# SOME EXACT RESULTS FOR THE MANY-BODY PROBLEM IN ONE DIMENSION WITH REPULSIVE DELTA-FUNCTION INTERACTION*

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The repulsive $\delta$ interaction problem in one dimension for $N$ particles is reduced, through the use of Bethe's hypothesis, to an eigenvalue problem of matrices of the same sizes as the irreducible representations $R$ of the permutation group $S_N$. For some $R$'s this eigenvalue problem itself is solved by a second use of Bethe's hypothesis, in a generalized form. In particular, the ground-state problem of spin-$\frac{1}{2}$ fermions is reduced to a generalized Fredholm equation.

(1) Consider the one-dimensional $N$-body problem

$$H=-\sum_1^N \partial^2/\partial x_i^2+2c\sum_{i<j}\delta(x_i-x_j),\quad c>0, \tag{1}$$

with no limitation on the symmetry of the wave function $\psi$. For a given irreducible representation $R_\psi$ of the permutation group $S_N$ of the $N$ coordinates $x_i$, we want to determine the wave function $\psi$. Assume Bethe's hypothesis[1] to be valid: Let $p_1,\cdots,p_N$ = a set of unequal numbers. For $0<x_{Q1}<x_{Q2}<\cdots<x_{QN}<L$,

$$\psi=\sum_P[Q,P]\exp i[p_{P1}x_{Q1}+\cdots+p_{PN}x_{QN}], \tag{2}$$

where $P=[P1,P2,\cdots,PN]$ and $Q=[Q1,Q2,\cdots,QN]$ are two permutations of the integers $1,2,\cdots,N$. $[Q,P]$ can be arranged as a $N!\times N!$ matrix. Denote the columns of this matrix by $\xi_P$. To satisfy the continuity of $\psi$ and the proper discontinuity of its derivative as required by (1) at $x_{Q3}=x_{Q4}$, it is sufficient to have

$$\xi_{\cdots ij\cdots}=Y_{ji}^{34}\xi_{\cdots ji\cdots}, \tag{3}$$

where the subscripts for $\xi$ on the two sides represent any two permutation $P$ and $P'$ so that $P1=P'1$, $P2=P'2$, $P3=i=P'4$, $P4=j=P'3$, etc. The operator $Y$ is defined by

$$Y_{ij}^{34}=(y_{ij}^{-1}-1)+y_{ij}^{-1}P_{34}=Y_{ij}^{43}, \tag{4}$$

where

$$y_{ij}=1+x_{ij}, \tag{5}$$

$$x_{jk}=ic(p_j-p_k)^{-1}=-x_{kj}, \tag{6}$$

and $P_{34}$ = the permutation operator on $\xi$ so that it interchanges $Q3$ and $Q4$. Altogether there are $N!(N-1)$ equations of the form (3). Are they mutually consistent? The answer is yes for any set of unequal $p$'s. This can be seen with the aid of the following identities:

$$Y_{ij}^{ab}Y_{ji}^{ab}=1, \tag{7}$$

and

$$Y_{jk}^{ab}Y_{ik}^{bc}Y_{ij}^{ab}=Y_{ij}^{bc}Y_{ik}^{ab}Y_{jk}^{bc}, \tag{8}$$

which are easily verified. Thus given a set of unequal $p$'s, and $\xi_0=\xi_P$ for $P$ = identity, all $\xi_P$'s are determined.

(2) The imposition of the periodic boundary conditions leads to equations which, upon expressing $\xi_P$ in terms of $\xi_0$, become

$$\lambda_j\xi_0=X_{(j+1)j}\times X_{(j+2)j}\cdots X_{Nj}X_{1j}X_{2j}\cdots X_{(j-1)j}\xi_0,\quad j=1,\cdots,N, \tag{9}$$

where

$$\lambda_j=\exp(ip_jL), \tag{10}$$

and

$$X_{ij}=P_{ij}Y_{ij}^{ij}=(1-P_{ij}x_{ij})(1+x_{ij})^{-1}. \tag{11}$$

The $N$ Eqs. (9) say that $\xi_0$ is simultaneously an eigenvector of $N$ operators. These $N$ operators can be shown to commute with each other, using

$$X_{ij}X_{ji}=1,\quad X_{jk}X_{ik}X_{ij}=X_{ij}X_{ik}X_{jk}X_{kj}X_{ki}X_{ji}=1,$$

$$X_{ij}X_{kl}=X_{kl}X_{ij};\quad i,j,k,\text{ and } l \text{ all unequal}. \tag{12}$$

