Lect 5: Fermi liquid theory (I)

Landau: 1956 ^3He normal state:

hard core radius $\sim 2.5 \AA$, average interparticle distance $\sim 3.5 \AA$

we assume that there are no phase transitions: crystalline order, magnetic order, superfluidity, ...

§1 Concept of quasi-particles

Suppose we have a N-body ground state $|G\rangle$. At time $t=0$, an extra particle is inserted in the plane wave state $C_k^\dagger$. After a time period of T, we check the amplitude of such a particle still in the state of $\vec{k}$ at time T.

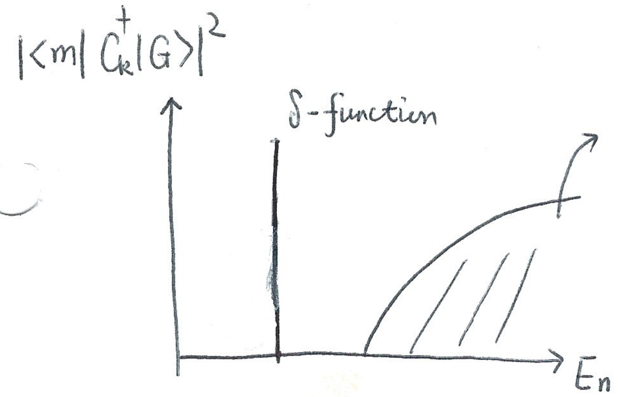
$$e^{-iHT} C_k^\dagger |G\rangle, \quad \text{the inner product with} \quad C_k^\dagger e^{-iHT} |G\rangle$$

$$\rightarrow G_k(T) = \langle G| e^{iHT} C_k e^{-iHT} C_k^\dagger | G\rangle = \langle G| C_k(T) C_k^\dagger(0) | G\rangle$$

we expand $G_k(T)$ in terms of Lehman representation as

$$G_k(T) = \sum_m \langle G| C_k(T)|m\rangle\langle m| C_k^\dagger(0)| G\rangle = \sum_m |\langle G| C_k|m\rangle|^2 e^{-i(E_m - E_g)T}$$

In many cases, the spectra weight of $|\langle m| C_k^\dagger |G\rangle|$ behaves like



$$G_k(T) = z\, e^{-i E_k' T} + \sum_n |\langle G | C_k | n \rangle|^2 e^{-i E_n T}.$$

The second part represents a continuum in the energy space, this leads to a rapid decaying function.

The long time behavor is controlled by the one particle-like excitation. In the Fermi liquid state $0 < z < 1$. The δ-function spike can be considered as quarsiparticle state.

§2: Adiabatical continuity:

Each state of the free fermi gas corresponds to a state of the interacting system by turning on the interaction adiabatically.

At zero temperature, the quasiparticle distribution satisfies

$$n^0_{p\sigma} = \begin{cases} 1 & k \le k_F \\ 0 & k > k_F . \end{cases}$$

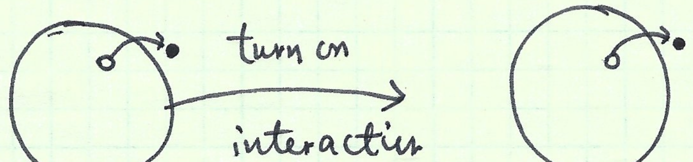
We can creat excitations by moving some quarsi-particles inside the fermi sphere into states outside the fermi surface. The quasi-particle energy is defined as

$$E - E_0 = \sum_{p\sigma} \epsilon(p)\, \delta n_{\vec{p},\vec{\sigma}}, \quad \text{so do}$$

$$\vec{P} = \sum_{p\sigma} \vec{p}\, \delta n_{p,\sigma}, \quad \vec{S} = \sum_\sigma \vec{\sigma}\, \delta n_{p\sigma}$$

Please note that $\vec{P}_{tot}$ and $\vec{S}_{tot}$ are conserved quantities through the process of turning interaction.

Fermi gas



Fermi liquid

quasi-particle / quasi-hole

bare electron (particle) / hole

let us expand $\boxed{\varepsilon(k) = \left(\frac{d\varepsilon}{dk_f}\right)\Big|_{k=k_f}(k-k_f), \quad \text{i.e. } v_F = \left(\frac{d\varepsilon}{dk}\right)_{k_f}}$

define effective mass $m^* = P_F / v_F$

§ Interactions among quasi-particles

We expand the variation of the ground state energy to the 2nd order of $\delta n_{p\sigma}$,

$$\delta E = \sum_{p\sigma} \frac{\delta E}{\delta n_{p\sigma}} \delta n_{p\sigma} + \frac{1}{2} \sum_{p\sigma, p'\sigma'} \frac{\delta^2 E}{\delta n_{p\sigma}\, \delta n_{p'\sigma'}} \delta n_{p\sigma}\, \delta n_{p'\sigma'}$$

$$\varepsilon_{p,\sigma} = \frac{\delta E}{\delta n_{p\sigma}},$$

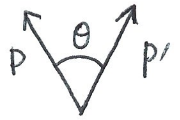
$$\frac{1}{V} f(\vec{p}, \vec{p}\,'; \sigma\sigma') = \frac{\delta^2 E}{\delta n_{p\sigma}\, \delta n_{p'\sigma'}} \longrightarrow \text{the unit of } f = \text{energy} \times \text{volume}$$

i.e Fourier component of interaction

$$\Rightarrow \boxed{\delta E = \sum_{p\sigma} \varepsilon_{p\sigma}\, \delta n_{p\sigma} + \frac{1}{2V} \sum_{p\sigma, p'\sigma'} f(\vec{p}\,\vec{p}\,'; \sigma\sigma')\, \delta n_{p\sigma}\, \delta n_{p'\sigma'}}$$

$\boxed{f(\vec{p}\,\vec{p}\,'; \sigma\sigma')}$: Landau interaction function we have

$$f(pp';\sigma\sigma') = f_s(\cos\theta) + f_a(\cos\theta)\,\sigma\sigma'$$



where $f_s = (f_{\uparrow\uparrow} + f_{\uparrow\downarrow})/2$

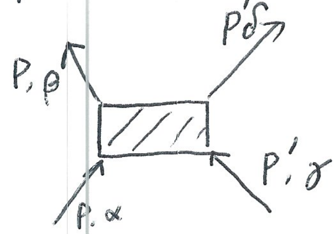
$f_a = (f_{\uparrow\uparrow} - f_{\uparrow\downarrow})/2$

More generally, spin does not need to be diagnoal, and should be represented as density matrix $\delta n_{p,\alpha\beta}$, and physical quantities, such as spin, should be represented as $S = tr[\vec{\sigma}\,\delta n_p] = (\vec{\sigma})_{\beta\alpha}\,\delta n_{p,\alpha\beta}$.

