Lect 21 Path integral for quantum mechanics

§1 propagator of a point particle.

Consider $H = \frac{p^2}{2m} + V(x)$, and thus the time evolution operator is

$\mathcal{U}(t_b, t_a) = e^{-i(t_b - t_a)H}$. Define the propagator

$$i\, G(x_b t_b ; x_a t_a) \equiv \langle x_b | \mathcal{U}(t_b, t_a) | x_a \rangle,$$ the it satisfies the following Schrödinger Eq as

$$i \partial_t G(xt; x_a t_a) = \langle x | i \partial_t e^{-i(t-t_a)H} | x_a \rangle = \langle x | H\, \mathcal{U}(t, t_a) | x_a \rangle.$$

H in the coordinate Rep is a function of x and ∂_x, thus

$$\langle x | H\, \mathcal{U}(t, t_a) | x_a \rangle = \int dx'\, \delta(x - x')\, H(x', \partial_{x'})\, \mathcal{U}(t, t_a)\, \delta(x' - x_a)$$

$$= H(x, \partial_x) \int dx'\, \delta(x - x')\, \mathcal{U}(t, t_a)\, \delta(x' - x_a) = H(x, \partial_x) \langle x | \mathcal{U}(t, t_a) | x_a \rangle$$

$$\Rightarrow \boxed{i \partial_t G(xt, x_a t_a) = H(x, \partial_x)\, G(xt, x_a t_a)}$$

Ex: for 1D free space, with the initial condition $G(x t_a; x_a t_a) = -i\delta(x - x_a)$, we have $G(x_b t; x_a t_a) = (-i) \left(\frac{m}{2\pi i t} \right)^{1/2} \exp\left[\frac{i\, m (x_b - x_a)^2}{2t} \right]$.

Hint: Solve the differential Eq. $i \partial_t G(xt, x_a t_a) = \frac{-\hbar^2 \partial_x^2}{2m} G(xt, x_a t_a)$.

§2 Path integral representations of the propagator

$$\mathcal{U}(t_b, t_a) = \mathcal{U}(t_b\, t)\, \mathcal{U}(t, t_a) \Rightarrow i\, G(x_b t_b; x_a t_a) = \int dx\; i G(x_b t_b; x t)\, i G(xt; x_a t_a)$$

let us divide the time interval $[t_b, t_a]$ into N equal segments

$$iG(x_b t_b; x_a t_a) = \int dx_1 \cdots dx_{N-1}\, iG(x_b t_b; x_{N-1} t_{N-1}) \cdots\quad iG(x_1 t_1; x_a t_a)$$

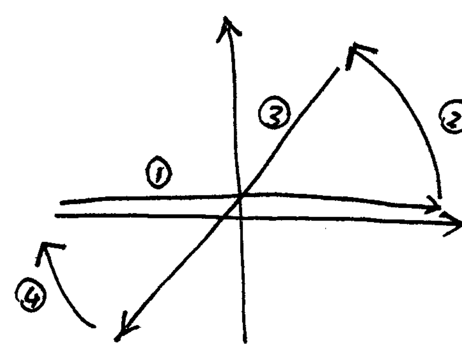
$$= A^N \int \prod_{i=1}^{N-1} dx_i \ \exp\left[i \sum \Delta t\, L\left(t_j, \frac{x_j + x_{j-1}}{2}, \frac{x_j - x_{j-1}}{\Delta t}\right)\right]$$

$$iG(x_i t_i, x_{i-1} t_{i-1}) = \langle x_i |\, e^{-i\frac{p^2}{2m}\Delta t}\ e^{-iV(x)\Delta t}\, | x_{i-1}\rangle$$

$$= \int dP_i\ \langle x_i |\, e^{-\frac{i p^2}{2m}\Delta t} | P_i\rangle \langle P_i |\, e^{-iV(x)\Delta t}\, | x_{i-1}\rangle$$

$$= \int_{-\infty}^{+\infty} \frac{dP_i}{2\pi}\ e^{-i\frac{p^2}{2m}\Delta t - iV(x)\Delta t}\ e^{+iP_i(x_i - x_{i-1})}$$

using Gauss integral $\displaystyle\int_{-\infty}^{+\infty} dx\ e^{-ax^2} = \sqrt{\frac{\pi}{a}}$



then what's the vaule of $\displaystyle\int_{-\infty}^{+\infty} dx\ e^{-iax^2} = ?$