(3) The operators $P_{ij}$ on $\xi$ form a $N!\times N!$ representation of $S_N$. To find the eigenfunctions $\xi_0$ in (9) we can first reduce this representation to irreducible ones. Choosing one specific irreducible representation $R$ reduces the

eigenvalue problem (9) to one of smaller dimensions. It can be shown that the resultant wave function (2) would have a permutation symmetry $R_\psi$ which is the same as $R$. For example, if $R$ = identity representation = $[N]$, then $P_{ij}=1$, and (9) becomes $1\times 1$ matrix equations and the result is precisely the well-known boson result.[2] If $R$ = antisymmetric representation = $[1^N]$, then $P_{ij}=-1$, and $X_{ij}=1$, so that (9) and (10) reduce to $\exp(ip_j L)=1$, showing there is no interaction, a result to be expected for the antisymmetrical wave function.

(4) The $\lambda_j$'s are functions of the $p$'s, $c$, and $R$. It is easily seen (that $R$ and $\tilde{R}$ being conjugate representations)

$$\lambda_j(p;c;R)=\prod_{i\neq j}\left(\frac{1-x_{ij}}{1+x_{ij}}\right)\lambda_j(p;-c;\tilde{R}). \qquad (13)$$

(5) Define $\mu_j(p;c;R)$ by

$$\mu_j\Phi = X_{(j+1)j}{}'X_{(j+2)j}{}'\cdots \times X_{Nj}{}'X_{1j}{}'X_{2j}{}'\cdots X_{(j-1)j}{}'\Phi, \qquad (14)$$

where

$$X_{ij}{}'=(1+P_{ij}x_{ij})(1+x_{ij})^{-1}. \qquad (15)$$

Clearly

$$\mu_j(p;c;\tilde{R})=\lambda_j(p;c;R). \qquad (16)$$

(6) We now evaluate $\lambda_j$ for $R_\psi=R=[2^M 1^{N-2M}]$. By (16) we need to find $\mu_j(p;c;[N-M,M])$. To do this we first define a convenient representation for $P_{ij}$ of (15):

Consider $N$ spin-$\frac{1}{2}$ particles, and consider the spin wave functions $\Phi$ for total $z$ spin $=\frac{1}{2}(N-2M)$. These spin wave functions transform under $S_N$ according to a sum of irreducible representations,

$$[N]+[N-1,1]+[N-2,2]+\cdots+[N-M,M]. \qquad (17)$$

We consider the $P_{ij}$'s of (15) as operating on these spin wave functions $\Phi$. The eigenvalue equations (14) for $\mu_j$ are then to be solved for a $\Phi$ that belongs to the symmetry $[N-M,M]$.

(7) Consider the $N$ spins as forming a cyclic chain. The wave function $\Phi$ has $C_M{}^N$ components [$N-M$ spins up, $M$ spins down]. The eigenvalue problem (14) can be solved with a generalized Bethe's hypothesis:

$$\Phi=\sum_P A_P F(\Lambda_{P1},y_1)\times F(\Lambda_{P2},y_2)\cdots F(\Lambda_{PM},y_M), \qquad (18)$$

where $y_1<y_2<\cdots<y_M$ are the "coordinates," along the chain, of the $M$ down spins, and $\Lambda_1$, $\Lambda_2,\cdots,\Lambda_M$ are a set of unequal numbers. With this hypothesis, one finds

$$F(\Lambda,y)=\prod_{j=1}^{y-1}\frac{ip_j-i\Lambda-c'}{ip_{j+1}-i\Lambda+c'} \quad (c'=\tfrac{1}{2}c); \qquad (19)$$

$$-\prod_j\frac{ip_j-i\Lambda_\alpha-c'}{ip_j-i\Lambda_\alpha+c'}=\prod_\beta\frac{-i\Lambda_\beta+i\Lambda_\alpha+c}{-i\Lambda_\beta+i\Lambda_\alpha-c}; \qquad (20)$$

and

$$\mu_j(p;c;[N-M,M])=\prod_\beta\frac{ip_j-i\Lambda_\beta-c'}{ip_j-i\Lambda_\beta+c'}. \qquad (21)$$