the interaction function most generally shoulel be

$$\frac{1}{2V}\sum_{\substack{pp'\\ \alpha\beta,\gamma\delta}} \left\{ f_s(\vec{p};\vec{p}')\,\delta_{\alpha\beta}\delta_{\gamma\delta} + f_a(\vec{p};\vec{p}')\,\vec{\sigma}_{\alpha\beta}\cdot\vec{\sigma}_{\gamma\delta} \right\} \cdot \delta n_{p,\beta\alpha} \cdot \delta n_{p',\delta\gamma}$$

$f_{\alpha\beta,\gamma\delta}(p\,p')$ describes the forward scattering amplitude of quasi particles near the Fermi surface.



symmetry constraint:

orbital rotational symmetry $f(p,p')$ can only be a function of $\hat{p}\cdot\hat{p}'$,

spin-rotational symmetry: $\delta_{\alpha\beta}\delta_{\gamma\delta}$; $\vec{\sigma}_{\alpha\beta}\cdot\vec{\sigma}_{\gamma\delta}$

for particle p and p'

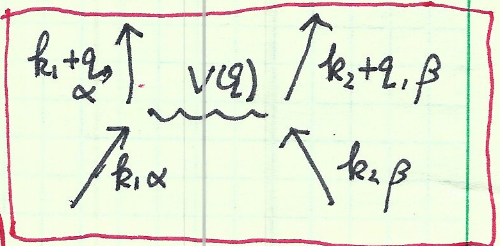
§ Calculation of Landau-interaction function at the tree level

Conside a spin-independent potential, in the 2nd quantization form, we have:

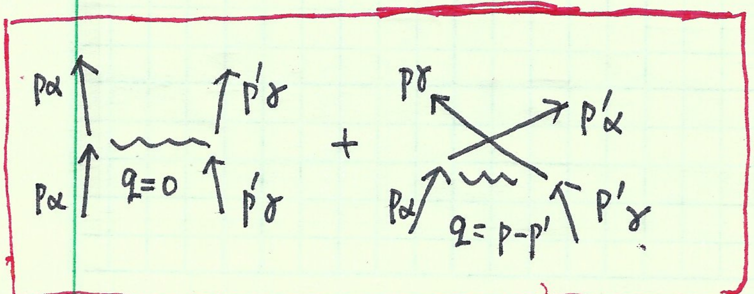
The interaction $H_{int} = \frac{1}{2}\int dr\, dr'\, \psi^{\dagger}_{\alpha}(r)\, \psi^{\dagger}_{\beta}(r')\, V(r-r')\, \psi_{\beta}(r')\, \psi_{\alpha}(r)$

$\xrightarrow{\text{Fourier transform}}$ $= \frac{1}{2V} \sum_{\substack{\alpha\beta \\ \widehat{V(q)}}} C^{\dagger}_{\alpha}(k_1+q)\, C^{\dagger}_{\beta}(k_2-q)\, C_{\beta}(k_2)\, C_{\alpha}(k_1)$

where $V(q) = \int dr\, e^{-iqr}\, V(r)$ → interaction vertex



Fermi liquid interaction function corresponds to forward-scattering, i.e $q \to 0$. However, due to indistinguishble processes, we have



$$f_{\alpha\beta,\gamma\delta}(\vec{p},\vec{p}') = V(q=0)\,\delta_{\alpha\beta}\,\delta_{\gamma\delta} - V(\vec{p}-\vec{p}')\,\delta_{\alpha\delta}\,\delta_{\beta\gamma}$$

using the identity

$$\frac{1}{2}\left[\vec{\sigma}_{\alpha\beta}\cdot\vec{\sigma}_{\gamma\delta} + \delta_{\alpha\beta}\delta_{\gamma\delta}\right] = \delta_{\alpha\delta}\delta_{\beta\gamma}$$

Ex: please prove it

Hint: express $\delta_{\alpha\delta}\delta_{\beta\gamma} = f_{\gamma\delta}\delta_{\alpha\beta} + \vec{g}_{\gamma\delta}\cdot\vec{\sigma}_{\alpha\beta}$ via trace

we have at the tree level

$$f_{\alpha\beta,\gamma\delta}(\vec{p},\vec{p}\,') = \left[V(0) - \frac{1}{2}V(p-p')\right]\delta_{\alpha\beta}\delta_{\gamma\delta} - \frac{1}{2}V(p-p')\,\vec{\sigma}_{\alpha\beta}\cdot\vec{\sigma}_{\gamma\delta}$$

Generally speaking, for system with spin conservation, the Landau interaction function can be represented as

SU(2)

$$\boxed{f_{\alpha\beta,\gamma\delta}(\vec{p},\vec{p}\,') = f^s(p,p')\,\delta_{\alpha\beta}\delta_{\gamma\delta} + f^a(p,p')\,\vec{\sigma}_{\alpha\beta}\cdot\vec{\sigma}_{\gamma\delta}.}$$

f^s and f^a describes the forward scattering amplitude which marks the fixed points of Fermi liquid in the RG language. f^s is in the density channel interaction, while f^a is in the spin channel.

Fermi liquid theory (II) Renormalization to physical quantities, and Pomeranchuk instability

§ Fermi liquid corrections to physical ~~others~~ quantities.

dimensionless Landau interaction function

$$f_{s,a}(\cos\theta) = \sum_l f_{l;\,s,a}\, P_l(\cos\theta)$$

$$F_{s,a} = N_0 f_{l;s,a} \quad ; \quad N_0 \text{ density of state}$$

The interaction effects are summarized in the two sets of Landau parameters.

☆ S-wave channel : molecular method

Spin-susceptibilies :

$$f_0^a\, \sigma\sigma' = N^{-1}(0)\, F_0^a\, \sigma\sigma'$$

$$\delta\mathcal{E}^{(2)} = \frac{1}{2} N^{-1}(0)\, F_0^a \sum_{pp'\sigma\sigma'} \sigma\sigma'\,\delta n_{p\sigma}\,\delta n_{p\sigma'} = \frac{1}{2} N^{-1}(0)\, F_0^a\, (S_z)^2$$

define molecule field $\quad E = -\int \vec{h}_{mol}\cdot d\vec{S}$

$$\Rightarrow h_{mol}(S) = -\frac{\delta\mathcal{E}}{\delta S_z} = -N^{-1}(0)\, S_z\, F_0^a$$

$$h_{tot} = h_{ex} + h_{mol} = h_{ex} - N^{-1}(0)\, S_z\, F_0^a$$

$$S_z = \chi_0\, h_{tot} = \chi_0\, h_{ext} - \chi_0 N^{-1}(0)\, S_z\, F_0^a$$

$$S_z(1 + \chi_0 F_0^a N^{-1}(0)) = \chi_0\, h_{ext} \quad \Rightarrow \quad \boxed{\chi = \frac{\chi_0}{1 + \chi_0 F_0^a (N(0))^{-1}}}$$

