\nu}F^{\mu\nu}) - m\bar{\psi}\psi}_{\text{QED Langrangian}} \; \underbrace{-\, i\,\frac{\theta e^2}{32\pi^2\hbar c}\,\epsilon^{\alpha\beta\mu\nu}F_{\alpha\beta}F_{\mu\nu}}_{\text{axion}}$$

total derivative

$$\varepsilon^{\alpha\beta\mu\nu}F_{\alpha\beta}F_{\mu\nu} = 4\,\varepsilon^{\alpha\beta\mu\nu}\partial_\alpha(A_\beta\partial_\mu A_\nu)$$

$$\to S_{surface} = \int \underset{dt\,dx\,dy}{d^3x}\;\frac{\theta e^2}{8\pi^2\hbar c}\,\varepsilon^{z\nu\rho\lambda}A_\nu\partial_\rho A_\lambda$$

Quantization of the E&M field

$$\int DA\, e^{iS[A]} \quad \text{where } S = \int d^4x\, (-\tfrac{1}{4}(F_{\mu\nu})^2]$$

$$S = \int d^4x \{-\tfrac{1}{4}(\partial^\mu A^\nu - \partial^\nu A^\mu)(\partial_\mu A_\nu - \partial_\nu A_\mu)\}$$

$$= -\tfrac{1}{2}\int d^4x\, A^\nu(\partial^\mu\partial_\mu A_\nu - \partial^\mu \partial_\nu A_\mu) = -\tfrac{1}{2}\int d^4x\, A_\nu(\partial^2 A_\nu) - \partial^\mu\partial^\nu \overset{A_\nu}{;} A_\mu$$

$$= -\tfrac{1}{2}\int d^4x\, A_\mu(\partial^2 g^{\mu\nu} - \partial^\mu\partial^\nu)A_\nu \qquad \leftarrow A_\mu = A(k)\, e^{-ikx}$$

$$= \frac{1}{2}\int \frac{d^4k}{(2\pi)^4} A_\mu(k)(-k^2 g^{\mu\nu} + k^\mu k^\nu) A_\nu(-k)$$

The kaernal $\det[-k^2 g^{\mu\nu} + k^\mu k^\nu] = 0$, since $(-k^2 g^{\mu\nu} + k^\mu k^\nu)\; k_\nu \alpha(k)$

$$= (k^2 - k^2) k^\mu \alpha(k) = 0$$

$\Rightarrow$ pure gauge $A_\nu = k_\nu \alpha(k)$ is unphysical, which should be excluded from the physical configuration. This is the difficulty of quantizing gauge field. In fact

$$(\partial^2 g_{\mu\nu} - \partial_\mu\partial_\nu) D_F^{\nu\rho}(x-y) = i\delta_\mu{}^\rho \delta^{(4)}(x-y)$$

or $(-k^2 g_{\mu\nu} + k_\mu k_\nu) D_F^{\nu\rho}(k) = i\delta_\mu{}^\rho$ has no unique solution.

All configurations of $A^\alpha_\mu(x) = A_\mu(x) + \frac{1}{e}\partial_\mu \alpha(x)$, no matter the configuration of $\alpha(x)$, should be identified as one physical field! Hence, we should separate $\int DA\, e^{iS[A]}$ the physical field contribution and the unphysical reduandency!

Fadder - popov trick

Set $G(A)$ as a gauge fixing condition, say, $G(A) = \partial_\mu A^\mu$.

we require $G(A) = 0$, which can be done via

$$1 = \int D\alpha(x)\, \delta(G(A^\alpha(x))) \det\left(\frac{\delta G(A^\alpha)}{\delta \alpha}\right), \quad \text{where } A^\alpha_\mu(x) = A_\mu(x) + \frac{1}{e}\partial_\mu \alpha(x).$$

discrete version

$$1 = \int \prod_i d\alpha_i \; \delta^{(n)}(g_i(\alpha_j)) \det\left[\frac{\partial g_i}{\partial \alpha_j}\right]$$

$G(A^\alpha) = \partial^\mu A_\mu + 1/e\, \partial^2 \alpha$, hence $\det\left(\frac{\delta G(A^\alpha)}{\delta \alpha}\right) = \det(1/e\, \partial^2)$

which is a constant independent of α and A.

$$\int DA\, e^{iS(A)} = \det\left(\frac{\delta G(A^\alpha)}{\delta \alpha}\right) \int D\alpha \int DA\, e^{iS[A]}\, \delta(G(A^\alpha))$$

Change variable A. $DA = DA^\alpha$, since $A^\alpha(x)$ compared to $A(x)$ just a shift. $S[A] = S[A^\alpha]$ since $S[A]$ is gauge invariant.

$$\int DA\, e^{iS(A)} = \det\left(\frac{\delta G(A^\alpha)}{\delta \alpha}\right) \int D\alpha \int DA^\alpha\, e^{iS[A^\alpha]}\, \delta(G(A^\alpha))$$

$$= \underbrace{\det\left(\frac{\delta G(A^\alpha)}{\delta \alpha}\right) \int D\alpha}_{\text{redunancy}} \int DA\, e^{iS[A]}\, \delta(G(A))$$

Now we conside $G(A) = \partial_\mu A^\mu - \omega(x) = 0$, $\det\left(\frac{\delta G(A^\alpha)}{\delta \alpha}\right) = \det\left(\frac{\partial^2}{e}\right)$

where $\omega(x)$ is an arbitany function as before

$$\int DA\, e^{iS[A]} = \det\left[\frac{1}{e}\partial^2\right] \int D\alpha \int DA\, e^{iS[A]}\, \delta(\partial^\mu A_\mu - w(x))$$

This result doesn't depend on $w(x)$, we can $\int Dw(x)\, e^{-i\int d^4x \frac{w^2}{2\xi}}$

$$\Rightarrow N(\xi) \int Dw\, e^{-i\int d^4x \frac{w^2}{2\xi}} \det\left(\frac{1}{e}\partial^2\right) \int D\alpha \int DA\, e^{iS[A]}\, \delta(\partial^\mu A_\mu - w(x))$$

$$= \underbrace{N(\xi) \det\left[\frac{1}{e}\partial^2\right] \int D\alpha}_{\text{redunancy factor}} \int DA\, \underbrace{e^{iS[A] - i\int d^4x \frac{1}{2\xi}(\partial^\mu A_\mu)^2}}_{\text{regular part}}$$

$$\left[-k^2 g_{\mu\nu} + \left(1-\frac{1}{\xi}\right) k_\mu k_\nu\right] \tilde{D}^{\nu\rho}(x) = i\delta_\mu{}^\rho$$

$$\Rightarrow \boxed{\tilde{D}^{\mu\nu}(k) = \frac{-i}{k^2+i\epsilon}\left(g^{\mu\nu} - (1-\xi)\frac{k^\mu k^\nu}{k^2}\right)}$$

