⟦78a⟧ # Generalization of Dirac's monopole to SU_2 gauge fields

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Dirac's monopole is generalized to SU_2 gauge fields in five-dimensional flat space or four-dimensional spherical space. The generalized fields have SO_5 symmetry. The method used is related to the concept of orthogonal gauge fields which is developed in an appendix. The angular momenta operators for the SO_5 symmetrical fields are given.

I. INTRODUCTION

The Dirac[1] monopole, which is singular only at the origin in three-dimensional space, satisfies the following properties:

(a) The magnetic flux through *any* closed surface around the origin, is $4\pi g \neq 0$. I. e.,

$$\tfrac{1}{2}\oint\!\!\oint f_{\mu\nu}\, dx^{\mu\nu} = 4\pi g \neq 0, \tag{1}$$

where $dx^{\mu\nu}$ is the *surface element* and is *antisymmetrical* in μ and ν.

(b) It is spherically symmetrical.

The Dirac monopole field is uniquely determined by (a) and (b) for each (allowed) value of g. We remark that if we remove condition (b) then the field is not uniquely determined by (a), since, e.g., the addition of a dipole or any higher order pole to the origin does not change (a).

A further remark is useful. The integral in (1) is independent of any distortion of the closed surface since[2]

$$f_{\mu\nu,\lambda} + f_{\nu\lambda,\mu} + f_{\lambda\mu,\nu} = 0,$$

or

$$\sum^{a} f_{\mu\nu,\lambda} = 0. \tag{2}$$

We want to generalize the Dirac monopole field to SU_2 gauge field. Consider a five-dimensional space with metric

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 + dx_5^2. \tag{3}$$

Consider an SU_2 gauge field which is singular only at the origin. The generalization of (1), which is the first Chern class number, is the second Chern class[3] number,

$$\tfrac{1}{24}\oint\!\!\oint\!\!\oint\!\!\oint f^i_{\mu\nu} f^i_{\alpha\beta}\, dx^{\mu\nu\alpha\beta} = (8\pi^2/3)C_2, \tag{4}$$

where the integral is taken over a closed four-dimensional surface enclosing the origin. This integral is also independent of any distortion of the surface. To see this we use the rules of the gauge Riemannian calculus of Ref. 4 and find in a straightforward manner,

$$\sum^{a} (f^i_{\mu\nu} f^i_{\alpha\beta})_{,\gamma} = \sum^{a} (f^i_{\mu\nu} f^i_{\alpha\beta})_{\|\gamma} = \sum^{a} (f^i_{\mu\nu\|\gamma} f^i_{\alpha\beta} + f^i_{\mu\nu} f^i_{\alpha\beta\|\gamma}) = 0,$$

which is a natural generalization of (2). It follows immediately that (4) is independent of any distortion of the surface provided it always encloses the origin.

We thus search for an SU_2 gauge field satisfying

(a′) $C_2 \neq 0$;

(b′) It is SO_5 symmetrical.

As before, condition (b′) is needed to make the field unique for a given value of C_2.

We shall prove that there are *two and only two* solutions α and β satisfying (a′) and (b′). They are respectively characterized by

$$C_2 = +1 \quad \text{and} \quad C_2 = -1. \tag{5}$$

Furthermore, fields α and β will be defined so that in orthogonal coordinates ξ_1, ξ_2, ξ_3, θ, r, where r is the radial variable and ξ_1, ξ_2, ξ_3, θ are five-dimensional angular coordinates (to be defined later),

$$b^j_r = 0, \quad b^j_\theta = 0, \tag{6}$$

$$b^j_{(\xi_i)} = \text{function only of } \theta,\ \xi. \tag{7}$$

Thus $b^j_r dr + b^j_\theta d\theta + b^j_{(\xi_i)} d\xi^i =$ independent of r and dr.

This means that the radial coordinate r and the angular coordinates can be separated, and the gauge fields α and β are only really dependent on the latter. One can thus view α and β as gauge fields confined to any sphere S_4 with its center at the origin. In this S_4 *viewpoint*, the field α is *self-dual* and *orthogonal* everywhere, and the field β is *self-antidual* and *orthogonal* everywhere. These concepts are defined in Appendix A.

In the five-dimensional viewpoint, α and β are both sourceless and analytic everywhere except at the origin. It is SO_5 symmetrical.

In the S_4 viewpoint, α and β are both sourceless and analytic everywhere, and is SO_5 symmetrical. We define a total "action"

$$L = \oint\!\!\oint\!\!\oint\!\!\oint f^i_{\mu\nu} f^{i\mu\nu}\, d(\text{surface}) \tag{8}$$

over S_4. We shall prove that solutions α and β have the least "action" among fields with their respective second Chern class numbers C_2.

The fields α and β will be defined in Secs. II and III in terms of nonintegrable phase factors.[4,6] A reader unfamiliar with this geometrical concept can take Eqs. (34) as the algebraic definition of the fields.

The concepts of orthogonal self-dual and self-antidual fields seem to be very useful. These fields are defined and discussed in Appendix A, where the relationship between these concepts and SO_4 symmetry is also discussed.

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The angular momentum operators in five-dimensional space are exhibited in Sec. X. They contain extra terms to take into account the angular momenta that reside in the field, just as the angular momentum operators for a charged particle in a Dirac monopole field contain extra terms.

II. CONSTRUCTION OF THE SOLUTIONS

We first recall the vector potential **A** for the Dirac monopole. They can be chosen in two[5] regions R_a and R_b which are defined by (in spherical coordinates r, θ, ϕ):

$$R_a:\ 0 \leqslant \theta < (\pi/2) + a,$$

$$R_b:\ \pi \geqslant \theta > (\pi/2) - a \quad (0 < a < \pi/2).$$

In the two regions they are respectively

$$A_r^{(a)} = A_\theta^{(b)} = 0, \quad A_\phi^{(a)} = g(1 - \cos\theta), \tag{9}$$

$$A_r^{(b)} = A_\theta^{(b)} = 0, \quad A_\phi^{(b)} = -g(1 + \cos\theta), \tag{10}$$

where we use tensor notation for the components, so that the expression for A_ϕ in these formulas is $(r\sin\theta)$ times the corresponding A_ϕ of Ref. 5.

It has been emphasized[4,6] that a more intrinsic concept than **A** is the phase factor. For an infinitesimal path from $P: (r, \theta, \phi)$ to $P + dP: (r + dr, \theta + d\theta, \phi + d\phi)$, the phase factor is (in R_a)

$$\Phi^{(a)}_{(P+dP)P} = 1 + ieA_\mu\, dx^\mu$$

$$= 1 + ieg(1 - \cos\theta)\, d\phi$$

$$\approx [\exp(+2ieg\, d\phi)]^{p(\theta)}, \tag{11}$$

where

$$p(\theta) = \tfrac{1}{2}(1 - \cos\theta). \tag{12}$$

In R_b we obtain

$$\Phi^{(b)}_{(P+dP)P} = [\exp(-2ieg\, d\phi)]^{1-p(\theta)}. \tag{13}$$

Now consider a function, defined everywhere except along the z axis,

$$T(r, \theta, \phi) = \exp(2ige\phi), \tag{14}$$

which is single valued in view of Dirac's quantization condition

$$2ge = \text{integer}. \tag{15}$$

Thus,

$$\Phi^{(a)}_{(P+dP)P} = (T_{P+dP}T_P^{-1})^{p(\theta)}, \tag{16}$$

$$\Phi^{(b)}_{(P+dP)P} = (T^{-1}_{P+dP}T_P)^{1-p(\theta)}. \tag{17}$$

Since $p(0) = 0$, Eq. (16) is applicable near $\theta = 0$. Similarly, Eq. (17) is applicable near $\theta = \pi$ since $1 - p(\pi) = 0$. Equations (16) and (17) define the Dirac field.