(8) Thus for the $R_\psi=[2^M 1^{N-2M}]$ symmetry, we need to solve

$$\exp(ip_j L)=\text{right-hand side of (21)}, \qquad (22)$$

together with (20). In taking the logarithm of (20) and (22) care must be taken to add terms $2\pi i$(integer). The value of the integer can be determined by going to the limit $c\to+\infty$. One obtains, for the ground state with the symmetry $R_\psi=[2^M 1^{N-2M}]$, for the case $N$ = even, $M$ = odd,

$$-\sum_p\theta(2\Lambda-2p)=2\pi J_\Lambda-\sum_{\Lambda'}\theta(\Lambda-\Lambda'), \qquad (23a)$$

$$Lp=2\pi I_p+\sum_\Lambda\theta(2p-2\Lambda), \qquad (23b)$$

where the $p$'s are a set of $N$ ascending real numbers, the $\Lambda$'s a set of $M$ ascending real numbers,

$$\theta(p)=-2\tan^{-1}(p/c)\quad(-\pi\leq\theta<\pi), \qquad (24)$$

and

$J_\Lambda$ = successive integers from

$$-\tfrac{1}{2}(M-1) \text{ to } +\tfrac{1}{2}(M-1), \qquad (24a)$$

$\frac{1}{2}+I_p$ = successive integers from

$$1-\tfrac{1}{2}N \text{ to } \tfrac{1}{2}N. \qquad (24b)$$

Equation (23a) differs from that given in a recent paper,[3] in the definition of $\theta$ and our introduction of $J_\Lambda$. The present equation allows for a natural discussion of the limit $c \to +\infty$ (not $c \to 0$!) and hence the values of $J_\Lambda$.

(9) We can now approach the limit $N \to \infty$, $M = \infty$, $L \to \infty$ proportionally, obtaining

$$-\int_{-Q}^{Q}\theta(2\Lambda-2p)\rho(p)dp = 2\pi g - \int_{-B}^{B}\theta(\Lambda-\Lambda')\sigma(\Lambda')d\Lambda', \quad (25a)$$

$$p = 2\pi f + \int_{-B}^{B}\theta(2p-2\Lambda)\sigma(\Lambda)d\Lambda, \quad (25b)$$

$$dg/d\Lambda = \sigma, \quad df/dp = \rho. \quad (25c)$$

Or, after differentiation,

$$2\pi\sigma = -\int_{-B}^{B}\frac{2c\sigma(\Lambda')d\Lambda'}{c^2+(\Lambda-\Lambda')^2} + \int_{-Q}^{Q}\frac{4c\rho dp}{c^2+4(p-\Lambda)^2}, \quad (26a)$$

$$2\pi\rho = 1 + \int_{-B}^{B}\frac{4c\sigma d\Lambda}{c^2+4(p-\Lambda)^2}, \quad (26b)$$

$$N/L = \int_{-Q}^{Q}\rho dp, \quad M/L = \int_{-B}^{B}\sigma d\Lambda, \quad (27a)$$

and

$$E/L = \int_{-Q}^{Q}p^2\rho(p)dp. \quad (27b)$$

(10) Equations (26) are generalized Fredholm equations with a symmetrical kernel. It is easy to show that the equations are nonsingular by first studying the eigenvalues of the kernel in the limit $B = Q = \infty$.

(11) Equations (26) and (27) yield the ground-state energy per particle for spatial wave functions with the symmetry $[2^M 1^{N-2M}]$, at a given density $N/L$. For $N$ fermions with spin $\frac{1}{2}$ interacting through the Hamiltonian (1), this spatial wave function is coupled to a spin wave function of conjugate symmetry $[N-M, M]$, i.e., the total spin of the system is $\frac{1}{2}N-M$.

(12) For $B = \infty$, integration of (26a) over all $\Lambda$ yields $N = 2M$. Thus for the fermion problem with spin $\frac{1}{2}$, $B = \infty$ gives the ground state for states with total spin = 0. This state is also the absolute ground state for the problem, by a theorem due to Lieb and Mattis.[4]

(13) For the case $B \cong 0$, $M/L$ is proportional to $B$. One can readily expand all quantities in powers of $B$, obtaining, for fixed $r = N/L$,

$$\frac{E}{L} = \text{const.} + \frac{M}{L}\left[cr - \left(\frac{c^2}{2\pi}+2\pi r^2\right)\tan^{-1}\frac{2\pi r}{c}\right] + \cdots. \quad (28)$$

This result is in agreement with results already obtained by McGuire[5] for the case $M = 1$ and by Flicker and Lieb[6] for the case $M = 2$.

(14) For each symmetry $R_\psi$ of spatial wave function $\psi$, the excited states near the ground state can be obtained in a similar way as in the boson case.[7] More quantum numbers are, however, necessary to designate the excitations than in the boson case, because of the existence of the integers $J_\Lambda$ (which are in fact quantum numbers). Details will be published elsewhere.