Anderson - Higgs mechanism $U(1)$ version (superconductivity)

$$\mathcal{L} = -\tfrac{1}{4}(F_{\mu\nu})^2 + |D_\mu \phi|^2 - V(\phi), \quad D_\mu = \partial_\mu + ie A_\mu$$

$U(1)$ gauge sym: $\phi(x) \to e^{i\alpha(x)}\phi(x)$

$$A_\mu \to A'_\mu(x) = A_\mu(x) - \tfrac{1}{e}\partial_\mu \alpha(x)$$

Condensation $V(\phi) = -\mu^2|\phi|^2 + \frac{\lambda}{2}|\phi|^4 \Rightarrow \langle\phi\rangle = \phi_0 = \left(\frac{\mu^2}{\lambda}\right)^{1/2}$

$$\phi(x) = \phi_0 + \frac{1}{\sqrt{2}}(\phi_1(x) + i\phi_2(x))$$

$$\Rightarrow V(\phi) = \frac{-1}{2\lambda}\mu^4 + \frac{1}{2}(2\mu^2)\phi_1^2 + \cdots$$

$\boxed{\phi_2 \text{ is the massless Goldstone boson.}}$

$$|D_\mu\phi|^2 = \left|\partial_\mu \tfrac{1}{\sqrt{2}}\phi_1(x) + i\partial_\mu\tfrac{1}{\sqrt{2}}\phi_2(x) + ieA_\mu\left(\phi_0 + \tfrac{1}{\sqrt{2}}\phi_1 + i\phi_2\right)\right|^2$$

$$= \left|\partial_\mu\tfrac{1}{\sqrt{2}}\phi_1(x) - eA_\mu\phi_2\right|^2 + \left|\partial_\mu\tfrac{1}{\sqrt{2}}\phi_2(x) + eA_\mu\left(\phi_0 + \tfrac{1}{\sqrt{2}}\phi_1\right)\right|^2$$

$$= \frac{1}{2}(\partial_\mu\phi_1)^2 + \underbrace{\frac{1}{2}(\partial_\mu\phi_2)^2 + e^2\phi_0^2 A_\mu^2 + \sqrt{2}e A_\mu \phi_0 \partial^\mu\phi_2}_{\left(\frac{1}{\sqrt{2}}\partial_\mu\phi_2 + e\phi_0 A_\mu\right)^2} + \cdots$$

$$+ \frac{e A_\mu^2 \phi_1 \phi_0}{\sqrt{2}}$$

$$\Rightarrow \mathcal{L} = \frac{1}{2}(\partial_\mu\phi_1)^2 + \frac{1}{2}(2\mu^2)\phi_1^2$$

$$-\tfrac{1}{4}(F_{\mu\nu})^2 + (e\phi_0)^2\Big(\underbrace{A_\mu + \frac{1}{\sqrt{2}e}\frac{1}{\phi_0}\partial_\mu\phi_2}\Big)^2$$

Goldstone boson becomes the longitudinal component of A_μ.

Gauge field acquires mass.

(*) unitary gauge $\quad \phi(x) = (\phi_0 + \delta\phi_0)\, e^{i\theta(x)}$

$$V(\phi) = -\mu^2 (\phi_0 + \delta\phi_0)^2 + \frac{\lambda}{2}(\phi_0 + \delta\phi_0)^4 = \frac{-1}{2\lambda}\mu^4 + \frac{1}{2}(2\mu^2)(\delta\phi_0$$

$$\partial_\mu \phi = \partial_\mu \delta\phi_0\, e^{i\theta(x)} + (\phi_0 + \delta\phi_0)\, e^{i\theta(x)}\, i\,\partial_\mu \theta(x)$$

$$(\partial_\mu + ieA_\mu)\phi = e^{i\theta(x)}\,[\,\partial_\mu \delta\phi + \phi_0\, i\, e[A_\mu + \frac{1}{e}\partial_\mu \theta(x)] + i\,\delta\phi_0\, \partial_\mu\theta$$

$$|D_\mu \phi|^2 = (\partial_\mu \delta\phi)^2 + e^2\phi_0^2 \underline{(A_\mu + \frac{1}{e}\partial_\mu \theta(x))}^2 + \cdots\cdots$$

Goldstone boson becomes gauge field's longitudinal component.

Yang - Mills

Consider an SU(2) doublet $\psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix}$, a global SU(2) rotation $\psi \to V\psi$ where $V = e^{i\vec{\alpha}\cdot\frac{\vec{\sigma}}{2}}$.

Now we promote the global SU(2) symmetry to the local one

$$\psi(x) \to \psi'(x) = V(x)\psi(x) \quad \text{where } V(x) = e^{i\vec{\alpha}(x)\cdot\frac{\vec{\sigma}}{2}}.$$

The consider two points $y - x = \varepsilon \to 0$, where we take the difference we need consider the connection

$$\boxed{\psi(y) - \mathcal{U}(y,x)\psi(x)}\ ,$$

$\mathcal{U}(y,x)$ transfer the phase at x to the SU(2) phase at y

under local SU(2) transformation

$$\psi'(y) - \mathcal{U}'(y,x)\psi'(x) = V(y)\psi(y) - \mathcal{U}'(y,x)V(x)\psi(x)$$

$$= V(y)\left(\psi(y) - V^\dagger(y)\mathcal{U}'(y,x)V(x)\psi(x)\right)$$

hence, we require $\mathcal{U}(y,x) = V^\dagger(y)\mathcal{U}'(y,x)V(x)$

$$\Rightarrow \boxed{\mathcal{U}'(y,x) = V(y)\,\mathcal{U}(y,x)\,V^\dagger(x)}$$

$$\mathcal{U}(y,x) = \exp\left[i\, g/_{\hbar c}\, A^i_\mu \cdot \frac{\sigma^i}{2} \cdot \varepsilon^\mu \right] \sim 1 + i g/_{\hbar c}\, \vec{A}_\mu \cdot \frac{\vec{\sigma}}{2}\, \varepsilon^\mu$$ ②

then $\lim\limits_{y\to x} \psi(y) - \mathcal{U}(y,x)\, \psi(x)$

$$= \psi(x) + \epsilon^\mu \cdot \partial_\mu \psi(x) - \left[1 + i g/_{\hbar c}\, \vec{A}_\mu \cdot \frac{\vec{\sigma}}{2}\, \varepsilon^\mu \right] \psi(x)$$

$$= \varepsilon^\mu \cdot \left[\partial_\mu - i g/_{\hbar c}\, \vec{A}_\mu \cdot \frac{\vec{\sigma}}{2} \right] \psi(x)$$

$$\Rightarrow \boxed{D_\mu \psi(x) = \left(\partial_\mu - i g/_{\hbar c}\, \vec{A}_\mu \cdot \frac{\vec{\sigma}}{2} \right) \psi(x)}$$

Do it again, $\vec{A}_\mu \to \vec{A}'_\mu$, $\psi(x) \to \psi'(x) = e^{i \vec{\alpha}(x) \frac{\vec{\sigma}}{2}} \psi(x)$