We are now ready to generalize to a SU_2 gauge field in five-dimensional space. We shall choose coordinates ξ_i, θ, r $(i = 1, 2, 3)$ such that

$$r = (x_1^2 + \cdots + x_5^2)^{1/2}, \quad x_5 = r\cos\theta \quad (0 \leqslant \theta \leqslant \pi). \tag{18}$$

ξ_1, ξ_2, ξ_3 parametrize the three-dimensional sphere (Fig. 1)

$$r = \text{fixed}, \quad \theta = \text{fixed}, \quad \text{or } r = \text{fixed}, \quad x_5 = \text{fixed}. \tag{19}$$

For the time being we shall not specify how to choose ξ_1, ξ_2, ξ_3 to avoid unnecessary distraction at this stage. Consider *any*

$$T(\xi_1, \xi_2, \xi_3, \theta, r)$$

which is an element of SU_2 and is defined and differentiable at all points in the five-dimensional space except on the x_5 axis. Consider any $p(\theta)$ which satisfies

$$p(0) = 1 - p(\pi) = 0. \tag{20}$$

Then (16) and (17) define a gauge field in R_a and R_b respectively. In the region of overlap, we find the following relationship between Eqs. (16) and (17)

$$T^{-1}_{P+dP}\Phi^{(a)}_{(P+dP)P}T_P = \Phi^{(b)}_{(P+dP)P}. \tag{21}$$

To prove this we start from

$$\Phi^{(a)}_{(P+dP)P} = (T_{P+dP}T_P^{-1})^{p(\theta)-1}(T_{P+dP}T_P^{-1}).$$

Thus,

$$T^{-1}_{P+dP}\Phi^{(a)}_{(P+dP)P}T_P = T^{-1}_{P+dP}(T_{P+dP}T_P^{-1})^{p-1}T_{P+dP}$$

$$= [T^{-1}_{P+dP}(T_{P+dP}T_P^{-1})T_{P+dP}]^{p-1}$$

$$= [T_P^{-1}T_{P+dP}]^{p-1}$$

which leads to Eq. (21).

Equation (21) shows that $\Phi^{(a)}$ and $\Phi^{(b)}$ define the same gauge field. T_p is thus the "transition function" S for the overlap.[6] *It defines the gauge transformation from region b to region a.*

III. CONSTRUCTION OF THE SOLUTIONS (CONTINUED)

For any T and $p(\theta)$ satisfying Eq. (20) we have a gauge field. It remains to choose an explicit form for T as a function of the coordinates and a $p(\theta)$ that satisfies Eq. (20) so that conditions (a) and (b) are satisfied. For $p(\theta)$ we choose Eq. (12), the same as in Dirac's case. For T, we endeavor to define it as a function independent of r and θ, again imitating (14) for the Dirac case. Thus $T = T(\xi_1, \xi_2, \xi_3)$. Since the sphere (19) has the *same geometry* as the SU_2 group manifold itself, it is natural to define T as the group element represented by the point ξ_1, ξ_2, ξ_3 on the sphere (19). [We observe that this is an exact generalization of the

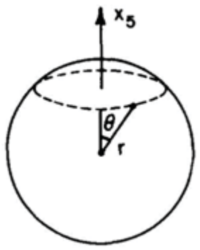

FIG. 1. The coordinates $\xi_1, \xi_2, \xi_3, \theta$, and r in five dimensions. r is the radius $r\cos\theta = x_5$ as illustrated. The equations r = const, θ = const is a three-dimensional sphere symbolized by the dotted curve. It is the generalization of the azimuthal circle in the usual spherical coordination system r, θ, ϕ. ξ_1, ξ_2, ξ_3 parametrizes this S_3, as ϕ parametrizes the usual azimuthal circle.

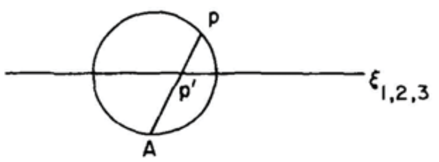

FIG. 2. Projective coordinate for three-dimensional sphere S_3 (in four dimensions). $\xi_{1,2,3}$ is symbolically three-dimensional flat space. A is the "south" pole of the unit sphere S_3. The point p on S_3 is projected to the point p' whose coordinates $\boldsymbol{\xi}$ parametrize the point p. The point A on S_3 corresponds to the point at ∞ in $\boldsymbol{\xi}$ space.

Dirac case where (19) reduces to a circle (Fig. 1). In that case, if $2eg=1$, T as defined by (14) *is* the group element that corresponds geometrically to the point ϕ on the circle.]

The above description gives a geometrical definition of T. To translate it into an explicit formula, we adopt a projective coordinate system ξ_1, ξ_2, ξ_3 for the sphere (19) (Fig. 2):

$$x_i = (r\sin\theta)2\xi_i(1+\xi^2)^{-1} \quad \text{where } \xi = \left(\sum_1^3 \xi_i^2\right)^{1/2} \geqslant 0, \qquad i=1,2,3, \tag{22}$$

$$x_4 = (r\sin\theta)(1-\xi^2)(1+\xi^2)^{-1}.$$

For fixed $r>0$, and $0<\theta<\pi$, the complete ξ_1, ξ_2, ξ_3 plane plus the point at ∞ maps through Eq. (22) onto the sphere (19) in a one-to-one mapping. The transformation $x_1, \ldots, x_5 \leftrightarrow \xi_1, \xi_2, \xi_3, \theta, r$ defined by Eqs. (22) and (18) has the metric

$$ds^2 = dr^2 + r^2\,d\theta^2 + r^2(\sin^2\theta)4(1+\xi^2)^{-2}\,d\xi^2. \tag{23}$$

Furthermore, the Jacobian of the transformation is positive,

$$\frac{\partial(x_1x_2x_3x_4x_5)}{\partial(\xi_1\xi_2\xi_3\theta r)} > 0.$$

We now define the SU_2 *monopole gauge field* α by Eqs. (12), (16), and (17) together with[7] the following definition of T,

$$R(T) = (1+\xi^2)^{-1}(1-\xi^2+2i\boldsymbol{\xi}\cdot\boldsymbol{\sigma}) \quad [\xi=(\xi_1^2+\xi_2^2+\xi_3^2)^{1/2}], \tag{24α}$$

where $\boldsymbol{\sigma}$ are the Pauli matrices satisfying $\sigma_1\sigma_2 = i\sigma_3$. $R(T)$ means the 2×2 representative of T. Similarly, we define the SU_2 *monopole gauge field* β by Eqs. (12), (16), and (17) together with

$$R(T) = (1+\xi^2)^{-1}(1-\xi^2-2i\boldsymbol{\xi}\cdot\boldsymbol{\sigma}). \tag{24β}$$