Compressibility $\quad f_0^s \cancel{\ }' = N^{-1}(0)\, F_0^s$

$$\delta \mathcal{E}^{(2)} = \frac{N^{-1}(0)}{2} F_0^s \sum_p \delta n_p \delta n_{p'} = \frac{1}{2} (N(0))^{-1} F_0^s (\delta n)^2$$

$$h_{mol} = - N^{-1}(0)\, F_0^s\, \delta n \quad \Rightarrow \quad \boxed{\frac{dn}{d\mu} = \frac{N(0)}{1+F_0^s}}$$

☆ p-wave channel : effective mass.

define $\quad n(r,t) = \sum_\sigma \int \frac{d^3p}{(2\pi)^3}\, n_{p,\sigma}(r,t) \qquad$ allow a slow spatial variation.

$$\vec{j}(r,t) = \sum_\sigma \int \frac{d^3p}{(2\pi)^3}\, \vec{\nabla}_p \mathcal{E}_{p\sigma}(r,t)\, n_{p\sigma}(r,t)$$

linearizing the expression of $\vec{j}(r,t)$, by using

$$\mathcal{E}_{p\sigma}(r,t) = \mathcal{E}_p^0 + \int \frac{d^3p'}{(2\pi)^3}\, f^s_{\sigma\sigma'}(p p')\, \delta n_{p'\sigma'}(r,t)$$

$$n_{p\sigma}(r,t) = n_p^0 + \delta n_{p,\sigma}(r,t)$$

$$\vec{j}(r,t) = \sum_\sigma \int \frac{d^3p}{(2\pi)^3}\, \nabla_p \mathcal{E}^0_{p\sigma}\, \delta n_{p\sigma}(r t) + \nabla_p \delta \mathcal{E}_{p\sigma}(r t) . n_p^0$$

↙ partial derivative

$$= \sum_\sigma \int \frac{d^3p}{(2\pi)^3}\, \nabla_p \mathcal{E}_p^0\, \delta n_{p\sigma}(r,t) - \nabla_p n_p^0\, \delta \mathcal{E}_{p\sigma}(r,t)$$

$$= \sum_\sigma \int \frac{d^3p}{(2\pi)^3}\, v_p \left[\delta n_{p\sigma}(r,t) - \frac{\partial n^0_{p\sigma}}{\partial \mathcal{E}_p} \int \frac{d^3p'}{(2\pi)^3}\, f_{\sigma\sigma'}(p p')\, \delta n_{p'\sigma}(r t) \right]$$

$$= \int \frac{d^3p}{(2\pi)^3}\, v_p\, \delta n_p(r,t) + \int \frac{d^3p}{(2\pi)^3}\, v_p \left(-\frac{\partial n^0_{p\sigma}}{\partial \mathcal{E}_p}\right) \int \frac{d^3p'}{(2\pi)^3}\, f^s(p p')\, \delta n_{p'}(r t)$$

$$\int \frac{d^3p}{(2\pi)^3} \vec{v}_p \left(-\frac{\partial n^0_{p\sigma}}{\partial \varepsilon_p}\right) f^s(p,p') = N(0) \int \frac{d\Omega}{4\pi} \sum_\ell f^s_\ell P_\ell(\cos\theta)\, v_F \cos\theta\, \hat{z}$$

↳ set p' along z-axis in the p-space

$$= \frac{N(0)}{3} f_1^s \vec{v}_F \hat{z},$$

other two direction average to zero

$$\Rightarrow \int \frac{d^3p}{(2\pi)^3} \vec{v}_p \left(-\frac{\partial n^0_{p\sigma}}{\partial \varepsilon_p}\right) f^s(\vec{p},\vec{p}') = \frac{N(0)}{3} f_1^s \vec{v}_{p'} = \frac{F_1^s}{3} \vec{v}_{p'}$$

$$\vec{j}(r,t) = \int \frac{d^3p}{(2\pi)^3} \vec{v}_p\, \delta n_p(r,t) + \frac{F_1^s}{3} \int \frac{d^3p'}{(2\pi)^3} \vec{v}_{p'}\, \delta n_{p'}(r,t)$$

$$= \int \frac{d^3p}{(2\pi)^3} \vec{v}_p \left(1 + \frac{F_1^s}{3}\right) \delta n_p(r,t) = \int \frac{d^3p}{(2\pi)^3} \frac{\vec{p}}{m^*} \left(1 + \frac{F_1^s}{3}\right) \delta n_p(r,t)$$

on other hand, by adiabatic continuity

$$\vec{j}(r,t) = \int \frac{d^3p}{(2\pi)^3} \frac{\vec{p}}{m} \delta n_p(r,t) \Rightarrow \boxed{\frac{1}{m} = \frac{1}{m^*}\left(1 + \frac{F_1^s}{3}\right)}$$

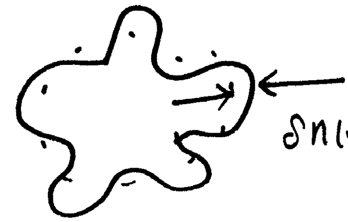
Similarly, we can derive spin current

$$j_i^\mu = 2 \int \frac{d^3p}{(2\pi)^3} \left(1 + \frac{F_1^a}{3}\right) \frac{p_i}{m^*} \sigma_p^\mu(r,t)$$

we can define spin-effective mass $\frac{1}{m_s^*} = \frac{1}{m^*}\left(1 + \frac{F_1^a}{3}\right)$

$$\boxed{\frac{m_s^*}{m} = \frac{1 + \frac{1}{3} F_1^s}{1 + \frac{1}{3} F_1^a}}$$

§ For general channels $F_l^{a,s}$



$$\delta n = V\int \frac{p^2 dp}{(2\pi)^3}\int d\Omega_p\, \delta n(p,\Omega_p) = V\int d\Omega\, \delta n(\hat{\Omega}_p)$$

where $\delta n(\hat{\Omega})$ is defined as $\int \frac{p^2 dp}{(2\pi)^3}\delta n(p,\Omega_p)$, i.e. integrate over radius direction.

we expand the angular distribution in terms of harmonic oscillators

$$\delta n(\hat{\Omega}_p) = \sum_{lm} \delta n_{lm} Y_{lm}(\hat{\Omega}_p)$$

$$E^{(2)} = \frac{1}{2V}\sum_{pp'} f_{\sigma\sigma'}(\hat{p}\,\hat{p}')\,\delta n_{p\sigma}\,\delta n_{p'\sigma'} = \frac{V}{2}\int d\Omega_p\, d\Omega_{p'}\, f_{\sigma\sigma'}(p\,p')\,\delta n_\sigma(\Omega_p)\,\delta n_{\sigma'}(\Omega_p')$$

$$= \frac{V}{2}N^{-1}(0)\int d\Omega_p\, d\Omega_{p'} \underline{\sum_{lm} F_l^s \frac{4\pi}{2l+1} Y^*_{lm}(\Omega_p)\, Y_{lm}(\Omega_{p'})} \leftarrow \text{addition theorem}$$

$$\Big[\Big(\sum_{l_1m_1} Y_{l_1m_1}(\Omega_p)\,\delta n^s_{l_1m_1}\Big)\Big(\sum_{l_2m_2} Y_{l_2m_2}(\Omega_p')\,\delta n^s_{l_2m_2}\Big) + (s\to a)\Big]$$

where $F_{\sigma\sigma'} = F^s + F^a\sigma\sigma'$, $\delta n_{s,a} = \delta n_\uparrow \pm \delta n_\downarrow$

$$E^{(2)} = \frac{V}{2}N^{-1}(0)\Big[\sum_{lm} F_l^s \frac{4\pi}{2l+1}\,\delta n^{*(s)}_{lm}\,\delta n^{(s)}_{lm} + (s\to a)\Big]$$