$= V(x)\, \psi(x)$

$$\left(\partial_\mu - \frac{ig}{\hbar c} \vec{A}'_\mu \cdot \frac{\vec{\sigma}}{2} \right) \widehat{V(x)}\, \psi(x)$$

$$= V(x) \left[V^\dagger(x)\, \partial_\mu V(x) - \frac{ig}{\hbar c} V^\dagger(x)\, \vec{A}'_\mu \cdot \frac{\sigma}{2}\, V(x) \right] \psi(x)$$

$$V^\dagger(x)\, \partial_\mu \left(V(x)\, \psi(x) \right) = \partial_\mu \psi(x) + \left(V^\dagger(x)\, \partial_\mu V(x) \right) \psi(x)$$

$$\Rightarrow \left(\partial_\mu - \frac{ig}{\hbar c} \vec{A}'_\mu \cdot \frac{\vec{\sigma}}{2} \right) \left(V(x)\, \psi(x) \right)$$

$$= V(x) \left[\partial_\mu - \frac{ig}{\hbar c} V^\dagger \vec{A}'_\mu \cdot \frac{\vec{\sigma}}{2} V + V^\dagger(x)\, \partial_\mu V(x) \right] \psi(x)$$

$$\Rightarrow -\frac{ig}{\hbar c} \vec{A}_\mu \cdot \frac{\vec{\sigma}}{2} = -\frac{ig}{\hbar c} V^\dagger \vec{A}'_\mu \cdot \frac{\vec{\sigma}}{2} V + V^\dagger(x)\, \partial_\mu V(x)$$

$$\boxed{\vec{A}'_\mu \cdot \frac{\vec{\sigma}}{2} = V\, \vec{A}_\mu \cdot \frac{\vec{\sigma}}{2} V^\dagger - \frac{i\hbar c}{g} \partial_\mu V\, V^\dagger = V\left[\vec{A}_\mu \cdot \frac{\vec{\sigma}}{2} + \frac{i\hbar c}{g} \partial_\mu \right] V^\dagger}$$

Field strength

$$[D'_\mu, D'_\nu]\,\psi'(x) = \left[\partial_\mu - \frac{ig}{\hbar c}\vec{A}'_\mu\cdot\frac{\vec{\sigma}}{2},\ \partial_\nu - \frac{ig}{\hbar c}\vec{A}'_\nu\cdot\frac{\vec{\sigma}}{2}\right] V(x)\psi(x)$$

$$= V(x)\,\underbrace{V^\dagger\left[\partial_\mu - \frac{ig}{\hbar c}\vec{A}'_\mu\cdot\frac{\vec{\sigma}}{2},\ \partial_\nu - \frac{ig}{\hbar c}\vec{A}'_\nu\cdot\frac{\vec{\sigma}}{2}\right]V}\ \psi(x)$$

$$[D_\mu, D_\nu] = \left[\partial_\mu - \frac{ig}{\hbar c}\vec{A}_\mu\cdot\frac{\vec{\sigma}}{2},\ \partial_\nu - \frac{ig}{\hbar c}\vec{A}_\nu\cdot\frac{\vec{\sigma}}{2}\right]$$

$$= -\frac{ig}{\hbar c}\left[\partial_\mu\vec{A}_\nu - \partial_\nu\vec{A}_\mu\right]\cdot\frac{\vec{\sigma}}{2} + \left(\frac{-ig}{\hbar c}\right)^2 i\,\epsilon_{ijk}\,A^i_\mu A^j_\nu\left(\frac{\sigma}{2}\right)^k$$

$$= -\frac{ig}{\hbar c}\left[\partial_\mu A^i_\nu - \partial_\nu A^i_\mu - \frac{ig}{\hbar c}i\,\varepsilon_{ijk}\,A^j_\mu A^k_\nu\right]\left(\frac{\sigma}{2}\right)^i$$

$$= -\frac{ig}{\hbar c}\,F^i_{\mu\nu}\cdot\left(\frac{\sigma}{2}\right)^i$$

$$\Rightarrow\quad V^\dagger F'^i_{\mu\nu}\left(\frac{\sigma}{2}\right)^i V = F^i_{\mu\nu}\left(\frac{\sigma}{2}\right)^i$$

$$\Rightarrow\quad \boxed{F'^i_{\mu\nu}\left(\frac{\sigma}{2}\right)^i = V\left(F^i_{\mu\nu}\left(\frac{\sigma}{2}\right)^i\right)V^\dagger}$$

$$F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + \frac{g}{\hbar c}\left(\varepsilon_{ijk}\right)A^j_\mu A^k_\nu$$

or $$\boxed{F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \frac{ig}{\hbar c}\left[A_\mu, A_\nu\right]}$$

Yang-Mills Lagrangian

$$\mathcal{L} = \bar{\psi}(i\not{D})\psi - \frac{1}{2}\mathrm{tr}\left(F^i_{\mu\nu}\frac{\sigma^i}{2}\right)^2 - m\bar{\psi}\psi, \text{ where } \not{D} = D_\mu\gamma^\mu$$

$$\searrow \quad -\frac{1}{4}\sum_i F^{i,\mu\nu}F^i_{,\mu\nu}$$

Classic equation of motion:

$$\frac{\delta\mathcal{L}}{\delta A^{i\mu}} = ① + ②$$

$$①: \quad \bar{\psi}\, i\left(\partial^\mu\gamma_\mu - ig A^{i,\mu}\gamma_\mu \cdot \frac{\sigma^i}{2}\psi\right) \rightarrow$$

$$① = g\bar{\psi}\gamma_\mu\frac{\sigma^i}{2}\psi$$

$$② \quad -\frac{1}{2}\sum_{i'} \delta F^{i',\mu'\nu'}/\delta A^{i,\mu} \cdot F^{i'}_{\mu'\nu'}$$

$$F^{i',\mu'\nu'} = \partial^{\mu'}A^{\nu',i'} - \partial^{\nu'}A^{\mu',i'} + g\,\varepsilon_{i'j'k'}A^{\mu',j'}A^{\nu',k'}$$

$$\delta F^{i',\mu'\nu'}/\delta A^{\mu,i} = \partial^{\mu'}(\delta_{\mu\nu'}\delta_{ii'}) - \partial^{\nu'}(\delta_{\mu\mu'}\delta_{ii'})$$
$$- g\,\varepsilon_{i'j'k'}\left(\delta_{\mu\mu'}\delta_{ij'}A^{\nu'k'} + A^{\mu'j'}\delta_{\mu\nu'}\delta_{ik'}\right)$$

$$\sum_{i'} \delta F^{i',\mu'\nu'}/\delta A^{i,\mu}\; F^{i'}_{\mu'\nu'}$$

$$= \delta_{\mu\nu'}\delta_{ii'}\,\partial^{\mu'}F^{i'}_{\mu'\nu'} - \delta_{\mu\mu'}\delta_{ii'}\,\partial^{\nu'}F^{i'}_{\mu'\nu'}$$
$$-\left(\delta_{\mu\mu'}\delta_{ij'}\,g\,\varepsilon_{i'j'k'}A^{\nu'k'} + \delta_{\mu\nu'}\delta_{ik'}\,g\,\varepsilon_{i'j'k'}A^{\mu'j'}\right)F^{i'}_{\mu'\nu'}$$