IV. POTENTIALS $b_\mu^{i(a)}$ AND $b_\mu^{i(b)}$

Defining the gauge potential b_μ^i by

$$\Phi_{(P+dP)P} = 1 - b_\mu^i X_i \cdot dx^\mu, \tag{25}$$

we can compute in R_a and R_b respectively $b_\mu^{i(a)}$ and $b_\mu^{i(b)}$ from Eqs. (24), (12), (16), and (17). We shall use tensor notation[8] and write

$$b_1^i, b_2^i, b_3^i, b_\theta^i, b_r^i$$

for

$$b_{(\xi_1)}^i, b_{(\xi_2)}^i, b_{(\xi_3)}^i, b_\theta^i, b_r^i.$$

By putting $d\theta = d\xi^i = 0$ in Eqs. (16), (24), and (25) we obtain b_r^i. Since T is independent of r, $\Phi_{(P+dP)P}=$ identity, we have

$$b_r^i = 0 \quad \text{in both } R_a \text{ and } R_b. \tag{26}$$

Similarly,

$$b_\theta^i = 0 \quad \text{in both } R_a \text{ and } R_b. \tag{27}$$

Putting $d\theta = dr = 0 = d\xi^2 = d\xi^3$, we obtain by Eqs. (25) and (16),

$$-b_1^i\left(-\frac{i}{2}\sigma_j\right) = p(\theta)\left(\frac{\partial T}{\partial \xi^1}\right)T^{-1} \quad \text{in } R_a, \tag{28}$$

where we have substituted $-i\sigma_j/2$ for X_j which it represents, and we write T for $R(T)$.

Substituting Eq. (24) into (28) we can calculate b_j^i. To present the results we define

$$B_j^i \equiv \langle i|B|j\rangle, \quad D_j^i \equiv \langle i|D|j\rangle, \tag{29}$$

and

$$B = -8(1+\xi^2)^{-2}[\psi\tilde{\psi} + \tfrac{1}{2}(1-\xi^2) + N], \tag{30}$$

$$D = -\tilde{B}, \tag{31}$$

where ~ means transposed, and

$$\psi = \begin{pmatrix}\xi_1\\ \xi_2\\ \xi_3\end{pmatrix}, \quad N = \begin{pmatrix}0 & -\xi_3 & \xi_2\\ \xi_3 & 0 & -\xi_1\\ -\xi_2 & \xi_1 & 0\end{pmatrix}. \tag{32}$$

The following formulas are useful:

$$N\psi = 0, \quad N^2 = \psi\tilde{\psi} - \xi^2, \quad B\tilde{B} = 16(1+\xi^2)^{-2}. \tag{33}$$

Using these definitions, we find

$$b_r^i = b_\theta^i = 0 \quad \text{for both solutions } \alpha \text{ and } \beta, \text{ in both } R_a \text{ and } R_b. \tag{34}$$

For[7] solution α:

$$b_j^{i(a)} = (1-\cos\theta)D_j^i/2, \quad b_j^{i(b)} = (1+\cos\theta)B_j^i/2. \tag{34$'\alpha$}$$

For solution β:

$$b_j^{i(a)} = (1-\cos\theta)B_j^i/2, \quad b_j^{i(b)} = (1+\cos\theta)D_j^i/2. \tag{34$'\beta$}$$

These equations are obtained from Eq. (28). We notice

$$B_k^j\left(-\frac{i}{2}\sigma_j\right) = -R_{,k}R^{-1},$$

where R stands for $R(T)$ of Eq. (24β). Similarly

$$D_k^j\left(-\frac{i}{2}\sigma_j\right) = -R_{,k}R^{-1},$$

where R stands for $R(T)$ of Eq. (24α).

In the overlap region R_{ab}, $b_j^{i(a)}$ and $b_j^{i(b)}$ are related by a gauge transformation, since they were computed from (25) using (16) and (17), which are related by the gauge transformation (21). The gauge transformation (i.e., transition function) from $b^{(b)}$ to $b^{(a)}$ is thus T which is given by (24α) or (24β).

In the rest of the paper, we concentrate on R_a. *Results for R_b are obtained by applying the gauge transformation* (21) *to that for R_a.*

V. FIELD STRENGTHS $f^i_{\mu\nu}$ FOR FIELD β

Applying Eq. (34) to the definitions

$$f^i_{\mu\nu} = b^i_{\mu,\nu} - b^i_{\nu,\mu} - C^i_{jk} b^j_\mu b^k_\nu,$$
$$C^1_{23} = C^2_{31} = C^3_{12} = -C^1_{32} = -C^2_{13} = -C^3_{21} = 1,$$

we obtain for field β, in region R_a,

$$f^i_{r\theta} = f^i_{rj} = 0, \tag{35}$$

$$f^i_{\theta j} = -(\sin\theta) B^i_j / 2, \tag{35$'\beta$}$$

and

$$f^i_{jk} = -p(p-1) C^i_{lm} B^l_j B^m_k. \tag{35$''\beta$}$$

We have used the relation

$$B^i_{j,k} - B^i_{k,j} - C^i_{lm} B^l_j B^m_k = 0 \tag{36}$$

which can be verified directly from definition (30) of B^i_j. It can also be verified by putting $p(\theta) = 1$ in Eq. (16). $\Phi^{(a)}_{(P+dP)P}$ is then obviously gauge transformable to unity. Thus the field strength in such a case should vanish. Now when $p(\theta) = 1$, Eq. (28) states that $b^i_j = B^i_j$. Equation (36) is then the statement that the field strengths vanish, which we already proved.

Explicit evaluation starting from Eq. (35) gives in R_a for field β:

$$\begin{aligned}
f^1_{\theta 1} &= 4(\sin\theta)(1+\xi^2)^{-2}[\xi_1\xi_1 + (1-\xi^2)/2],\\
f^1_{\theta 2} &= 4(\sin\theta)(1+\xi^2)^{-2}[\xi_1\xi_2 - \xi_3],\\
f^1_{\theta 3} &= 4(\sin\theta)(1+\xi^2)^{-2}[\xi_1\xi_3 + \xi_2],\\
f^1_{23} &= 8(\sin^2\theta)(1+\xi^2)^{-3}[\xi_1\xi_1 + (1-\xi^2)/2],\\
f^1_{31} &= 8(\sin^2\theta)(1+\xi^2)^{-3}[\xi_1\xi_2 - \xi_3],\\
f^1_{12} &= 8(\sin^2\theta)(1+\xi^2)^{-3}[\xi_1\xi_3 + \xi_2].
\end{aligned} \tag{37β}$$

Other components of $f^i_{\theta j}$ and f^i_{jk} can be obtained from Eq. (37β) by cyclic permutation of all indices 1, 2, and 3 (i.e., simultaneously of the SU_2 index and the ξ subscript). The field strengths $f^i_{\mu\nu}$ in R_b are obtained from (37β) by a gauge transformation as discussed in Sec. IV. In the rest of the paper we shall concentrate on R_a.

The field strengths for field α are similar to these. They are discussed in Appendix B.

Since conditions (6) and (7) are satisfied, we can take the S_4 viewpoint mentioned in the Introduction. We apply then the concepts of Appendix A to field β. Obviously, by (A1),

$$\eta_{\mu\nu\alpha\beta} = (8\sin^3\theta) r^4 (1+\xi^2)^{-3} \epsilon_{\mu\nu\alpha\beta}. \tag{38}$$

One can then evaluate f^* from Eq. (37β), arriving at

$$f^* = -f. \tag{39β}$$

Similarly for field α one proves this way that

$$f^* = f. \tag{39α}$$

Thus fields α and β are respectively self-dual and self-antidual.