The kinetic energy increase

$$\delta E^{(1)} = \sum \varepsilon_p\,\delta n_p = V\int d\Omega\int \frac{p^2 dp}{(2\pi)^3}\,\varepsilon_p\,\delta n(p,\Omega_p)$$

$$\int \frac{p^2 dp}{(2\pi)^3} \, \varepsilon_p \, \delta n(p, \hat{\Omega}_p) = \frac{P_F^2}{(2\pi)^3} v_F \cdot \frac{1}{2} (\delta P_F)^2 \leftarrow \quad \begin{array}{l} \varepsilon_p = v_F \cdot \delta p \\ p^2 \to P_F^2 \end{array}$$

Compare with $\int \frac{p^2 dp}{(2\pi)^3} \delta n(p, \hat{\Omega}_p) = \frac{P_F^2}{(2\pi)^3} \delta P_F = \delta n(\Omega_p)$

$$\Rightarrow \int \frac{p^2 dp}{(2\pi)^3} \varepsilon_p \, \delta n(p, \hat{\Omega}_p) = \frac{v_F}{2} [\delta n(\Omega_p)]^2 \Big/ \frac{P_F^2}{(2\pi)^3} = 4\pi N^{-1}(0) [\delta n_p(\Omega_p)]^2$$

$$\delta E^{(1)} = V N^{-1}(0) \int d\Omega \, [\delta n(\Omega_p)]^2 = 2\pi V N^{-1}(0) \sum_{lm} |\delta n^s_{lm}|^2 + |\delta n^a_{lm}|^2$$

$$\Rightarrow \quad \Delta E = 2V N^{-1}(0) \sum_{lm} \left\{ \left(1 + \frac{F^s_l}{2l+1}\right) |\delta n^s_{lm}|^2 + (s \to a) \right\}$$

From thermodynamic properties. we know

$$\Delta E = \sum_{lm} \frac{1}{2\chi^s_l} |\delta n^s_{lm}|^2 + (s \to a)$$

$$\Rightarrow \quad \frac{1}{\chi_{l\,FL}^{s,a}} = \frac{1}{\chi^{s,a}_{l,0}} \left(1 + \frac{F^{s,a}_l}{2l+1}\right)$$

i.e.
$$\boxed{\chi^{s,a}_{FL,l} = \frac{\chi^{s,a}_{l,}}{1 + \frac{F^{s,a}}{2l+1}}}$$

in ^{3}He $\quad F^s_0 \simeq 10.8. \quad F^0_a \approx -0.75$

Compressibility is greatly reduced

spin-susceptibility is greatly enhanced!

③

§: effective mass renormalization

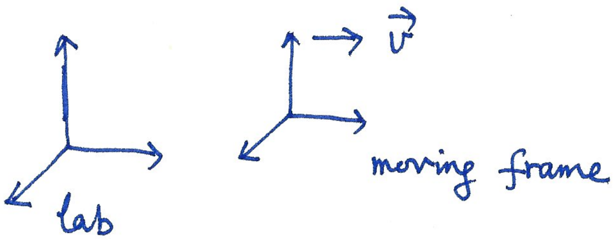
a moving frame with velocity $\vec{V}$

Consider that we do a Galilean transformation,

$$H = \sum_i \frac{P_i^2}{2m} \quad \rightarrow \quad \sum_i \frac{(\vec{P}_i + m\vec{V})^2}{2m}$$

where P_i is momentum in the moving frame

$P_i + m\vec{V}$ is the momentum in the lab frame.



P_i is cannonical momentum

$P_i + mV$ is mechanical momentum

In the lab frame, the current reads

$j(r,t) = \sum \delta(r - r_i)(\vec{P}_i + m\vec{V})$, which is zero because the system remains at rest.

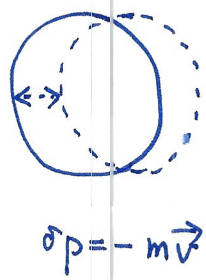
$$\langle j \rangle = \frac{1}{m}\langle p(t)\rangle + \vec{V} = 0 \Rightarrow \langle \vec{P}_i \rangle = -\vec{V}m$$

total mometum $\langle \vec{Q} \rangle = -N m \vec{V}$ in the moving frame.

Now let us calculate $\langle \vec{Q} \rangle$ in the moving frame by another method.

Let us conside $-\vec{v}$ as an external field, and $\vec{Q}$ as the response

$$\langle \vec{Q} \rangle = -\frac{\delta E}{\delta \vec{V}}$$



$\delta p = -m\vec{v}$

④

$$\langle\vec{Q}\rangle = \sum_{\sigma} p\,\delta n_{p\sigma} = V\cdot P_F \int d\Omega_p \int \frac{dp}{(2\pi)^3} P_F^2\,\delta n_{p\sigma} = V\cdot P_F \int d\Omega_p \cos\theta_p\,\delta n(\Omega_p)$$

$$= V\cdot P_F \sqrt{\frac{4\pi}{3}}\,\delta n_{10}^s$$

$$E(\vec{v}) = E(\vec{v}=0) + V P_F\left(\sqrt{\frac{4\pi}{3}}\,v\right)\delta n_{10}^s$$

$$\frac{E(\vec{v})}{V} = 2\pi N^{-1}(0)\left(1+\frac{F_1^s}{3}\right)(\delta n_{10}^s)^2 + P_F\left(\sqrt{\frac{4\pi}{3}}\,v\right)\delta n_{10}^s$$

$$\Rightarrow \langle\delta n_{10}^s\rangle = -\frac{N(0)\,P_F\left(\sqrt{\frac{4\pi}{3}}\right)}{4\pi\left(1+\frac{F_1^s}{3}\right)}\,v = -\frac{k_F^2\,P_F}{4\pi^3\hbar v_F}\,\frac{\sqrt{\frac{4\pi}{3}}}{1+\frac{F_1^s}{3}}\,v$$

$$\langle Q\rangle = -V\frac{P_F}{\hbar}\,\frac{k_F^2\,m^* v}{3\pi^2\left(1+\frac{F_1^s}{3}\right)} = -V\,\frac{k_F^3}{3\pi^2}\,\frac{m^* v}{1+\frac{F_1^s}{3}} = -\frac{N m^* v}{1+\frac{F_1^s}{3}}$$

$$\Rightarrow \quad m = \frac{m^*}{1+\frac{F_1^s}{3}} \qquad \text{i.e.}\quad \boxed{\frac{m^*}{m} = 1+\frac{F_1^s}{3}}$$

§ Pomeranchuk instability

Consider the Fermi surface as an elastic membrane in momentum space. The deformation of the Fermi surface not only changes the Kinetic energy, but also changes the interaction energy.