$$\xrightarrow{\text{sum}} \partial^{\mu'}F^i_{\mu'\mu} - \partial^{\nu'}F^i_{\mu\nu'} - g\,\varepsilon_{i'ik'}A^{\nu'k'}F^{i'}_{\mu\nu'} - g\,\varepsilon_{i'j'i}A^{\mu'j'}F^{i'}_{\mu'\mu}$$

$$= 2\left(\partial^\nu F^i_{\nu\mu} - g\,\varepsilon_{ijk}\, A^{\nu,j} F^k_{\mu\nu}\right)$$

$$= 2\left[\partial^\nu F^i_{\nu\mu} + g\,\varepsilon_{ijk}\, A^{\nu,j} F^k_{\nu\mu}\right]$$

$$\Rightarrow \; 2 = -\left[\partial^\nu F^i_{\nu,\mu} + g\,\varepsilon_{ijk}\, A^{\nu,j} F^k_{\nu\mu}\right]$$

$$\Rightarrow \; \partial^\nu F^i_{\nu,\mu} + g\,\varepsilon^{ijk} A^{\nu,j} F^k_{\nu\mu} = g\,\bar\psi \gamma_\mu \frac{\sigma^i}{2}\psi$$

or $\boxed{\partial^\mu F^i_{\mu\nu} + g\,\varepsilon^{ijk} A^{\mu,j} F^k_{\mu\nu} = g\,\bar\psi \gamma_\nu \frac{\sigma^i}{2}\psi}$

Lect 12. Non-abelian Berry phase / holonomy

In this lecture, we generalize the Berry phase to systems with energy level degeneracy. We will see the Berry connection becomes a matrix, not just a phase, and non-abelian structure appears. Suppose $|\eta_\alpha(R)\rangle$ $(\alpha = 1, \cdots N)$ is an N-fold degenerate set of ortho-normal instaneous eigenstates.

Let us write the eigenstate

$$|\psi_a(t)\rangle = \sum_b |\eta_b(R(t))\rangle\, u_{ba}(t)\, e^{-i\int_0^t dt'\, E(t')/\hbar}$$

at $t=0$, $|\psi_a(0)\rangle = |\eta_a(R(0))\rangle$ and $u_{ba}(0) = \delta_{ba}$.

$$i\hbar\frac{\partial}{\partial t}|\psi_a(t)\rangle = \sum_b i\hbar\frac{\partial}{\partial t}|\eta_b(R(t))\rangle\, u_{ba}(t)\, e^{-i\int_0^t dt' E(t')/\hbar}$$

$$+\sum_b i\hbar\, |\eta_b(R(t))\rangle\, \frac{\partial}{\partial t}u_{ba}(t)\, e^{-i\int_0^t dt' E(t')/\hbar}$$

$$+\sum_b |\eta_b(R(t))\rangle\, u_{ba}(t)\, E(t')\, e^{-i\int\cdots} = H(t)|\psi_a(t)\rangle = \sum_b |\eta_b(R(t))\rangle\, u_{ba}\, E(t')\, e^{-i\int_0^t dt' E/\hbar}$$

$$\Rightarrow \sum_b \frac{\partial}{\partial t}|\eta_b(R(t))\rangle\, u_{ba}(t) = -\sum_b |\eta_b(R(t))\rangle \frac{\partial}{\partial t}u_{ba}(t)$$

$$\sum_b \langle\eta_c(R(t))|\eta_b(R(t))\rangle \frac{\partial}{\partial t}u_{ba}(t) = \sum_b \langle\eta_c(R(t))|\frac{\partial}{\partial t}|\eta_b(R(t))\rangle\, u_{ba}(t)$$

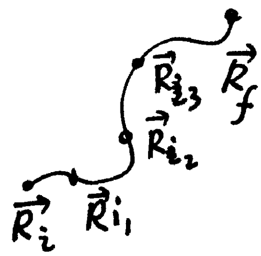
$$\Rightarrow \frac{\partial}{\partial t}\dot{u}_{ca} = -\sum_b \langle\eta_c|\frac{d}{dt}|\eta_b\rangle\, u_{ba}$$

define non-Abeliean gauge field

$$A_{ab,\mu} = -i \langle \eta_a(\vec{R}) | \nabla_{R_\mu} | \eta_b(\vec{R}) \rangle$$

$\Rightarrow \quad \frac{\partial}{\partial R_\mu} \mathcal{U} = -i A_\mu \cdot \mathcal{U}$ where $\mathcal{U}$, A_μ are $N \times N$ matrix

$$\Rightarrow \quad \mathcal{U}(\vec{R}_f) = T_R \exp\left[-i \int_{\vec{R}_i}^{\vec{R}_f} dR_\mu A_\mu(\vec{R}) \right] \mathcal{U}(\vec{R}_i)$$



path ordered operator

$$T_R \exp\left[-i \int_{\vec{R}_i}^{\vec{R}_f} d\vec{R}\, A_\mu(\vec{R})\right]$$

$$= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{\vec{R}_i}^{\vec{R}_f} dR_{n,\mu_n} \int_{\vec{R}_i}^{\vec{R}_f} dR_{\mu_{n-1}} \cdots \int_{\vec{R}_i}^{\vec{R}_f} dR_{1,\mu_1}\, T_R \left[A_{\mu_n}(\vec{R}_n) A_{\mu_{n-1}}(\vec{R}_{n-1}) \cdots A_{\mu_1}(\vec{R}_1) \right]$$

where $T_R[A_{\mu_n}(\vec{R}_n) \cdots A_{\mu_1}(\vec{R}_1)] = A_{\mu_{in}}(\vec{R}_{in}) A_{\mu_{in-1}}(\vec{R}_{in-1}) \cdots A_{\mu_{i1}}(\vec{R}_{i1})$

and along the path from $\vec{R}_i$ to $\vec{R}_f$, $\underbrace{\vec{R}_{in} > \vec{R}_{in-1} > \cdots > \vec{R}_{i1}}_{\text{the sequence is as}}$,

which is a permutation of $\vec{R}_n, \cdots \vec{R}_1$ in the right order.