Using Eq. (37β) we can prove (Appendix C) that *field β is self-antidual and orthogonal everywhere*, and

$$\mathcal{H} = -\mathcal{E} = -(1+\xi^2)\tilde{B}(4r^2)^{-1}, \tag{40}$$

$$a = r^{-2}. \tag{41}$$

One can similarly show that *field α is self-dual and orthogonal everywhere*, with the same amplitude a given by (41). The inverse square dependence of a on r is the same as in Dirac's monopole.

VI. ANALYTICITY AT $\theta = 0$ AND $\theta = \pi$

For the Dirac monopole, the choice (9) of b_μ in R_a has an apparent singularity at $\theta = 0$, since A_ϕ cannot be defined there. However, in Cartesian coordinates at a point on the $+z$ axis,

$$\begin{aligned}
A_x &= -g\frac{1-\cos\theta}{r\sin\theta}\sin\phi = -g\frac{1}{1+\cos\theta}\frac{y}{r^2},\\
A_y &= g\frac{1-\cos\theta}{r\sin\theta}\cos\phi = g\frac{1}{1+\cos\theta}\frac{x}{r^2},\\
A_z &= 0.
\end{aligned} \tag{42}$$

Thus $A^{(a)}$ is analytic at $\theta = 0$. Similarly we can prove that $A^{(b)}$ is analytic at $\theta = \pi$.

For the fields α and β in five-dimensional space we can, by using Cartesian coordinates at a point on the $+x_5$ axis, in exactly the same manner, prove that $b^{i(a)}_\mu$ in Cartesian coordinates is analytic at $\theta = 0$. Similarly we find that $b^{i(b)}_\mu$ in Cartesian coordinates is analytic at $\theta = \pi$. Thus the fields α and β are *both everywhere analytic* except at the origin.

In the S_4 viewpoint, fields α and β are analytic everywhere.

VII. PROOF OF SO_5 SYMMETRY

An SO_5 rotation around the origin generates a new field α' from field α. We shall prove now that field α can be gauge transformed into field α'. This can be done by considering infinitesimal SO_5 rotations. We shall, however, present a different and better proof in the following steps:

(a) Since fields α and α' are both self-dual orthogonal everywhere in the S_4 view, and they have the same amplitude (41), their $f^i_{\mu\nu}$ can both be gauge transformed into the same standard form (A10). [Equation (35) insures that $f^i_{r\mu} = 0$ always.] Thus their field strengths are gauge transformable into each other.

(b) Now adopt gauges for α and α' so that $(f^i_{\mu\nu})_\alpha = (f^i_{\mu\nu})_{\alpha'} =$ standard form (A10). It remain to be proved that $(b^i_\mu)_{\alpha'} = (b^i_\mu)_{\alpha}$. To do this we observe that $(b^i_r)_\alpha = (b^i_r)_{\alpha'} = 0$ by definition (since the gauge transformants that we used are independent of r). Next we write down the Bianchi identities for α and α' and subtract the corresponding equations from each other, resulting in

$$C^i_{jk}(\Delta b^j_\mu) f^k_{\nu\lambda} + (\text{cyclic permutations of } \mu\nu\lambda) = 0, \tag{43}$$

where

$$\Delta b^i_\mu = (b^i_\mu)_{\alpha'} - (b^i_\mu)_\alpha.$$

Choose μ, ν, λ to be three of the coordinates $\xi_1\xi_2\xi_3\theta$. There are four ways of doing this. Since $i = 1$, 2, or 3 in Eq. (43), we thus have 12 equations in the 12 unknowns Δb^i_μ. The determinant of these two 12 equations are known from the standard form (A11). It is simple to evaluate and it is equal to $16a^{12}=16r^{-24}\neq 0$. Thus

$$\Delta b^i_\mu=0$$

and we have proved that α and α' are gauge equivalent. The proof for the gauge equivalence of β and β' is similar.

VIII. ADDITIONAL PROPERTIES

(a) Fields α and β are both sourceless, i.e.,

$$f^{i\mu\nu}_{\mu\nu}=0. \tag{44}$$

This is true both when the field is viewed from the five-dimensional viewpoint, if we exclude the origin, or from the S_4 viewpoint. To prove it for the latter view, we use Eqs. (A2), (A3), and $f^*=\pm f$,

$$f^{i\mu\nu}_{\mu\nu}=\eta_{\mu\nu\alpha\beta}f^{*i\alpha\beta\mu\nu}/2=\pm\,\eta_{\mu\nu\alpha\beta}f^{i\alpha\beta\mu\nu}/2$$

which is zero because of the Bianchi identity. The proof of Eq. (44) for the five-dimensional view follows from this easily.

(b) $f^i_{\mu\nu}f^{i\mu\nu}$ can be evaluated, using Eqs. (A12) and (41),

$$f^i_{\mu\nu}f^{i\mu\nu}=12r^{-4}. \tag{45}$$

(c) In the S_4 viewpoint we can evaluate, via Eq. (A13),

$$f^i_{\mu\nu}f^{*i\mu\nu}=\pm\,12r^{-4}, \tag{46}$$

where + is for field α, − for field β. We can now evaluate (4) on a sphere $r=$ const,

$$\tfrac{1}{24}\oint\!\!\oint\!\!\oint\!\!\oint f^i_{\mu\nu}f^i_{\alpha\beta}\,dx^{\mu\nu\alpha\beta}=\pm\,r^{-4}\oint\!\!\oint\!\!\oint\!\!\oint d(\text{area})=\pm\,\tfrac{8}{3}\pi^2, \tag{47}$$

verifying (4) with $C_2=\pm 1$.

(d) Consider the S_4 viewpoint. The sourceless condition (44) which we just proved is[4] the condition that the "action"

$$L=\oint\!\!\oint\!\!\oint\!\!\oint f^i_{\mu\nu}f^{i\mu\nu}d(\text{surface})$$

over the sphere is stationary against changes of the gauge field. But we can prove a stronger statement. Consider any SU_2 gauge field on the sphere. Using the notation of (A6) we see that

$$f^i_{\mu\nu}f^{i\mu\nu}=2(\mathbf{E}^i\cdot\mathbf{E}^i+\mathbf{H}^i\cdot\mathbf{H}^i), \tag{48}$$

$$\tfrac{1}{24}f^i_{\mu\nu}f^i_{\alpha\beta}\eta^{\mu\nu\alpha\beta}=\tfrac{1}{3}\mathbf{E}^i\cdot\mathbf{H}^i. \tag{49}$$

Integrating Eq. (48) over the sphere we get L. Integrating Eq. (49) over the sphere we get by definition (4), $(8\pi^2/3)C_2$. Thus

$$L\geq 12\,|8\pi^2C_2/3|=32\pi^2|C_2|. \tag{50}$$

Since the Chern class number C_2 is always an integer, we find that for all fields for which $C_2\neq 0$, L attains an *absolute minima* $32\pi^2$ for fields α and β. This conclusion is the same as a corresponding one in Ref. 10.

For fields α and β, Eq. (45) leads to $L=32\pi^2$.

IX. FIELDS α AND β AS THE ONLY SO_5 SYMMETRICAL FIELDS

We shall now prove that fields α and β are the only SO_5 symmetrical SU_2 gauge fields other than the trivial case of all $f^i_{\mu\nu}=0$.