As we showed before,

$$\delta E \propto \left(1+\frac{F_\ell^{s,a}}{2\ell+1}\right)|\delta n_{\ell m}^{s,a}|^2 + O(\delta n^s_{\ell m})^4 + \cdots$$

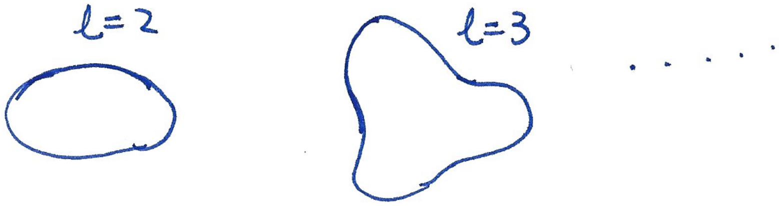
if $F_\ell^{s,a} < -(2\ell+1)$, then the Fermi surface will not be spheric stable, but develop distortions.

$F_0^s \to$ phase seperation : diveragence of compressibility

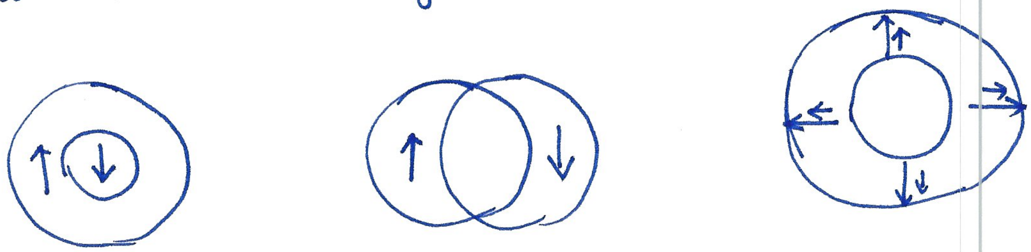
$F_0^a \to$ Ferromagnetism : divergence of spin-susceptibility

for $\ell \geq 1$, Fermi surface anisotropic distortions.

electronic liquid crystal phase



* unconventional magnetism



F_0^a F_1^a

p-wave magnetism

J. Hirsch PRB 41, 6820 (1990)
PRB 41 6828 (1990)
C. Wu et al PRL 93, 36403 (2004)
PRB 75, 115103 (2007)

Lect 7 Boltzmann transport (I)

§ Wigner distribution

we need to consider a spatially inhomogenous (slowly varing) system. We will talk about the particle with momentum $\vec{p}$ at point $\vec{r}$, and the distribution function $n_{\alpha\beta}(p;r,t)$ in the phase space. A more rigours definition is as follows

define $f(p;k;t) = \langle C^\dagger_{p+k/2} C_{p-k/2} \rangle(t)$, and perform the Fourier transform over small variable k, and arrive at

$$n_{\alpha\beta}(p;r,t) = \sum_k f(p;k;t)\, e^{ikr}$$

another way to express $n_{\alpha\beta}(p;r,t) = \int dr' e^{ip\cdot r'} \langle \psi^\dagger_\alpha(r+\frac{r'}{2})\, \psi_\beta(r-\frac{r'}{2}) \rangle$

where "r" is the center of mass coordinate, r' is the ralative coordinate.

$n_{\alpha\beta}(p;r,t)$ is a semiclassical distribution, The resolution of Δp and Δr need to satisfy $\Delta p \cdot \Delta r \geq \hbar/2$.

§ Boltzmann equation:

Let us study the equation of motion of $n(p;r;t)$. It has three

Contributions: ① Flow in the momentum space

② Flow in the real space

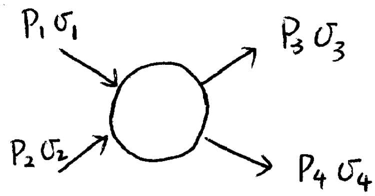
③ Collisions

$$\frac{\partial n(p;r;t)}{\partial t} + \nabla_r \left[\underset{\uparrow\, dr/dt}{v_p(r,t)}\, n(p;r,t) \right] + \nabla_p \left[\underset{\uparrow\, dp/dt}{f_p(r,t)}\, n(p;r,t) \right] = I_{collision}$$

where $v_p(r,t) = \nabla_p \mathcal{E}_p(r,t)$, $f_p(r,t) = -\nabla_r \mathcal{E}_p(r,t)$

$$\Rightarrow \boxed{\frac{\partial}{\partial t} n(p;r;t) + \nabla_p \mathcal{E}(p,r,t)\, \nabla_r n(prt) - \nabla_r \mathcal{E}(prt)\, \nabla_p n(prt) = I_{collision}}$$

Collision integral



$$\frac{2\pi}{\hbar} |\langle 34|t|12\rangle|^2 \, \delta(\varepsilon_1+\varepsilon_2-\varepsilon_3-\varepsilon_4) \, n_1 n_2 (1-n_3)(1-n_4)$$

$$\frac{2\pi}{\hbar} |\langle 12|t|34\rangle|^2 \, \delta(\varepsilon_1+\varepsilon_2-\varepsilon_3-\varepsilon_4) \, n_3 n_4 (1-n_1)(1-n_2)$$

define $\frac{2\pi}{\hbar} |\langle 34|t|12\rangle|^2 = \frac{1}{V^2} W(12;34) \, \delta_{p_1+p_2,\, p_3+p_4} \, \delta_{\sigma_1+\sigma_2 = \sigma_3+\sigma_4}$

$$\Rightarrow I_{coll}[n_p] = \frac{1}{V^2} \sum_{p_2\sigma_2} \sum_{p_3\sigma_3} \sum_{p_4\sigma_4} W(12;34) \, \delta_{p_1+p_2,\, p_3+p_4} \, \delta_{\sigma_1+\sigma_2,\, \sigma_3+\sigma_4}$$

$$\delta(\varepsilon_1+\varepsilon_2-\varepsilon_3-\varepsilon_4) \left[n_3 n_4 (1-n_1)(1-n_2) - n_1 n_2 (1-n_3)(1-n_4) \right]$$

In most situations, we will use a relaxation time approximation

$$[I(n_p)]_{RT} = -\frac{\delta n}{\tau}$$

In the case of $\omega\tau \gg 1$, we can neglect the collision integral.