For a close loop, and suppose we state from $\mathcal{U}(0) = 1 \Rightarrow$

The ~~flux~~ non-abelian phase gained is $\boxed{\mathcal{U} = T_R \exp\left[-i \oint d\vec{R}\, A_\mu(\vec{R})\right] \quad \text{Wilson loop}}$

Gauge transformation: For degenerate states $|\eta_a(R)\rangle$

$$\rightarrow \quad |\eta_a(R)\rangle \rightarrow |\tilde{\eta}'_a(R)\rangle = |\eta_b(R)\rangle \; W_{ba}(R)$$

$$\Rightarrow \langle \tilde{\eta}_a(R)| = \langle \eta_b | W^*_{ba} = \langle \eta_b | W^\dagger_{ab}$$

$$\Rightarrow \quad \tilde{A}_{ab,\mu} = -i \langle \tilde{\eta}_a(R) | \nabla_{R\mu} | \tilde{\eta}_b(R)\rangle$$

$$= -i \langle \eta_{a'} | W^\dagger_{aa'} | \nabla_{R\mu} \{ |\eta_{b'}(R)\rangle W_{b'b}(R) \}$$

$$= W^\dagger_{aa'} (-i) \langle \eta_{a'} | \nabla_{R\mu} | \eta_{b'} \rangle W_{b'b}$$

$$+ \; W^\dagger_{aa'} (-i) \langle \eta_{a'} | \eta_{b'} \rangle \nabla_{R\mu} W_{b'b}$$

$$= W^\dagger_{aa'} A_{a'b'} W_{b'b} + (-i) W^\dagger_{aa'} \nabla_{R\mu} W_{a'b}$$

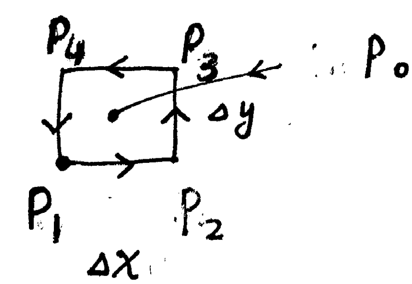
$$\Rightarrow \quad \boxed{\tilde{A}_\mu = W^\dagger A_\mu W - i \, W^\dagger \nabla_{R\mu} W}$$

non-abelian gauge transformation

W is an unitary matrix

Non-abelian gauge field strength - Curvature

$$T_R \exp[-i \oint d\vec{R}\, A] = \exp[-i F_{xy} \Delta x \Delta y]$$



$$1 - i \oint dR_\mu A_\mu + \frac{(-i)^2}{2!} \oint dR_{2,\mu_2} \oint dR_{1,\mu_1} T[A_{\mu_2}(\vec{R}_2) A_{\mu_1}(\vec{R}_1)]$$

$$= 1 - i F_{xy} \Delta x \Delta y$$

The $\oint dR_\mu A_\mu(\vec{R}) = \Delta x A_x(P_0 - \frac{\Delta y}{2}\hat{e}_y) + \Delta y\, A_y(P_0 + \frac{\Delta x}{2}\hat{e}_x) - \Delta x\, A_x(P_0 + \frac{\Delta y}{2}\hat{e}_y) - \Delta y\, A_y(P_0 - \frac{\Delta x}{2}\hat{e}_x)$

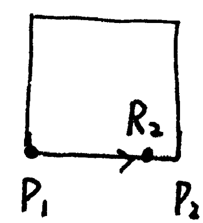
$$\Delta X = \left[A_x(P_0) - \partial_{R_y} A_x \frac{\Delta y}{2}\right] + \left[A_y(P_0) + \partial_{R_x} A_y \frac{\Delta X}{2}\right] \Delta y$$

$$\Delta X\left[- A_x(P_0) - \partial_{R_y} A_x \frac{\Delta y}{2}\right] + \left[A_y(P_0) + \partial_{R_x} A_y \frac{\Delta X}{2}\right] \Delta y$$

$$= \left[\partial_{R_x} A_y - \partial_{R_y} A_x\right] \Delta x \Delta y$$

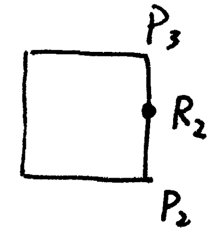
The second term $= (-i)^2 \cdot$

$$\oint dR_{2,\mu_2} \int_{P_1}^{R_2} dR_1 \, A_{\mu_2}(R_2) A_{\mu_1}(R_1)$$



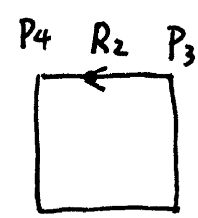
① if R_2 is from $P_1 \to P_2$ $\Rightarrow$ $\int_{P_1}^{P_2} dR_{2,x} \int_{P_1}^{R_2} dR_1 A_x A_x = \frac{(\Delta X)^2}{2} A_x^2$

② if R_2 is from $P_2 \to P_3$



$$\int_{P_2}^{P_3} dR_{2,y} \left[\int_{P_1}^{P_2} dR_{1,x} A_y A_x + \int_{P_2}^{R_2} dR_{1,y} A_y A_y \right]$$

$$= \Delta y \Delta X \, A_y A_x + \frac{(\Delta y)^2}{2} A_y^2$$

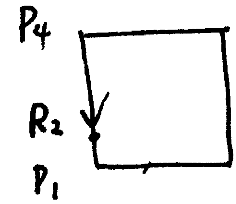


③ if R_2 is from $P_3 \to P_4$

$$\int_{P_3}^{P_4} dR_{2,x} \left[\int_{P_1}^{P_2} dR_{1,x} A_x A_x + \int_{P_2}^{P_3} dR_{1,y} A_x A_y + \int_{P_3}^{P_4} dR_{1,x} A_x A_x \right]$$

$$\ldots (\Delta X)(\Delta y) A_y A_x + \left(\frac{\Delta X}{2}\right) A_x^2$$

4 if R_2 is from $R_4 \to R_1$



$$\int_{P_4}^{P_1} dR_{2y} \Big[\int_{P_1}^{P_2} dR_{1,x} A_y A_x + \int_{P_2}^{P_3} dR_{1y} A_y A_y$$

$$+ \int_{P_3}^{P_4} dR_{1x} A_y A_x + \int_{P_4}^{P_1} dR_{1y} A_y A_y \Big]$$

$$= -\Delta y \Delta x \; A_y A_x + -(\Delta y)^2 A_y^2 + (\Delta x \Delta y) A_y A_x + \frac{(\Delta y)^2}{2} A_y^2$$

$\Rightarrow$ Add together $\Rightarrow$ $-\Delta x \Delta y \, [A_x A_y - A_y A_x]$

$$\Rightarrow -i [\partial_{R_x} A_y - \partial_{R_y} A_x] \Delta x \Delta y + \Delta x \Delta y (A_x A_y - A_y A_x) = -i F_{xy} \Delta x \Delta y$$

$$\Rightarrow \boxed{F_{\mu\nu} = \partial_{R_\mu} A_\nu - \partial_{R_\nu} A_\mu + i [A_\mu, A_\nu]}$$

under gauge transformation $\tilde{A}_\mu = W^\dagger A_\mu W - i W^\dagger \nabla_\mu W$

$$\Rightarrow \tilde{F}_{\mu\nu} = \partial_\mu \tilde{A}_\nu - \partial_\nu \tilde{A}_\mu + i [\tilde{A}_\mu, \tilde{A}_\nu]$$

$$= (\partial_\mu W^\dagger) A_\nu W + W^\dagger \partial_\mu A_\nu W + W^\dagger A_\nu \partial_\mu W - i \, \partial_\mu (W^\dagger \partial_\nu W)$$