To be precise, we assume that field γ, whose strength is not equal to 0, defined on all five-dimensional space except the origin, can be gauge transformed to become any SO_5 rotation of itself. We shall prove that γ is gauge transformable to either field α or field β.

(a) Use the coordinates $\xi_1, \xi_2, \xi_3, \theta, r$. Consider a point P and write the twelve elements f^j_{ri} ($i=1,2,3,\theta$; $j=1,2,3$) as a 4×3 matrix M. An SO_4 rotation around the r axis at P generates a transformation A on the i index. SO_4 invariance requires that there is a compensating gauge transformation R so that

$$AMR=M.$$

Thus,

$$AM\tilde{M}\tilde{A}=M\tilde{M}.$$

Since A is an irreducible representation of SO_4 and the rank of $M\tilde{M}\leq 3$, we find $M\tilde{M}=0$, i.e., $M=0$. Thus

$$f^j_{ri}=0. \tag{51}$$

(b) Consider a sphere S_4: $r=$ const. The sphere is geometrically SO_5 symmetrical. γ is clearly pointwise SO_4 symmetric at any point P on the sphere. Using Lemma 3 of Appendix A we conclude that γ is orthogonal and self-dual or self-antidual at P. Thus it can be gauge transformed to the standard form (A10) or (A11). SO_5 symmetry implies that a is a function of r alone. Thus $a=a(r)$. Since the gauge transformation can be made independently at every point in five-dimensional space, we conclude that in a proper gauge,

$$(f^i_{\mu\nu})_\gamma=(ar^2)(f^i_{\mu\nu})_{\alpha\text{ or }\beta}. \tag{52}$$

(c) Now we can imitate the arguments of Sec. VII (b) and write down the Bianchi identities for the field γ, and for α or β. Multiplying the latter by (ar^2) and subtracting from the former we obtain for $\mu,\nu,\lambda\neq r$,

$$C^i_{jk}(\Delta b^j_\mu)(ar^2)(f^k_{\nu\lambda})_{\alpha\text{ or }\beta}+(\text{cyclic permutation of }\mu\nu\lambda)=0, \tag{53}$$

where

$$\Delta b^j_\mu=(b^j_\mu)_\gamma-(b^j_\mu)_{\alpha\text{ or }\beta}.$$

Just as in Sec. VII (b), Eq. (53) implies $\Delta b^j_\mu=0$ $(\mu\neq r)$. Substitution into Eq. (52) further leads to

$$ar^2=1. \tag{54}$$

(d) We need only prove now that $b^j_r=0$. To do this we subtract the Bianchi identity again, like in Eq. (53), but in addition use $ar^2=1$ and take one of μ,ν,λ to be r. Because of Eq. (35) we get

$$C^i_{jk}(\Delta b^j_r)(f^k_{\nu\lambda})_{\alpha\text{ or }\beta}=0. \tag{55}$$

It follows trivially that $\Delta b^j_r=0$, i.e., $b^j_r=0$.

X. ANGULAR MOMENTUM OPERATORS

In Dirac's monopole field, the angular momentum of a particle of charge Ze is[5,9]

$$\mathbf{L}=\mathbf{r}\times(\mathbf{p}-Ze\mathbf{A})-Zeg\mathbf{r}r^{-1}. \tag{56}$$

We want to generalize this formula to the field α (or β).

Consider the motion of a particle of isospin I in field α or β in five-dimensional space. Let Y_1, Y_2, Y_3 be the representation of generators X_1, X_2, X_3 for isospin I. Then the generalization of Eq. (56) is

$$L_{\mu\nu} = x_\mu(\partial_\nu + Y_k b^k_\nu) - x_\nu(\partial_\mu + Y_k b^k_\mu) - r^2 f^k_{\mu\nu} Y_k \\ = -L_{\nu\mu} \quad (\mu,\nu = 1,2,3,4,5). \tag{57}$$

It is important to notice that the wavefunctions are *sections*.[5] The transition function S_{ab} can be read off from Eq. (21), so that a section is defined by

$$\psi^{(a)} = \mathrm{Rep}(T)\psi^{(b)}, \tag{58}$$

where Rep(T) is the representation of T in the representation generated by the Y's. This formula is exactly the same as the corresponding one in Ref. 5. As in that reference, we consider the *Hilbert space of sections*. Equation (57) is then a Hermitian operator in the Hilbert space. The commutation rules of $L_{\mu\nu}$ can be obtained by direct calculation. After some algebra we obtain

$$[L_{\mu\nu}, L_{\alpha\beta}] = \delta_{\nu\alpha}L_{\mu\beta} - \delta_{\mu\alpha}L_{\nu\beta} - \delta_{\nu\beta}L_{\mu\alpha} + \delta_{\mu\beta}L_{\nu\alpha}, \tag{59}$$

which shows that $L_{\mu\nu}$ are the angular momentum operators.

One can now generalize the monopole harmonies of Ref. 5 to SU_2 monopole harmonies which are harmonic sections on a sphere r = const in five-dimensional space. We shall return to this problem in a later paper.

One notices that if we make the replacement

$$Y_k \to -iZ, \quad b^k_\nu \to eA_\nu, \quad -iL_{12} \to L_z, \tag{60}$$

then Eq. (57) reduces to Eq. (56).

XI. REMARKS

(a) The fields α and β on a sphere S_4 (in five dimensions) exhibit SO_5 symmetry. The sphere has a nonflat geometry. Does there exist corresponding solutions on a flat four-dimensional space with $ds^2 = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2$? The answer is no if we require maximum symmetry consistent with the geometry, i.e., if we require SO_4 symmetry plus displacement symmetry. (The symmetry group, which we shall call the Poincaré group, has 10 generators and is the natural extension to flat R^4 space of the SO_5 group for S_4 geometry.)

To prove the nonexistence in R^4 of a SU_2 gauge field δ with Poincaré symmetry we proceed exactly as in Sec. IX. If δ exists, it is pointwise SO_4 symmetrical at every point P. Lemma 3 of Appendix A then shows that it is orthogonal and self-dual or self-antidual at every point. Lemmas 1α and 1β then lead to the conclusion that δ can be gauge transformed to the standard form (A10) or (A11), where $a^2 = G$. Now Eq. (A5) and displacement symmetry imply G = numerical constant. Thus a = const in Eq. (A10) or (A11). i.e., $f^i_{\mu\nu}$ is independent of $x_1x_2x_3x_4$. The Bianchi identity then reads

$$C^i_{jk} b^j_\mu f^k_{\nu\lambda} + (\text{cyclic permutation of } \mu\nu\lambda) = 0. \tag{61}$$

If $a \neq 0$, this is 12 equations in the 12 numbers b^j_μ. The determinant is not equal to 0, as in Eq. (43). Thus $b^j_\mu = 0$. Therefore,

$$f^k_{\mu\nu} = 0. \tag{62}$$

If $a = 0$, then automatically Eq. (62) also holds. Thus in R^4 there is no SU_2 gauge field with strengths not equal to 0 that is SO_4 invariant and displacement invariant.

Belavin, Polyakov, Schwartz, and Tyupkin[10] have exhibited a solution which they call a pseudoparticle solution. It is a sourceless SU_2 gauge field on R^4 which is everywhere analytic. It has a second Chern class number $C_2 = \pm 1$. It does not have displacement invariance, in agreement with the conclusion above. The relationship between this pseudoparticle solution and O(5) symmetry has been discussed by Jackiw and Rebbi,[11] who found that the pseudoparticle when conformally mapped to a sphere S_4 is O(5) symmetrical. According to Sec. IX above, the conformally mapped solution is, exactly, the O(5) symmetrical SU_2 gauge field which is the generalization of Dirac's monopole. Further comments on this relationship will be communicated in a separate paper.