$\boxed{\text{Warm up.}}$ — the Boltzman Eq for the density channel, for ordinary

- FL. — only density mode

$$\frac{\partial}{\partial t} n(rpt) + \nabla_p \mathcal{E}(rpt)\nabla_r n(rpt) - \nabla_r \mathcal{E}(rpt)\nabla_p n(rpt) = I_{coll}$$

linearing the equation

$$\mathcal{E}(rpt) = \mathcal{E}_0(p) + \frac{1}{V}\sum_{p'} f^s(p,p')\,\delta n_{p'}(r,t)$$

$$n(rpt) = n_0(p) + \delta n(r,p,t)$$

$\nabla_r n(r,p,t)$ and $\nabla_r \mathcal{E}(r,p,t)$ are already at the order of δn

thus we keep $\nabla_p \mathcal{E}(r,pt) = \nabla_p \mathcal{E}_0(p)$ and $\nabla_p n(rpt) = \nabla_p n_0(p)$

at the zeroth order.

- $$\Rightarrow \frac{\partial}{\partial t}\delta n(r,p,t) + \vec{v}_p\cdot\nabla_r\,\delta n(r,p,t) - \nabla_p n_0(p)\int\frac{d\vec{p}'}{(2\pi)^3} f^s_{\vec{p}\vec{p}'}\,\nabla_r\,\delta n_{p'}(r,t) = 0$$

$$\frac{\partial}{\partial t}\delta n(r,p,t) + \vec{v}_p\cdot\nabla_r\left[\delta n(r,p,t) - \frac{\partial n_0(p)}{\partial \mathcal{E}}\int\frac{d\vec{p}'}{(2\pi)^3} f^s_{\vec{p}\vec{p}'}\,\delta n(r,p',t)\right] = 0$$

do Fourier transform

$$\delta n(rpt) = \sum_q \delta n(\vec{p})\, e^{i(qr-\omega t)}$$ $\leftarrow$ $\boxed{\text{Collision neglected at } \omega\tau \gg 1}$

$$\Rightarrow (-i\omega + i\vec{v}_p\cdot\vec{q})\,\delta n(\vec{p}) - \frac{\partial n_0(p)}{\partial \mathcal{E}}\,\vec{v}_p\cdot i\vec{q}\left(\int\frac{d\vec{p}'}{(2\pi)^3} f^s_{\vec{p}\vec{p}'}\,\delta n(p'\ldots)\right] = 0$$

- $$(\omega - v_f q\cos\theta_p)\,\delta n(\vec{p}) + \frac{\partial n_0(p)}{\partial \mathcal{E}}\, v_f q\cos\theta_p\left[\int\frac{d\vec{p}'}{(2\pi)^3} f^s(\vec{p}\vec{p}')\,\delta n(p')\right] = 0$$

$$(s-\cos\theta_p)\,\delta n(\vec p) - \left(-\frac{\partial n_0(p)}{\partial \varepsilon_p}\right)\cos\theta_p \int \frac{d^3p'}{(2\pi)^3}\, f^s(\vec p,\vec p')\,\delta n(\vec p') = 0$$

define $\delta n(\hat p) = \int \frac{p^2\,dp}{(2\pi)^3}\,\delta n(\vec p)$ — integrate over radial

$$\Rightarrow (s-\cos\theta_p)\overbrace{\int \frac{p^2\,dp}{(2\pi)^3}}\delta n(\vec p) - \cos\theta_p \overbrace{\int\frac{p^2\,dp}{(2\pi)^3}}\int \frac{p'^2\,dp'\,d\Omega_{p'}}{(2\pi)^3}\left(-\frac{\partial n_0(p)}{\partial \varepsilon_p}\right) f^s(\hat p,\hat p')\,\delta n(\vec p') = 0$$

$$(s-\cos\theta_p)\,\delta n(\hat p) - \cos\theta_p \underbrace{\int \frac{p^2\,dp}{(2\pi)^3}\left(-\frac{\partial n_0(p)}{\partial \varepsilon_p}\right)}\int d\Omega_{p'}\, f^s(\hat p,\hat p')\,\delta n(\hat p') = 0$$

$$\int \frac{p^2\,dp}{(2\pi)^3}\,\delta(\varepsilon_p-\varepsilon_f) = \frac{N_0}{4\pi}$$ ← single component DOS

$$\Rightarrow \boxed{(s-\cos\theta_p)\,\delta n(\hat p) - N_0\cos\theta_p\int\frac{d\Omega_{p'}}{4\pi}\, f^s(\hat p,\hat p')\,\delta n(\hat p') = 0}$$

next step is another Fourier ~~transform~~

$$\delta n(\hat p) = \sum_l Y_{l0}(\hat p)\,u_l$$ (set the direction q as $\hat z$-axis)

$$\sum_l (s-\cos\theta_p)\,Y_{l0}(\hat p)\,u_l - N_0\cos\theta_p\int\frac{d\Omega_{p'}}{4\pi}\sum_{l'm'} f^s_{l'}\,\frac{4\pi}{2l'+1}\,Y_{l'm'}(\hat p)\,Y^*_{l'm'}(\hat p')\sum_l Y_{l0}(\hat p')\,u_l = 0$$

$$\boxed{\sum_l Y_{l0}(\hat p)\,u_l - \frac{\cos\theta_p}{s-\cos\theta_p}\sum_l \frac{F^s_l}{2l+1}\,Y_{l0}(\hat p)\,u_l = 0}$$

$$\int d\Omega_p\, Y^*_{l'm'}(\hat p) \sum_l Y_{l0}(\hat p)\, u_l - \sum_{\substack{ll'\\m'}} \int d\Omega_p \frac{\cos\theta_p}{s-\cos\theta_p} Y^*_{l'm'}(\hat p) \frac{F_l^s}{2l+1} Y_{l0}(\hat p)\, u_l = 0$$

exchange $l \leftrightarrow l'$

$$\Rightarrow u_l - \sum_{l'} \int d\Omega_p\, Y^*_{l0}(\hat p) \frac{F^s_{l'}}{2l'+1} Y_{l'0}(\hat p) \frac{\cos\theta_p}{s-\cos\theta_p} u_{l'} = 0$$

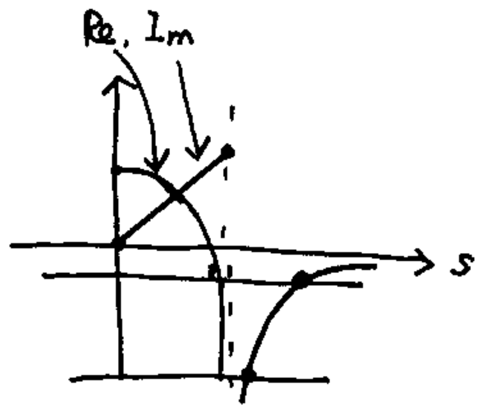
$$\boxed{\begin{aligned} &\frac{u_l}{\sqrt{2l+1}} + \sum_{l'} F^s_{l'}\, \Omega_{ll'}(s) \frac{u_{l'}}{\sqrt{2l'+1}} = 0 \\ &\text{where} \quad \Omega_{ll'}(s) = \frac{-1}{\sqrt{(2l'+1)(2l+1)}} \int d\Omega_p\, Y^*_{l0}(\hat p)\, Y_{l'0}(\hat p) \frac{\cos\theta_p}{s-\cos\theta_p} \end{aligned}}$$