(15) For the boson problem the thermodynamics and excitations for finite $T$ were treated by Yang and Yang.[8] Extension to the present problem presents no difficulty. Details will be published elsewhere.

(16) Using (13) one could generalize all the considerations above to the case of $R_\psi = [N-M, M]$. Details will be published elsewhere. The main change is that while all Eqs. (26) and (27) remain the same, (26b) is replaced by

$$2\pi\rho = 1 - \int_{-B}^{B}\frac{4c\sigma d\Lambda}{c^2+4(p-\Lambda)^2} + \int_{-Q}^{Q}\frac{2c\rho(p')dp'}{c^2+(p-p')^2}. \quad (26b')$$

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# EXACT DYNAMICS OF LANDAU ELECTRONS

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The availability of intense monochromatic radiation sources has stimulated both experimental and theoretical studies of strong-field effects in solids. The theoretical problem is that of calculating the dynamics of charge carriers in strong electric fields, for which simple perturbation theory is inadequate.

We treat exactly the dynamics of electrons in crossed electric and magnetic fields, the electric field being spatially uniform but otherwise arbitrary. We consider only the case of spherical constant-energy surfaces and neglect interband effects, thus limiting our analysis to intraband dynamics.

A recent paper by Hanamura, Lax, and Shin[1] has dealt with this problem for the special case of a sinusoidal electric field; however, their results are unfortunately incorrect. The exact solutions given here are readily understandable and are essentially identical to the results one would obtain classically.

We take the magnetic field along the $z$ direction, the electric field along the $x$ direction, and we work in the Landau gauge $\vec{A}=[0, +Hx, 0]$. The Hamiltonian in the presence of a spatially uniform electric field $E(t)$ is simply

$$\mathcal{H}=(P_x^{\,2}+P_z^{\,2})/2m+\tfrac{1}{2}m\omega_c^{\,2}(x-x_0)^2-eE(t)x, \qquad (1)$$

where $\omega_c=eH/mc$ and $x_0=-P_y/m\omega_c$ is the orbit center. We wish to calculate the evolution operator $U$ for the Hamiltonian (1), where

$$|t\rangle=U|i\rangle;\quad dU/dt=-\beta\mathcal{H}U;\quad \beta=i/\hbar. \qquad (2)$$

$|t\rangle$ is the state at the time $t\geqslant 0$, given that the initial state ($t=0$) is $|i\rangle$. Since $P_z$ and $P_y$ commute with $\mathcal{H}$, they are constants of motion and will be taken equal to zero for convenience. This makes $x=0$, and we then have the simple one-dimensional harmonic-oscillator Hamiltonian with a driving force,·

$$\mathcal{H}=P_x^{\,2}/2m+\tfrac{1}{2}m\omega_c^{\,2}x^2-eE(t)x. \qquad (3)$$

There are various methods available for obtaining the exact evolution operator corresponding to this Hamiltonian. Louisell[2] has presented a solution using normal ordering techniques, and similar results are obtainable from the generalized Baker-Hausdorff[3] formula. We present here a simple and physical method for obtaining $U$. We take $U$ in the following form:

$$U=e^{-\beta D}e^{\beta\alpha x}e^{-\beta\gamma P_x}e^{-\beta H_0 t}, \qquad (4)$$

where $\alpha$, $\gamma$, and $D$ are $c$ numbers depending on time and $H_0$ is the Hamiltonian of Eq. (3) with $E(t)=0$. Evaluating $dU/dt$ and requiring that Eq. (2) be satisfied, we obtain

$$\frac{d^2\gamma}{dt^2}+\omega_c^{\,2}\gamma=\frac{eE(t)}{m};\quad \alpha=m\frac{d\gamma}{dt};$$

$$\frac{dD}{dt}=\frac{\alpha^2}{2m}-\frac{m\omega_c^{\,2}}{2}\gamma^2. \qquad (5)$$

Here $\gamma(t)$ satisfies the classical driven-harmonic-oscillator equation and plays the role of the classical particle displacement, while $\alpha(t)$ plays the role of the classical momentum. It is immediately obvious that a sinusoidal driving field can only result in a time dependence of $\alpha$ and $\gamma$ which has frequency components at the driving frequency and at the resonant frequency. There are no sum and difference frequencies as has been reported in Ref. 1.

The requirement that $U(0)=1$ is readily met by choosing $\alpha(0)=\gamma(0)=D(0)=0$. The correspond-