(b) Does there exist a SO_n symmetrical SU_2 gauge field on the n-dimensional flat space (with positive signatures) minus the origin? (We do not consider the trivial case of $f^i_{\mu\nu} \equiv 0$.) We have seen in Sec. IX that for $n = 5$, there are two such fields α and β. We shall now prove that for $n \geq 6$, there are no such fields.

Take the case $n = 6$. Choose *orthogonal* coordinates $\xi_1\xi_2\xi_3\xi_4\xi_5 r$ where r is the radius. We can first easily prove the generalization of Eq. (51),

$$f^j_{ri} = 0 \quad (i = 1,2,3,4,\theta).$$

Next consider a point P and choose the scales of $\xi_1\xi_2\xi_3\xi_4\xi_5$ so that $g_{11} = g_{22} = g_{33} = g_{44} = g_{55} = 1$ at P. Consider $f^i_{\mu\nu}$ for $\mu,\nu = 1,2,3,4$. SO_4 symmetry in the directions of $\xi_1\xi_2\xi_3\xi_4$ leads to

$$f^i_{12} = \pm f^i_{34},$$

by an argument similar to that in Sec. IX. Similarly, if we consider $\mu,\nu = 1,2,3,5$ we obtain

$$f^i_{12} = \pm f^i_{35}.$$

Now take $\mu,\nu = 2,3,4,5$. SO_4 symmetry in the directions of $\xi_2\xi_3\xi_4\xi_5$ implies orthogonality, so that

$$f^i_{35} f^i_{34} = 0.$$

Thus

$$f^i_{12} f^i_{12} = 0,$$

i.e.,

$$f^i_{12} = 0.$$

We thus find all components of $f = 0$. The proof for the case $n > 6$ is similar.

(c) What happens for $n = 4$? One can find SO_4 symmetric solutions, singular only at the origin, in the following way:

Consider a path $A \to B$ not passing through the origin. Project the path radially onto the unit sphere $r = 1$. Let the projection be called $A'B'$. Let p be a real number. Each point $a, b, c, \ldots, z$ along the path $A'B'$ corresponds to an element of SU_2 which we shall denote by $\underline{a}, \underline{b}, \ldots, \underline{z}$. If the path $A'B'$ is $A'ab\cdots zB'$ we define

$$\Phi_{BA} = \Phi_{B'A'} = (\underline{B'}\underline{z}^{-1})^p(\underline{z}\underline{y}^{-1})^p \cdots (\underline{a}\underline{A'}^{-1})^p.$$

To show that this gauge field is SO_4 invariant, consider any two fixed elements of the group ξ, η. Let $\underline{z}_0 = \xi \underline{z} \eta$, $\underline{B}'_0 = \xi \underline{B}' \eta$, etc.

Then

$$\begin{aligned}\xi \Phi_{BA} \xi^{-1} &= (\xi \underline{B}' z^{-1} \xi^{-1})^p (\xi \underline{z} \underline{y}^{-1} \xi^{-1})^p \cdots (\xi \underline{a} \underline{A}'^{-1} \xi^{-1})^p \\ &= (\underline{B}'_0 \underline{z}_0^{-1})^p (\underline{z}_0 \underline{y}_0^{-1})^p \cdots (\underline{a}_0 \underline{A}_0'^{-1})^p \\ &= \Phi_{B'_0 A'_0}.\end{aligned}$$

Now the path $A'_0 \to B'_0$ is an SO_4 rotation of $A' \to B'$, since the transformation $z \to z_0 = \xi \underline{z} \eta$ is an SO_4 rotation. Furthermore, every SO_4 rotation is such a transformation. Thus an SO_4 rotation only produces a gauge transformation.

Notice that p is an arbitrary real number. So we have exhibited a 1 parameter family of SO_4 symmetrical SU_2 gauge fields in R^4 minus origin.

(d) For $n = 3$, we are in more familiar geometry. To construct an SO_3 symmetrical SU_2 gauge field ϵ we take a Dirac U(1) monopole b_μ and put $b^1_\mu = b^2_\mu = 0$, $b^3_\mu = b_\mu$. Such gauge fields are, however, not really interesting because the space does not have enough dimensions to develop the full complexity of the group. One consequence of this lack of enough dimensions is the fact, demonstrated in Ref. 6, that field ϵ is of the same gauge type (i.e., same fibre bundle) as the vacuum field $f^i_{\mu\nu} = 0$.

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APPENDIX A: SOME PROPERTIES OF SU_2 GAUGE FIELDS IN FOUR DIMENSIONS

Consider a SU_2 gauge field in four-dimensional space, with signature $++++$, flat or otherwise. We define the antisymmetrical tensor η by

$$\eta_{\alpha\beta\mu\nu} = \sqrt{g}\,\epsilon_{\alpha\beta\mu\nu}, \tag{A1}$$

where $\epsilon = \pm 1$ is the antisymmetrical symbol. We define the dual f^* of a field f by

$$f^*_{\alpha\beta} = \tfrac{1}{2}\eta_{\alpha\beta\mu\nu} f_{\mu\nu}. \tag{A2}$$

Clearly,

$$f^{**} = f. \tag{A3}$$

We only consider coordinate choices that leave $\eta_{1234} > 0$. In other words, a reflection in four-dimensional space is not considered a legitimate transformation. We adopt the terminology at any point P,

$$\begin{aligned}&f^* = f \text{ at } P \leftrightarrow f \text{ is self-dual at } P,\\ &f^* = -f \text{ at } P \leftrightarrow f \text{ is self-antidual at } P.\end{aligned} \tag{A4}$$

We further call a gauge field "orthogonal" at a point P if at that point

$$f^i_{\mu\nu} f^{j\lambda\nu} = a^2 \delta^{ij} \delta^\lambda_\mu + a\epsilon^{ijk} f^{k\lambda}_\mu. \tag{A5}$$

It is clear that the orthogonality and self duality properties of a field f at a point is independent of the choice of gauge or the choice of the coordinate system. We shall call the scalar a the *amplitude* of the orthogonal field at P. It is independent of the choice of coordinates and can be positive or negative.

Consider any field f at a point P. Adopt a coordinate system so that at P the metric $g_{\mu\nu} = \delta_{\mu\nu}$. Write the field strengths in the following form:

$$f^j_{\mu\nu} = \begin{bmatrix} 0 & H^j_3 & -H^j_2 & E^j_1 \\ -H^j_3 & 0 & H^j_1 & E^j_2 \\ H^j_2 & -H^j_1 & 0 & E^j_3 \\ -E^j_1 & -E^j_2 & -E^j_3 & 0 \end{bmatrix} \quad (f^j_{12} = H^j_3, \text{ etc.}). \tag{A6}$$

We shall consider f^j $(j = 1, 2, 3)$ as three 6-vectors. The matrices

$$H = \begin{bmatrix} H^1_1 & H^2_1 & H^3_1 \\ H^1_2 & H^2_2 & H^3_2 \\ H^1_3 & H^2_3 & H^3_3 \end{bmatrix}, \quad \mathcal{E} = \begin{bmatrix} E^1_1 & E^2_1 & E^3_1 \\ E^1_2 & E^2_2 & E^3_2 \\ E^1_3 & E^2_3 & E^3_3 \end{bmatrix} \tag{A7}$$

will be called magnetic and electric matrices. It is obvious that

$$\begin{aligned}&\mathcal{E} = H \leftrightarrow \text{self-duality},\\ &\mathcal{E} = -H \leftrightarrow \text{self-antiduality}.\end{aligned} \tag{A8}$$

By substituting (A6) into (A5) we find

$$\begin{aligned}&\mathcal{E} = H, \quad H = a\Gamma \leftrightarrow \text{self-duality} + \text{orthogonality},\\ &\mathcal{E} = -H, \quad H = a\Gamma \leftrightarrow \text{self-antiduality} + \text{orthogonality}.\end{aligned} \tag{A9}$$

In Eq. (A9), Γ is an orthogonal matrix with determinant $+1$.

