PSEUDOPARTICLE SOLUTIONS OF THE YANG-MILLS EQUATIONS

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We find regular solutions of the four dimensional euclidean Yang-Mills equations. The solutions minimize locally the action integrals which is finite in this case. The topological nature of the solutions is discussed.

In the previous paper by one of the authors [1] the importance of the pseudoparticle solutions of the gauge field equations for the infrared problems was shown. By "pseudoparticle" solutions we mean the long range fields A_μ which minimize locally the Yang-Mills actions S and for which $S(A) < \infty$. The space is euclidean and four-dimensional. In the present paper we shall find such a solution. Let us start from the topological consideration which shows the existence of the desired solutions.

All fields we are interested in satisfy the condition:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \underset{x\to\infty}{\to} 0 \,. \tag{1}$$

Consider a very large sphere S^3 in our 4-dimensional space. The sphere itself is of course 3-dimensional. From (1) it follows that

$$A_\mu|_{S^3} \approx g^{-1}(x) \left.\frac{\partial g(x)}{\partial x_\mu}\right|_{S^3} \tag{2}$$

where $g(x)$ are matrices of the gauge group. Hence every field $A_\mu(x)$ produce a certain mapping of the sphere S^3 onto the gauge group G. It is clear that if two such mappings belong to different homotopy classes then the corresponding fields $A_\mu^{(1)}$ and $A_\mu^{(2)}$ cannot be continuously deformed one into another. It is well known [2] that there exists an infinite number of different classes of mappings of $S^3 \to G$ if G is a nonabelian simple Lie group. Hence, the phase space of the Yang-Mills fields are divided into an infinite number of components, each of which is characterized by some value of q, where q is a certain integer.

Our idea is to search for the absolute minimum of the given component of the phase space. In order to do this we need the formula expressing the integer q through the field A_μ‡. It is easy to check that

$$q = \frac{1}{8\pi^2} \epsilon_{\mu\nu\lambda\gamma} \, \mathrm{Sp} \int F_{\mu\nu} F_{\lambda\gamma} \, \mathrm{d}^4 x. \tag{3}$$

To prove this let us use the identity:

$$\begin{gathered} \epsilon_{\mu\nu\lambda\gamma} \, \mathrm{Sp}\, F_{\mu\nu} F_{\lambda\gamma} = \partial_\alpha J_\alpha \\ J_\alpha = \epsilon_{\alpha\beta\gamma\delta} \, \mathrm{Sp}(A_\beta(\partial_\gamma A_\delta + \tfrac{2}{3} A_\gamma A_\delta)). \end{gathered} \tag{4}$$

From (4) follows:

$$\begin{aligned} q &= \frac{1}{8\pi^2} \oint_{S^3} J_\alpha \, \mathrm{d}^3\sigma^\alpha \\ &= \frac{1}{8\pi^2} \tfrac{4}{3} \epsilon_{\alpha\beta\gamma\delta} \oint \mathrm{Sp}(A_\beta A_\gamma A_\delta) \mathrm{d}^3 \sigma_\alpha \end{aligned} \tag{5}$$

where

$$A_\mu = g^{-1}(x) \partial g / \partial x_\mu \tag{6}$$

Now consider the case $G = \mathrm{SU}(2)$. In this case it is clear that:

$$\mathrm{d}\mu(g) = \mathrm{Sp}(g^{-1}\mathrm{d}g \times g^{-1}\mathrm{d}g \times g^{-1}\mathrm{d}g) \tag{7}$$

is just the invariant measure on this group, since it is the invariant differential form of the appropriate dimension. The meaning of the notation in (7) is as follows. Let $g(\xi_1 \xi_2 \xi_3)$ be some parametrization of SU(2), say, through the Euler angles. Then the invariant measure will be:

‡ Formulas like (3) are known in topology by the name of "Pontryagin class".

$$d\mu = \mathrm{Sp}\left(g^{-1}\frac{\partial g}{\partial \xi_1} g^{-1}\frac{\partial g}{\partial \xi_2} g^{-1}\frac{\partial g}{\partial \xi_3}\right) d\xi_1 \ldots d\xi_3 \,. \quad (8)$$

Comparing (8) with (5) we see that the integrand in (5) is precisely the Jacobian of the mapping of S^3 on SU(2). Hence q is the number of times the SU(2) is covered under this mapping. It is just the definition of the mapping degree. In the case of the arbitrary group G one should consider the mapping of S^3 on its SU(2) subgroup and repeat the above. There exists an important inequality which will be extensively used below. Consider the following relation:

$$\mathrm{Sp} \int (F_{\mu\nu} - \widetilde{F}_{\mu\nu})^2 d^4x \geqslant 0 \quad (9)$$

where $\widetilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\lambda\gamma}F_{\lambda\gamma}$. From (9) and (3) it follows that:

$$E \geqslant 2\pi^2 |q| \quad (10)$$

where

$$S(A) \equiv E(A)/g^2$$

and g^2 is a coupling constant.