① truncate at $l=0$ $\Rightarrow$

$$u_0 + F_0^s\, \Omega_{00}(s)\, u_0 = 0 \Rightarrow \frac{-1}{F_0^s} = \Omega_{00}(s)$$

$$\Omega_{00}(s) = \int \frac{d\Omega_p}{4\pi} \frac{-\cos\theta_p}{s-\cos\theta_p} = \int \frac{d\Omega_p}{4\pi}\left[\frac{-s}{s-\cos\theta_p+i\eta}+1\right] = +\left[1+\frac{s}{2}\ln\left|\frac{s-1}{s+1}\right|\right] + i\frac{\pi}{2} s\, \Theta(1-s)$$

$$\Rightarrow \boxed{\frac{1}{-F_0^s} = 1+\frac{s}{2}\ln\left|\frac{s-1}{s+1}\right|}$$



for $F_0^s \to 0^+$, $s \to 1^+$

$$1+\frac{1}{2}\ln\frac{s-1}{2} = -\frac{1}{F_0^s} \Rightarrow s \approx 1+2e^{\frac{-2}{F_0^s}}$$

for $F_0^s \gg 1$ $\Rightarrow$ $1+\frac{s}{2}\ln\left|\frac{s-1}{s+1}\right| \sim -\frac{1}{3s^2} \Rightarrow s = \sqrt{\frac{F_0}{3}}$ (s is the sound velocity)

§ Spin-related transport

define $n_p(r,t) = \frac{1}{2}\, tr[n_{p,\alpha\beta}]$, $\vec{\sigma}_p^{(r,t)} = \frac{1}{2}\, tr[n_{p,\alpha\beta}\, \vec{\tau}_{\beta\alpha}]$

$$\Rightarrow \boxed{n(p;rt) = n_p(r,t)\,\delta_{\alpha\alpha'} + \vec{\sigma}_p(r,t)\cdot\vec{\tau}_{\alpha\alpha'}}$$ (decompose into density and spin)

the quasi-particle energy can be written as

$$\boxed{\varepsilon(prt) = \varepsilon_p(r,t)\,\delta_{\alpha\alpha'} + \vec{h}_p(r,t)\cdot\vec{\tau}_{\alpha\alpha'}}.$$

Plug in the Boltzman Eq

$$\frac{\partial n(rpt)}{\partial t} + \underline{\frac{\partial}{\partial r}\left[\frac{\partial \varepsilon}{\partial p}\cdot n(rpt)\right] + \frac{\partial}{\partial p}\left[-\frac{\partial \varepsilon}{\partial r}\cdot n(r,pt)\right]}$$

Need symmetrizization

$$- \underbrace{\frac{1}{i\hbar}\left[n(r,pt),\ \varepsilon(r,p,t)\right]}_{\text{Larmor procession}} = I_{collision}$$

Separate variables

$$\boxed{\begin{aligned} &\frac{\partial n_p(rt)}{\partial t} + \frac{\partial}{\partial r_i}\left[\frac{\partial \varepsilon_p}{\partial p_i} n_p + \frac{\partial \vec{h}_p}{\partial p_i}\cdot\vec{\sigma}_p\right] + \frac{\partial}{\partial p_i}\left[-\frac{\partial \varepsilon}{\partial r_i} n_p - \frac{\partial \vec{h}_p}{\partial r_i}\cdot\vec{\sigma}_p\right] = I_{coll} \\ &\frac{\partial \vec{\sigma}_p(rt)}{\partial t} + \frac{\partial}{\partial r_i}\left[\frac{\partial \varepsilon_p}{\partial p_i}\vec{\sigma}_p + \frac{\partial \vec{h}_p}{\partial p_i} n_p\right] + \frac{\partial}{\partial p_i}\left[-\frac{\partial \varepsilon}{\partial r_i}\vec{\sigma}_p - \frac{\partial \vec{h}_p}{\partial r_i}\cdot n_p\right] = \frac{2}{\hbar}\vec{h}_p\times\vec{\sigma}_p + I_{coll}. \end{aligned}}$$

Lamor precession: in an external magnet field $\mathcal{H}$, it couples to electron spin as $-\frac{\gamma}{2}\mathcal{H}\cdot\sigma \Rightarrow \frac{\partial \vec{\sigma}_p}{\partial t} = \vec{\sigma}_p \times \gamma \mathcal{H} \Rightarrow \omega_0 = \gamma \mathcal{H}$

In the Fermi liquid, $h_p = -\gamma\frac{\hbar}{2}\mathcal{H} + 2\int\frac{d^3p'}{(2\pi)^3} f^a(p,p')\,\sigma_{p'}$

↳ contribution from interaction

in the uniform system, we have

$$\frac{\partial \vec{\sigma}_p}{\partial t} = \gamma\,\vec{\sigma}_p \times \vec{\mathcal{H}} - \frac{4}{\hbar}\int\frac{d^3p'}{(2\pi)^3}\,(\vec{\sigma}_p \times \vec{\sigma}_{p'})$$

define $\delta\,\sigma(\hat{n}) = 2\int\frac{dp\cdot p^2}{(2\pi)^3}\sigma(p,\hat{n})$, (we integrate out radius direction).

$$\Rightarrow 2\int\frac{p^2dp}{(2\pi)^3}\frac{\partial \vec{\sigma}(p,\hat{n})}{\partial t} = 2\int\frac{p^2dp}{(2\pi)^3}\,\vec{\sigma}(p,\hat{n}) \times (\gamma\vec{\mathcal{H}}) - 8\int\frac{d\Omega'}{4\pi}\int\frac{p'^2dp'}{(2\pi)^3}\int\frac{p^2dp}{(2\pi)^3}\,\underbrace{\quad}_{f^a(pp')}\sigma_p \times \sigma_{p'}$$

$$\frac{\partial}{\partial t}\sigma(\hat{\Omega}_p) = \gamma\,\sigma(\hat{\Omega}_p)\times\vec{\mathcal{H}} - \frac{2}{\hbar}\int\frac{d\Omega'_p}{4\pi}\,f^a(p,p')\,\hat{\sigma}(\Omega_p)\,\hat{\sigma}(\Omega_{p'})$$