A gauge transformation multiplies $\mathcal{E}$ and H from behind by an orthogonal 3×3 matrix R of determinant unity. Thus if $H = a\Gamma$, there always exists a gauge transformation to make $H \to aI$. Hence, we have

Lemma 1α: Consider a gauge field which is self-dual and orthogonal at a point P. Consider any coordinate system so that at P, $g_{\mu\nu} = \delta_{\mu\nu}$. The field at P can be gauge transformed to a *standard form* for such fields:

$$f^1_{\mu\nu} = a\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \quad f^2_{\mu\nu} = a\begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix},$$

$$f^3_{\mu\nu} = a\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \tag{A10}$$

(e.g., $f^1_{14} = a$). Equation (A10) can also be written as

$$\mathcal{E} = H = aI \quad (I = \text{unit matrix}). \tag{A10$'$}$$

Lemma 1β: Consider a gauge field which is self-antidual and orthogonal at a point P. Consider any coordinate system so that at P, $g_{\mu\nu} = \delta_{\mu\nu}$. The field at P can be gauge transformed to a *standard form* for such fields,

$$f^1_{\mu\nu}=a\begin{bmatrix}0&0&0&-1\\0&0&1&0\\0&-1&0&0\\1&0&0&0\end{bmatrix},\quad f^2_{\mu\nu}=a\begin{bmatrix}0&0&-1&0\\0&0&0&-1\\1&0&0&0\\0&1&0&0\end{bmatrix},$$

$$f^3_{\mu\nu}=a\begin{bmatrix}0&1&0&0\\-1&0&0&0\\0&0&0&-1\\0&0&1&0\end{bmatrix}. \qquad \text{(A11)}$$

Equation (A11) can also be written as

$$-\mathcal{E}=\mathcal{H}=aI \quad (I=\text{unit matrix}). \qquad \text{(A11}')$$

Lemma 2: A field which is orthogonal and self-dual or self-antidual at P satisfies, at P,

$$f^i_{\mu\nu}f^{i\mu\nu}=12a^2, \qquad \text{(A12)}$$

$$f^i_{\mu\nu}f^{*i\mu\nu}=\tfrac{1}{2}f^i_{\mu\nu}\eta^{\mu\nu\alpha\beta}f^i_{\alpha\beta}=\pm 12a^2, \qquad \text{(A13)}$$

where the + sign is for the case of self-dual fields and the − sign is for the case of self-antidual fields, and a is the amplitude of f.

The proof is trivial.

Consider a four-dimensional space that has geometrically SO_4 symmetry at a point P. Examples are the flat space, a S_4 sphere, or the more general space M_4,

$$ds^2=d\rho^2+\rho^2[e(\rho)]^2\frac{4d\xi^2}{1+\xi^2}, \quad -\infty<\xi<\infty, \qquad \text{(A14)}$$

where $e(\rho)$ is any function of ρ. [If $\rho e=\sin\rho$, we get the sphere S_4, of Eq. (23).] M_4 has SO_4 symmetry at the point $\rho=0$.

For a gauge field γ defined on a space that has geometrically SO_4 symmetry at a point P, we can generate another field γ' by rotating the whole potential (and field around P by an SO_4 rotation. Is γ gauge equivalent to γ' as far as the field strength at P is concerned? If it is, we say that the field is *pointwise* SO_4 *symmetrical* at P.

We shall call a field self-dual orthogonal or self-antidual orthogonal at a point P, *regular* at P. We now have a geometrical meaning of regularity (we shall show later that orthogonality is equivalent to regularity).

Lemma 3: Consider a space that has geometrical SO_4 symmetry at a point P. Then for a SU_2 field,

$$\text{pointwise } SO_4 \text{ symmetry at } P \leftrightarrow \text{regularity at } P. \qquad \text{(A15)}$$

Proof:

(a) That the right-hand side implies the left-hand side follows from Lemmas 1α and 1β. To prove the converse, we start with the field $f^i_{\mu\nu}$ of Eq. (A6). An SO_3 rotation around P means, for the field strengths at P,

$$\mathcal{H}\to\Gamma\mathcal{H}, \quad \mathcal{E}\to\Gamma\mathcal{E},$$

where Γ is an orthogonal matrix with determinant unity. Pointwise SO_4 symmetry at P implies that there exists a compensating gauge transformation which causes a multiplication from the right by the 3×3 representation R of the compensating SU_2 gauge rotation at the point. i.e., for every Γ there exists an R so that

$$\Gamma\mathcal{H}R=\mathcal{H}, \quad \Gamma\mathcal{E}R=\mathcal{E}. \qquad \text{(A16)}$$

Thus

$$\Gamma\mathcal{H}\tilde{\mathcal{H}}\tilde{\Gamma}=\mathcal{H}\tilde{\mathcal{H}}. \qquad \text{(A17)}$$

Hence

$$\mathcal{H}\tilde{\mathcal{H}}=h^2I. \qquad \text{(A18)}$$

Similarly we find that $\mathcal{E}\tilde{\mathcal{E}}$ and $\mathcal{E}\tilde{\mathcal{H}}$ are proportional to the unit 3×3 matrix I. Thus $\mathcal{E}$ and $\mathcal{H}$ are proportional to each other.

(b) Now make a transformation at P that mixes the indices 1 and 4. It is easy to see that $\mathcal{E}+\mathcal{H}$ and $\mathcal{E}-\mathcal{H}$ are independently rotated:

$$\mathcal{E}+\mathcal{H}\to\Gamma_1(\mathcal{E}+\mathcal{H}),$$

$$\mathcal{E}-\mathcal{H}\to\Gamma_2(\mathcal{E}-\mathcal{H}).$$

For $\mathcal{E}$ and $\mathcal{H}$ to remain proportional we must have either

$$\mathcal{E}+\mathcal{H}=0 \quad\text{or}\quad \mathcal{E}-\mathcal{H}=0. \qquad \text{(A19)}$$

Equations (A18) and (A19) show that we can gauge transform the field at P to the standard form Eqs. (A10) or (A11). This completes the proof of the lemma.

Lemma 4: Choose coordinates so that $g_{\mu\nu}=\delta_{\mu\nu}$ at P. An orthogonal field at P satisfies at P

$$f^1f^1=f^2f^2=f^3f^3=-a^2, \qquad \text{(A20)}$$

$$f^1f^2=-f^2f^1=-af^3, \quad \text{and cyclic permutation}, \qquad \text{(A21)}$$

$$f^1f^2f^3=a^3, \qquad \text{(A22)}$$

where f^i is a 4×4 antisymmetrical matrix with elements

$$\langle\mu|f^i|\nu\rangle=f^i_{\mu\nu}.$$

The proof is easy, starting from definition (A5).