The formula (10) gives the lower bound for the energy of the quasiparticles in each homotopy class. We shall show now that for $q = 1$ this bound can be saturated. In other words one can search the solution of the equation, which replace the usual Yang-Mills one:

$$F_{\alpha\beta} = \pm\tfrac{1}{2}\epsilon_{\alpha\beta\gamma\delta}F_{\gamma\delta}$$
$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha + [A_\alpha A_\beta] \,. \quad (11)$$

Again it is sufficient to consider the case G = SU(2). In this case it is convenient though not necessary to extend this group up to SU(2) × SU(2) ≈ O(4). The gauge fields for O(4) are $A_\mu^{\alpha\beta}$ where A_μ are antisymmetric on $\alpha\beta$. The SU(2) gauge field are connected with $A_\mu^{\alpha\beta}$ by the formulas:

$$\pm A_\mu^i = \tfrac{1}{2}(A_\mu^{oi} \pm \tfrac{1}{2}\epsilon_{ikl}A_\mu^{kl}). \quad (12)$$

Now, two equations:

$$\pm F_{\mu\nu}^i = \pm\tfrac{1}{2}\epsilon_{\mu\nu\lambda\gamma} \pm F_{\lambda\gamma}^i$$

are equivalent to the following one:

$$\epsilon_{\alpha\beta\gamma\delta}F_{\mu\nu}^{\gamma\delta} = \epsilon_{\mu\nu\lambda\gamma}F_{\lambda\gamma}^{\alpha\beta}. \quad (13)$$

Let us search the solution of (13) which is invariant under simultaneous rotations of space and isotopic space. The only possibility is:

$$A_\mu^{\alpha\beta} = f(\tau)(x_\alpha\delta_{\mu\beta} - x_\beta\delta_{\mu\alpha}). \quad (14)$$

It is easy to calculate F:

$$F_{\mu\nu}^{\alpha\beta} = (2f - \tau^2 f^2)(\delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\mu\beta}\delta_{\nu\alpha})$$
$$+ (f'/\tau + f^2)(x_\alpha x_\mu \delta_{\nu\beta} - x_\alpha x_\nu \delta_{\mu\beta} \quad (15)$$
$$+ x_\beta x_\nu \delta_{\mu\alpha} - x_\beta x_\mu \delta_{\nu\alpha}).$$

It is evident that the first tensor structure (15) satisfies the equation (13) and the second does not. Hence we are to choose:

$$f'/\tau + f^2 = 0, \quad f(\tau) = \frac{1}{\tau^2 + \lambda^2} \quad (16)$$

where τ is an arbitrary scale. The quasi-energy E is given by

$$E = \tfrac{1}{4}\mathrm{Sp}\int_\pm F_{\mu\nu}^2 d^4x$$
$$= \tfrac{1}{32}\mathrm{Sp}\int (F_{\mu\nu}^{\alpha\beta})^2 d^4x = 2\pi^2. \quad (17)$$

Comparison of (17) and (10) shows that we find absolute minimum for $q = 1$.

Another representation for the solution (14) is given by the formulas:

$$A_\mu = \frac{\tau^2}{\tau^2 + \lambda^2} g^{-1}(x)\frac{\partial g(x)}{\partial x_\mu}$$

$$g(x) = (x_4 + i\boldsymbol{x}\cdot\boldsymbol{\sigma})(x_4^2 + \boldsymbol{x}^2)^{-1/2}$$

$$g^\dagger g = 1, \quad \tau^2 = x_4^2 + \boldsymbol{x}^2$$

($\boldsymbol{\sigma}$ are Pauli matrixes).

For arbitrary group G one should consider its subgroup SU(2) for which A_μ is given by (18) and all other matrix elements of A_μ let be zero.

Our solution, as is evident from the scale invariance, contains the arbitrary scale λ. Hence these fields are long range and are essential in the infrared problems.

We do not know whether any solutions of (13) exist with $q > 1$. One may consider of course several pseudoparticles with $q = 1$. However, we do not know whether they are attracted to each other and form the pseudoparticle with $q > 1$ or whether there exists repulsion and no stable pseudoparticle.

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References

[1] A.M. Polyakov, Phys. Lett. 59B (1975) 82.