In the external field $\sigma(\hat{\Omega}_p) = \sigma^0 + \delta\sigma(\hat{p}) \Rightarrow$

$$\frac{\partial}{\partial t}\delta\vec{\sigma}(\Omega_p) = \gamma\,\delta\vec{\sigma}(\Omega_p)\times\vec{\mathcal{H}} - \frac{2}{\hbar}\int\frac{d\Omega'_p}{4\pi}\,f^a(pp')\left[\delta\sigma(\hat{p})\times\sigma^0 + \sigma^0\times\delta\sigma(\hat{p}')\right]$$

define $\delta\sigma_+(\hat{\Omega}_p) = \delta\sigma_x(\hat{\Omega}_p) + i\,\delta\sigma_y(\hat{\Omega}_p)$

$$\frac{\partial}{\partial t}\delta\sigma_+(\Omega_p) = -i\left[\omega_0\,\delta\sigma_+(\Omega_p) - \frac{2\sigma^0}{\hbar}\int\frac{d\Omega'}{4\pi}\,f^a(p,p')\left(\delta\sigma_+(\Omega_p) - \delta\sigma_+(\Omega_{p'})\right)\right.$$

$$= -i\left[(\omega_0 - \frac{2}{\hbar}N^{-1}(0)\,F_0^a\,\sigma^0)\,\delta\sigma_+(\Omega_p) + \frac{2}{\hbar}\sigma^0\int\frac{d\Omega'}{4\pi}\,f^a(p,p')\,\delta\sigma_+(\Omega_{p'})\right]$$

expand $\delta\sigma_+(\Omega_p) = \sum_{lm}\delta\sigma_+(lm)\,Y_{lm}(\Omega_p)$

$$\int \frac{d\Omega'}{4\pi}\, f^a(p,p')\, \delta\sigma_+(\Omega_{p'}) = N^{-1}(0) \int \frac{d\Omega_{p'}}{4\pi}\, F_l^a \frac{4\pi}{2l+1}\, Y_{lm}(\Omega_p)\, Y^*_{lm}(\Omega_{p'}) \sum_{l'm'} Y_{l'm'}(\Omega_{p'})\, \delta\sigma_+(l'm')$$

$$= N^{-1}(0) \sum_{lm} \frac{F_l^a}{2l+1}\, Y_{lm}(\Omega_p)\, \delta\sigma_+(lm)$$

$$\Rightarrow \quad \frac{\partial}{\partial t}\delta\sigma_+(lm) = -i\left\{\omega_0 - \frac{2}{\hbar} N^{-1}(0)\, F_0^a\, \sigma^0 , \quad + N^{-1}(0)\frac{F_l^a}{2l+1}\right\}\delta\sigma_+(lm)$$

$$\Rightarrow \quad \omega_{l+} = \left\{\omega_0 - \frac{2}{\hbar}\sigma_0 N^{-1}(0)\left[F_0^a - \frac{F_l^a}{2l+1}\right]\right\}$$

$$\sigma^0 = \frac{\gamma\hbar}{2}\,\frac{N(0)}{1+F_0^a}\,\mathcal{H}, \qquad \omega_0 = \gamma\mathcal{H}$$

$$\Rightarrow \quad \boxed{\frac{\omega_{l+}}{\omega_0} = \frac{1+F_l^a/2l+1}{1+F_0^a}}$$

the $l=0$ channel Larmor frequency is not modified by interaction because Spin is conserved by interaction!

~~§ spin hydrodynamic equations~~

Integrate over momentum for the Boltzman transport equation $\Rightarrow$

$$\frac{\partial}{\partial t}\vec{\sigma}(r,t) + \frac{\partial}{\partial r_i}\vec{j}_i(\vec{r},t) = -\frac{2}{\hbar}\int\frac{d\vec{p}}{(2\pi)^3}\,\vec{\sigma}_p \times \Big[-\frac{\gamma}{2}\mathcal{H}^{(r,t)} + \int\frac{d\vec{p}\,'}{(2\pi)^3} f^a(p,p')\,\vec{\sigma}_{p'}\Big)$$

$$\boxed{\frac{\partial}{\partial t}\vec{\sigma}(r,t) + \frac{\partial}{\partial r_i}\vec{j}_i(\vec{r},t) = \gamma\,\vec{\sigma}(r,t)\times\mathcal{H}(r,t)}$$

(inteaction part cancels).

where
$$\vec{\sigma}(r,t) = 2\int\frac{d^3p}{(2\pi)^3}\,\vec{\sigma}(r,p,t)$$

$$\vec{j}_i(r,t) = 2\int\frac{d^3p}{(2\pi)^3}\left[\frac{\partial\varepsilon_p}{\partial p_i}\,\vec{\sigma}(r\,p\,t) + \frac{\partial\vec{h}_p}{\partial p_i}\, n_p(r,p,t)\right]$$

$$\vec{j}_i(r,t) = 2\int \frac{d^3p}{(2\pi)^3} \left[\frac{\partial \varepsilon_p^0}{\partial p_i} \vec{\sigma}(r,p,t) - \frac{\partial n_p^0}{\partial p}(r,p,t)\, \vec{h}_p \right] \quad (\text{linearize}).$$

$$= 2\int \frac{d^3p}{(2\pi)^3}\, v_{p_i} \left(\vec{\sigma}_p - \frac{\partial n_p^0}{\partial \varepsilon_p} \vec{h}_p \right)$$

v_pi is odd function, after integration, this term goes to zero

$$= 2\int \frac{d^3p}{(2\pi)^3}\, v_{p_i} \left(\vec{\sigma}_p - \frac{\partial n_p^0}{\partial \varepsilon_p} \left(-\frac{\gamma}{2}\frac{\hbar}{2}\mathcal{H} + 2\int \frac{d^3p'}{(2\pi)^3} f^a(p,p')\, \vec{\sigma}_{p'} \right) \right]$$

$$= 2\int \frac{d^3p}{(2\pi)^3}\, v_{p_i} \vec{\sigma}_p \; - \; 4\int \frac{d^3p}{(2\pi)^3}\, v_{p_i} \frac{\partial n_p^0}{\partial \varepsilon_p} \int \frac{d^3p'}{(2\pi)^3} f^a(p,p')\, \vec{\sigma}_{p'}$$

$$2\int \frac{d^3p}{(2\pi)^3}\, v_{p_i} \frac{\partial n_p^0}{\partial \varepsilon_p} f^a(p,p') = -N(0) \int \frac{d\Omega}{4\pi} \sum_l f_l^a \; P_l(\cos\theta)\; v_F \cdot \cos\theta \quad (\text{set } p' \text{ along } z \text{ axis}).$$

$$= -\frac{F_1^a}{3} v_F$$

$$\Rightarrow \vec{j}_i(r,t) = 2\int \frac{d^3p}{(2\pi)^3}\, v_{p_i} \vec{\sigma}_p + 2\int \frac{d^3p'}{(2\pi)^3} \frac{F_1^a}{3} v_{p_i'} \vec{\sigma}_{p'}$$

$$\boxed{\vec{j}_i(r,t) = 2\int \frac{d^3p}{(2\pi)^3}\, v_{p_i} \vec{\sigma}_p \left(1 + \frac{F_1^a}{3}\right)}$$