Lemma 5: A field orthogonal at P is either self-dual or self-antidual at P. Therefore, it is regular at P.

Proof: If $a=0$, $f^1=f^2=f^3=0$, and the lemma is proved. If $a\neq0$, f^3/a is an antisymmetrical real matrix whose square is -1, according to (A20). By a well-known theorem one can, by an SO_4 rotation of coordinates at P, bring f^3 into the form displayed in (A10) or (A11). Thus f^3 is either self-dual or self-antidual. (i) If f^3 is self-dual, $f^3=ai\sigma_2$ where $\sigma_1,\sigma_2,\sigma_3,\tau_1,\tau_2,\tau_3$, are the standard 4×4 Pauli matrices. $i(f^1)$ is imaginary Hermitian, and anticommutes with σ_2. Thus $i(f^1)$ is a sum of $\sigma_1\tau_2$ and $\sigma_3\tau_2$ with real coefficients. By another SO_4 rotation of coordinates we can make $f^1=a\sigma_1(i\tau_2)$, leaving $f^3=ai\sigma_2$. $i(f^2)$ is imaginary Hermitian and anticommutes with both f^1 and f^3. Thus $f^2=\xi(a^2)^{1/2}(-i\sigma_3\tau_2)$, $\xi=\pm1$. Thus f^1,f^2,f^3 are all self-dual. (ii) If f^3 is self-antidual, we can similarly prove that f^1 and f^2 are also self-antidual.

Lemma 6:

$$[f^i_{\mu\nu}f^{j\lambda\nu}+f^j_{\mu\nu}f^{i\lambda\nu}=2a^2\delta^{ij}\delta^\lambda_\mu]\to\text{(A5)}. \qquad \text{(A23)}$$

Proof: That (A5) implies the left-hand side is obvious. If the left-hand side holds, the sign of a is for us to choose. We choose coordinates at P so that $g_{\mu\nu}=\delta_{\mu\nu}$. Then (A20) holds. In fact

$$f^if^j+f^jf^i=-2a^2\delta^{ij}. \qquad \text{(A24)}$$

Now the proof of Lemma 5 depends only on this equation. Following that proof we conclude that there are only two possibilities.

(i) We can by an SO_4 rotation at P bring the f's into the following form,

$$f^1 = |a|\sigma_1(i\tau_2), \quad f^2 = \xi|a|(-i\sigma_3\tau_2), \quad f^3 = |a|i\sigma_2.$$

Thus f^1, f^2, f^3 satisfy (A20) and (A21) with $a = \xi|a|$. (A20) and (A21) together imply (A5).

(ii) f^3 is antiselfdual. The proof is similar.

APPENDIX B: FIELD STRENGTHS FOR FIELD α

Using Eq. (31) we find how to obtain $(f^i_{\mu\nu})_\alpha$ from $(f^i_{\mu\nu})_\beta$:

$$[f^1_{\theta i}(\xi, \theta)]_\alpha = -[f^1_{\theta i}(-\xi, \theta)]_\beta \quad (i = 1, 2, 3), \tag{B1}$$

$$[f^1_{ij}(\xi, \theta)]_\alpha = [f^1_{ij}(-\xi, \theta)]_\beta \quad (i, j = 1, 2, 3). \tag{B2}$$

APPENDIX C: PROOF THAT FIELD β IS ORTHOGONAL SELF-ANTIDUAL

One proof consists in evaluating the left-hand side of Eq. (A5) using Eq. (37β). The calculation is long but straightforward.

Another proof follows the steps in Appendix A by starting with a scale change from variable $\xi_1, \xi_2, \xi_3, \xi_4 \to y_1, y_2, y_3, y_4$,

$$\theta = \theta_0 + y_4 r^{-1}, \quad \xi_i = (\xi_i)_0 + y_i(1 + \xi_0^2)(2r_0 \sin\theta)^{-1}. \tag{C1}$$

Then *in the y variables*, at $y_\mu = 0$ the metric is unity. $f^i_{\mu\nu}$ in the y variables are easily obtained from Eq. (37β). We arrange it in the form of Eqs. (A6) and (A7), obtaining

$$\mathcal{H} = -\mathcal{E} = -(1 + \xi^2)\tilde{B}(4r^2)^{-1}, \tag{C2β}$$

where we have dropped the subscript 0 in all variables. Using Eq. (33) we find

$$\mathcal{H}\tilde{\mathcal{H}} = r^{-4}I \quad (I = \text{unit matrix}). \tag{C3}$$

To determine the value of det$\mathcal{H}$ we put $\xi = 0$. Then $B = -4I$. Thus

$$\det\mathcal{H} > 0 \quad \text{everywhere}. \tag{C4}$$

We can then make a gauge transformation to make

$$-\mathcal{E} = \mathcal{H} \to \mathcal{H}R = r^{-2}I.$$

Thus we have arrived at the standard form Eq. (A11) showing that field β is orthogonal self-antidual with $a = r^{-2}$, in agreement with Eq. (41).

The same calculations can be made for field α. Using Eqs. (B1) and (B2), we see that all formulas are unchanged, except Eq. (C2β) becomes in this case

$$\mathcal{H} = \mathcal{E} = -(1 + \xi^2)B(4r^2)^{-1}. \tag{C2α}$$

[1] P. A. M. Dirac, Proc. Roy. Soc. A **133**, 60 (1931).

[2] We use the notation of Chen Ning Yang, *Proceedings of the Sixth Hawaii Topical Conference*, 1975 (University of Hawaii Press, Hawaii, 1976). Specifically, $\sum^a$ means antisymmetrical sum with respect to all Greek indices. $,\lambda$ means $\partial/\partial\chi^\lambda$. Dummy indices are summed over. X_i are the generators of SU_2 satisfying $[X_1, X_2] = X_3$, etc. $C^1_{23} = 1$.

[3] Shiing-shen Chern, *Topics in Differential Geometry* (Inst. for Adv. Study, Princeton, 1951); Ann. Math. (N.Y.) **47**, 85 (1946).

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[7] We denote by ($xx\alpha$) an equation that is valid for field α. ($xx\beta$) denotes one that is valid for field β. An equation number without α or β is valid for both fields.

[8] $\xi^1 = \xi_1$, $\xi^2 = \xi_2$, and $\xi^3 = \xi_3$ are not tensors. $d\xi^1$, $d\xi^2$, and $d\xi^3$ are.

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Commentary I was fascinated in the mid-1970s by BPST's sourceless SU_2 gauge field on a S_4. It has beautiful SO_5 symmetry as well as interesting E↔H relations. It is in a sense a direct generalization of the U_1 Dirac monopole, as I explained in [78a]. I tried to further generalize it to gauge fields belonging to higher Lie groups, but failed. Thus I did not include [78a] in my earlier *Selected Papers*.

Such a generalization was recently found by Van-Hoang Le and Thanh-Son Nguyen in *J. Math. Phys.* 52, 032105 (2011). It is an SO(8) monopole with SO_9 symmetry. [Compare also B. A. Bernevig, J. Hu, N. Toumbas and S.-C. Zhang, *Phys. Rev. Lett.* 91, 236803 (2003).]