$$- (\partial_\nu W^\dagger) A_\mu W - W^\dagger \partial_\nu A_\mu W - W^\dagger A_\mu \partial_\nu W + i \, \partial_\nu (W^\dagger \partial_\mu W)$$

$$+ i [W^\dagger A_\mu W, W^\dagger A_\nu W] + [W^\dagger A_\mu W, W^\dagger \partial_\nu W] + [W^\dagger \partial_\mu W, W^\dagger A_\nu W]$$

$$- i [W^\dagger \partial_\mu W, W^\dagger \partial_\nu W]$$

check

$$\partial_\mu W^\dagger A_\nu W + W^\dagger A_\nu \partial_\mu W + [W^\dagger \partial_\mu W, \; W^\dagger A_\nu W]$$

$$- \partial_\mu W^\dagger A_\nu W - W^\dagger A_\nu \partial_\mu W$$

$$= 0$$

$$- \partial_\nu W^\dagger A_\mu W \mp W^\dagger \partial_\nu A_\mu W + [W^\dagger A_\mu W, \; W^\dagger \partial_\nu W]$$

$$W^\dagger A_\mu \partial_\nu W + \partial_\nu W^\dagger A_\mu W$$

$$= 0$$

$$- \partial_\mu (W^\dagger \partial_\nu W) + \partial_\nu (W^\dagger \partial_\mu W) - [W^\dagger \partial_\mu W, \; W^\dagger \partial_\nu W]$$

$$= - \partial_\mu W^\dagger \partial_\nu W + \partial_\nu W^\dagger \partial_\mu W + \partial_\mu W^\dagger \partial_\nu W - \partial_\nu W^\dagger \partial_\mu W = 0$$

$$\Rightarrow \boxed{\tilde{F}_{\mu\nu} = W^\dagger \left(\partial_\mu A_\nu - \partial_\nu A_\mu + i [A_\mu, A_\nu] \right) W = W^\dagger F_{\mu\nu} W .}$$

we used $\quad W^\dagger \partial_\mu W = - \partial_\mu W^\dagger W \quad$ above.

$$W \partial_\mu W^\dagger = - \partial_\mu W \, W^\dagger$$

Example: qudratic Zeeman for spin-$3/2$ system

$$H = (S\cdot B)^2.$$

each energy level is doubly degenerate due to time-reversal symmetry

PRA 38, 1 1988.

$$H = B^2\, e^{-i\varphi S_z} e^{-i\theta S_y}\, S_z^2\, e^{i\theta S_y} e^{i\varphi S_z}$$

we denote $|a\rangle = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$ $|b\rangle = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$ $|c\rangle = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}$ $|d\rangle = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$ eigenstates of S_z

$$|\eta_a\rangle = e^{-i\varphi S_z} e^{-i\theta S_y}|a\rangle$$

where θ, φ are the direction of B-field.

$$\frac{\partial}{\partial\theta}|\eta_b\rangle = e^{-i\varphi S_z} e^{-i\theta S_y}(-i)S_y|b\rangle$$

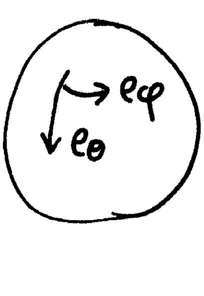
$$A_{ab,\theta} = -i\langle\eta_a|\frac{\partial}{\partial\theta}|\eta_b\rangle = -\langle a|e^{i\theta S_y} e^{i\varphi S_z} e^{-i\varphi S_z} e^{-i\theta S_y} S_y|b\rangle$$

$$= -\langle a|S_y|b\rangle$$

$$\frac{1}{\sin\theta\,\partial\varphi}|\eta_b\rangle = -i\frac{S_z}{\sin\theta}\, e^{-i\varphi S_z} e^{-i\theta S_y}|b\rangle$$

$$e^{i\theta S_y} e^{i\varphi S_z} S_z e^{-i\varphi S_z} e^{-i\theta S_y} = e^{i\theta S_y} S_z e^{-i\theta S_y} = -\sin\theta S_x + \cos\theta S_z$$

$$A_{ab,\varphi} = -\langle a|\frac{1}{\sin\theta}[\cos\theta S_z - \sin\theta S_x]|b\rangle$$



along $\hat{e}_\theta \Rightarrow A_{ab,\theta} = -\langle a|S_y|b\rangle$

$$= -\begin{pmatrix} 0 & -\frac{\sqrt{3}i}{2} & 0 & 0 \\ \frac{\sqrt{3}i}{2} & 0 & -i & 0 \\ 0 & i & 0 & -\frac{\sqrt{3}i}{2} \\ 0 & 0 & \frac{\sqrt{3}i}{2} & 0 \end{pmatrix}$$

non-abelian

along $\hat{e}_\varphi$: $A_{ab,\varphi} = \frac{-1}{\sin\theta} \langle a | \cos\theta S_z - \sin\theta S_x | b \rangle$

$$= \frac{-1}{\sin\theta} \begin{pmatrix} \frac{3}{2}\cos\theta & -\frac{\sqrt{3}}{2}\sin\theta & 0 & 0 \\ \frac{\sqrt{3}}{2}\sin\theta & \frac{1}{2}\cos\theta, & -\sin\theta & 0 \\ 0 & -\sin\theta & -\frac{1}{2}\sin\theta & -\frac{\sqrt{3}}{2}\sin\theta \\ 0 & 0 & -\frac{\sqrt{3}}{2}\sin\theta & 0 \end{pmatrix}$$

Take the $\pm 1/2$ part

$$\vec{A} = (-)\left[\frac{1}{\sin\theta} \left[-\sin\theta\, \sigma_1 + \frac{\cos\theta}{2} \sigma_3 \right] \hat{e}_\varphi + \sigma_2 \hat{e}_\theta \right] = -\vec{A}^i \left(\frac{\sigma^i}{2} \right)$$

$$\vec{A}^1 = -2\hat{e}_\varphi \qquad \vec{A}^2 = 2\hat{e}_\theta \; , \quad \vec{A}^3 = \text{ctg}\,\theta \; \hat{e}_\varphi$$

The non-abelian field strength

$$F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + \epsilon^{ijk} A^j_\mu A^k_\nu$$

define $F^i{}_\lambda = \frac{1}{2}\epsilon_{\mu\nu\lambda} F^i_{\mu\nu}$

$$= (\nabla \times \vec{A}^i)_\lambda + \frac{1}{2}\epsilon_{\lambda\mu\nu}\epsilon_{ijk} A^j_\mu A^k_\nu$$

$$\vec{F}^3 = F^3{}_r \hat{e}_r + F^3{}_\theta \hat{e}_\theta + F^3{}_\varphi \hat{e}_\varphi$$

$$= \nabla \times \vec{A}^3 + \hat{e}_r \frac{1}{2}\epsilon^{3jk}[A^j_\theta A^k_\varphi - A^j_\varphi A^k_\theta]$$
$$+ \hat{e}_\theta \frac{1}{2}\epsilon^{3jk}[A^j_\varphi A^k_r - A^j_r A^k_\theta]$$
$$+ \hat{e}_\varphi \frac{1}{2}\epsilon^{3jk}[A^j_r A^k_\theta - A^j_\theta A^k_r]$$

$$= \nabla \times \vec{A}^3 + \frac{\hat{e}_r}{2}[A^1_\theta A^2_\varphi - A^2_\theta A^1_\varphi - A^1_\varphi A^2_\theta + A^2_\varphi A^1_\theta]$$

$$\nabla \times \vec{A}^3 = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta \, ctg\theta)\,\hat{e}_r = -\hat{e}_r$$

$$\vec{F}^3 = -\hat{e}_r + \frac{\hat{e}_r}{2}[0 - 2(-2) - (-2)2 + 0] = 3\hat{e}_r$$

Similarly

$$\vec{F}^1 = (\nabla \times \vec{A}^1) + \frac{\hat{e}_r}{2}[A^2_\theta A^3_\varphi - A^3_\theta A^2_\varphi - A^2_\varphi A^3_\theta + A^3_\varphi A^2_\theta]$$

$$= \nabla \times (-2\hat{e}_\varphi) + \frac{\hat{e}_r}{2}[2ctg\theta + 2ctg\theta]$$

$$= -2\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\,\hat{e}_r + 2ctg\theta\,\hat{e}_r = 0$$

Similarly $\vec{F}^2 = (\nabla \times \vec{A}^2) + \frac{\hat{e}_r}{2}(A^3_\theta A^1_\varphi - A^1_\theta A^3_\varphi - A^3_\varphi A^1_\theta + A^1_\varphi A^3_\theta)$

$$= (\nabla \times 2\hat{e}_\theta) + \frac{\hat{e}_r}{2}(0 - 0 - ctg\theta \cdot 0 + (-2)\cdot 0)$$

$$= 0$$

Faddeev - popov Lagrangian

$\int DA \exp[i \int d^4x \, (-1/4 \, F^a_{\mu\nu})^2]$ by the same procedure, we introduce the gauge fixing condition $G(A) = \partial^\mu A^a_\mu - \omega^a(x)$.

Consider a gauge transformation:

$$(A^\alpha_\mu)^a \frac{\sigma^a}{2} = e^{i\alpha^a \frac{\sigma^a}{2}} \left[A^b_\mu \frac{\sigma^b}{2} + \frac{i}{g} \partial_\mu \right] e^{-i\alpha^a \frac{\sigma^a}{2}}$$

Consider $\alpha \rightarrow \alpha + \Delta\alpha$

$$(A^{\alpha+\Delta\alpha}_\mu)^a \frac{\sigma^a}{2} = (A^\alpha_\mu)^a + \frac{1}{g} \partial_\mu \cancel{\alpha}\Delta\alpha^a + \varepsilon^{abc} (A^\alpha_\mu)^b (\Delta\alpha)^c$$

$$= (A^\alpha_\mu)^a + 1/g \; D_\mu \Delta\alpha^a \;, \text{ where } D_\mu = \partial_\mu - ig T^b A^b_\mu$$

$T^b_{ac} = -i\varepsilon_{bac} = i\varepsilon_{abc}$

Adjoint Rep.

$$G[A^{\alpha+\Delta\alpha}] = \partial^\mu (A^\alpha_\mu)^a + 1/g \, \partial^\mu D_\mu (\Delta\alpha)^a - \omega^a(x)$$

$$\frac{\delta G[A^\alpha]}{\delta \alpha^a} = \frac{G[A^{\alpha+\Delta\alpha}] - G[A^\alpha]}{\delta \Delta\alpha^a} = 1/g \, \partial^\mu D_\mu$$

$$= 1/g \, \partial^\mu \left[\partial_\mu - ig T^b (A^b_\mu)^\alpha \right]$$

$$\int DA\, e^{iS[A]} = \int D\alpha \int DA\; e^{iS[A]}\; \delta(G[A^\alpha])\, \det\left[\frac{\delta G[A^\alpha]}{\delta \alpha}\right]$$

① $\int DA = \int DA^\alpha$, since $A \to A^\alpha$ is a shift followed by a unitary transformation

② $S[A] = S[A^\alpha]$

③ And $\dfrac{\delta G[A^\alpha]}{\delta \alpha} = \frac{1}{g}\, \partial^\mu \left[\partial_\mu - ig\, T^b (A^b_\mu)^\alpha \right]$ ← "α" please note

Then $$\int DA\, e^{iS[A]} = \int D\alpha \int DA^\alpha\, e^{iS[A^\alpha]}\, \frac{\delta(G[A^\alpha])}{\det\left[\frac{1}{g}\partial^\mu(\partial_\mu - ig T^b (A^b_\mu)^\alpha\right]}$$

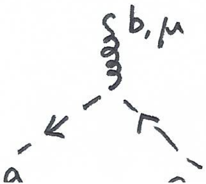
$$= \underbrace{\int D\alpha}_{\text{const}} \int DA\; e^{iS[A]}\, \delta(G[A])\, \det[\partial^\mu(\partial_\mu - ig T^b A^b_\mu)] \cdot \text{const}$$

$$\to \int DA\, e^{iS[A]} = \int DA \int Dc\, D\bar{c}\;\; e^{iS[A] - i\int d^4x \frac{1}{2\xi}(\partial^\mu A_\mu)^2} \cdot e^{i\int d^4x\, \bar{c}^a(-\partial^\mu D^{ac}_\mu c^c)}$$

$$\Rightarrow \mathcal{L}_{FP} = -\tfrac{1}{4}(F^a_{\mu\nu})^2 - \frac{1}{2\xi}(\partial^\mu A_\mu)^2 + \bar{c}^a(-\partial^\mu D^{ac}_\mu c^c)$$

$D^{ac}_\mu = \partial_\mu - ig\, T^b_{ac} A^b_\mu$

ghost  $= i\delta^{ab}/p^2$

 $= -g\, T^b_{ac}\, p^\mu$

Glashow - Weinberg - Salam theory

1. Scalar field of the spinor Rep of $SU(2)$,
Complex

$$\phi \to \phi' = e^{i\alpha^a \tau^a} e^{i\beta/2} \phi, \quad \tau^a = \sigma^a/2 .$$

$$V(\phi) = -\mu^2 \phi^\dagger \phi + \lambda/2 (\phi^\dagger \phi)^2,$$

Consider if ϕ acquires an expectation value $\langle \phi \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}$

then $\alpha^1 = \alpha^2 = 0$, $\alpha^3 = \beta$ leave the vaccum invariant $\rightarrow$ gapless boson.

$$D_\mu \phi = (\partial_\mu - ig A^a_\mu \tau^a - i \frac{1}{2} g' B_\mu) \phi$$

$$\mathcal{L} = D^\mu \phi^* D_\mu \phi - V(\phi)$$

Gauge boson mass

$$\Delta \mathcal{L} = \frac{1}{2} (0, v) \underline{(g A^a_\mu \tau^a + \frac{1}{2} g' B_\mu)(g A^{b,\mu} \tau^b + \frac{1}{2} g' B^\mu)} \begin{pmatrix} 0 \\ v \end{pmatrix}$$

$$\swarrow$$

$$g^2 A^a_\mu A^{b\mu} \tau^a \tau^b + 1/4 g'^2 B_\mu B^\mu + gg' A^a_\mu \tau^a B^\mu$$

$$= \frac{1}{4} g^2 A^a_\mu A^{a\mu} + 1/4 g'^2 B_\mu B^\mu + gg' A^a_\mu \tau^a B^\mu$$

$$(\ )_{22} = 1/4 g^2 [A^1_\mu A^{1\mu} + A^2_\mu A^{2,\mu}] + \frac{1}{4} (-g A^3_\mu A^{3\mu} + g' B_\mu B^\mu]^2$$

$$(\tau^3)_{22} = -1/2$$

$$\Rightarrow \Delta \mathcal{L} = \frac{v^2}{4} [g^2 W^+_\mu W^-_\mu \qquad \Rightarrow m_W = g \frac{v}{2}$$

$$+ \frac{v^2}{8} \sqrt{g^2 + g'^2} \; Z^0_\mu Z^{0,\mu} \qquad \Rightarrow m_Z = \frac{v}{2} \sqrt{g^2 + g'^2}$$

$$A_\mu = \frac{1}{\sqrt{g^2 + g'^2}} (g' A^3_\mu + g B_\mu) \leftarrow \text{photon zero mass.}$$

couple to fermion:

$$D_\mu = \partial_\mu - ig A_\mu^a T^a - ig' Y B_\mu$$

T^a: generator of a Representatation

$$D_\mu = \partial_\mu - \frac{ig}{\sqrt{2}} (W_\mu^+ T^+ + W_\mu^- T^-) - i \frac{1}{\sqrt{g^2+g'^2}} Z_\mu (g^2 T^3 - g'^2 Y)$$

$$- i \frac{gg'}{\sqrt{g^2+g'^2}} A_\mu \underline{(T^3 + Y)}$$ electric charge $Q = T^3 + Y$

$$W_\mu^\pm = \frac{1}{\sqrt{2}} (A_\mu^1 \mp i A_\mu^2)$$

$$Z = \frac{1}{\sqrt{g^2+g'^2}} (g A_\mu^3 - g' B_\mu) \quad , \quad A_\mu = \frac{1}{\sqrt{g^2+g'^2}} (g' A_\mu^3 + g B_\mu)$$

check $\frac{1}{g^2+g'^2} \{ (g A_\mu^3 - g' B_\mu)(g^2 T^3 - g'^2 Y) + gg'(g' A_\mu^3 + g B_\mu)(T^3 + Y) \}$

$$g(g^2+g'^2)(A_\mu^3 T^3) + g'(g^2+g'^2) B_\mu Y$$

$$A_\mu^3 Y (\cancel{-gg'^2 + gg'^2}) + B_\mu T^3 (\cancel{-g' g^2 + g^2 g'})$$

$$= g A_\mu^3 T^3 + g' B_\mu Y.$$

$$\begin{pmatrix} Z^0 \\ A \end{pmatrix} = \begin{pmatrix} \cos\theta_W & -\sin\theta_W \\ \sin\theta_W & \cos\theta_W \end{pmatrix} \begin{pmatrix} A^3 \\ B \end{pmatrix} \Rightarrow \cos\theta_W = \frac{g}{\sqrt{g^2+g'^2}} \quad \sin\theta_W = \frac{g'}{\sqrt{g^2+g'^2}}$$

$$e = \frac{gg'}{\sqrt{g^2+g'^2}} \qquad Q = T^3 + Y$$

$$\bar{\psi} i \not{\partial} \psi = \bar{\psi}_L i \not{\partial} \psi_L + \bar{\psi}_R i \not{\partial} \psi_R$$

$$E_L = \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L , \quad e_R , \quad Q_L = \begin{pmatrix} u \\ d \end{pmatrix}_L \quad u_R , d_R$$

$$y = -\tfrac{1}{2} \qquad y = -1 \qquad y = \tfrac{1}{6} \qquad \underbrace{y = \tfrac{2}{3} \quad y = -\tfrac{1}{3}}_{T^3 = 0}$$

$$T^3 = \begin{pmatrix} \tfrac{1}{2} \\ -\tfrac{1}{2} \end{pmatrix} \qquad T^3 = 0 \qquad T^3 = \begin{pmatrix} \tfrac{1}{2} \\ -\tfrac{1}{2} \end{pmatrix}$$

$$\mathcal{L} = \bar{E}_L (i \not{D}) E_L + \bar{e}_R (i \not{D}) e_R + \bar{Q}_L (i \not{D}) Q_L + \bar{u}_R (i \not{D}) u_R + \bar{d}_R (i \not{D}) d_R$$

example: $\bar{Q}_L (i \not{D}) Q_L = \bar{Q}_L i \gamma^\mu \left(\partial_\mu - i g A^a_\mu \tau^a - i \frac{1}{6} g' B_\mu \right) Q_L$

Fermion mass : SU(2) doublet Higgs.

$$\Delta \mathcal{L}_e = - \lambda_e \underbrace{\bar{E}_L \cdot \phi}_{\text{SU(2) doublet}} e_R + h.c.$$

$\leftarrow$ spinor $\leftarrow$ lepton

$$\phi = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix} \Rightarrow \bar{E}_L \cdot \phi = \frac{1}{\sqrt{2}} (\bar{\nu}_e , \bar{e}^-)_L \begin{pmatrix} 0 \\ v \end{pmatrix} = \frac{1}{\sqrt{2}} \bar{e}_L v$$

$$\Rightarrow \Delta \mathcal{L}_e = -\frac{1}{\sqrt{2}} \lambda_e v \, \bar{e}_L e_R + h.c \Rightarrow \boxed{m_e = \frac{1}{\sqrt{2}} \lambda_e v}$$

Quarks.

$$\Delta \mathcal{L}_q = - \lambda_d \bar{Q}_L \cdot \phi \, d_R - \lambda_u \epsilon^{ab} \bar{Q}_{La} \phi^\dagger_b u_R + h.c.$$

$\bar{Q}_L = (\bar{u}, \bar{d})$, $\bar{Q}_L \cdot \phi = \frac{v}{\sqrt{2}} \bar{d}_L$ $\qquad \epsilon^{ab} \bar{Q}_{La} \phi^\dagger_b = \frac{v}{\sqrt{2}} \bar{u}_L$

$$\Rightarrow \Delta \mathcal{L}_q = -\frac{1}{\sqrt{2}} \lambda_d v \, \bar{d}_L d_R - \frac{1}{\sqrt{2}} \lambda_u v \, \bar{u}_L u_R$$