For the countour $\underset{e^{-az^2}dz}{\oint} = \int_1 + \int_2 + \int_3 + \int_4 = 0$, the contribution from ② and ④ $\longrightarrow 0$

$$\int_{-\infty}^{+\infty} dx\, e^{-ax^2} = \int_{-\infty e^{i\pi/4}}^{+\infty e^{i\pi/4}} dz\ e^{-az^2} = \int_{-\infty}^{+\infty} dy\ e^{-ia\, y^2} \cdot e^{i\pi/4}$$

$$\Rightarrow \boxed{\int_{-\infty}^{+\infty} dy\, e^{-iay^2} = e^{-i\pi/4}\sqrt{\frac{\pi}{a}} = \sqrt{\frac{\pi}{ai}}}$$

$$\Rightarrow iG(x_i t_i, x_{i-1} t_{i-1}) = \left(\frac{m}{2\pi i \Delta t}\right)^{1/2} \exp\left[i\left(\frac{m}{2}\frac{(x_i - x_{i-1})^2}{(\Delta t)^2} - V\left(\frac{x_i + x_{i-1}}{2}\right)\right)\Delta t\right]$$

$$\Rightarrow \quad A = \left(\frac{m}{2\pi i \Delta t}\right)^{1/2} \quad \text{and} \quad \mathcal{L}(t, x, \dot{x}) = \frac{m}{2}\dot{x}^2 - V(x)$$

i.e.

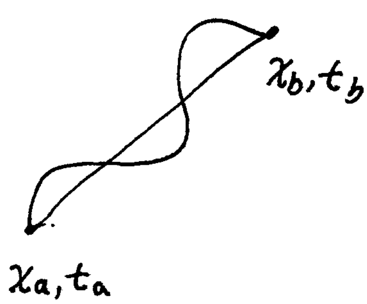
$$iG(x_b t_b, x_a t_a) = \left(\frac{m}{2\pi i \Delta t}\right)^{N/2} \int dx_1 \cdots dx_{N-1} \; e^{i\int_{t_a}^{t_b} dt \left[\frac{m}{2}\dot{x}^2 - V(x)\right]}$$

§ Evalution of the path integral:

We can first find the saddle point solution, which corresponds to the solution of the classic path, and then evaluate the fluctuation parts. Let us consider the free space propagater as an example.

The classic path

$$X_c(t) = X_a + \frac{X_b - X_a}{t_b - t_a}(t - t_a)$$



The action of this part $\int_{t_a}^{t_b} \mathcal{L}\, dt$

$$= \frac{m}{2}\dot{x}^2 (t_b - t_a) = \frac{m}{2}\frac{(X_b - X_a)^2}{t_b - t_a}$$

The fluctuations $\delta X_j = X_j - X_c(t_j)$, $\begin{cases} X_0 = X_a \\ X_N = X_b \end{cases}$ $\delta X_0 = \delta X_N = 0$

$$iG(x_b t_b; x_a t_a) = \left(\frac{m}{2\pi i \Delta t}\right)^{N/2} \int dx_1 \cdots dx_{N-1} \prod_{i=1}^{N} \exp\left[i\frac{m}{2}\frac{(X_i - X_{i-1})^2}{\Delta t}\right]$$

$$\frac{(X_i - X_{i-1})^2}{\Delta t} = \frac{(X_c(t_i) - X_c(t_{i-1}))^2}{\Delta t} + \frac{(\delta X_i - \delta X_{i-1})^2}{\Delta t} + 2(\delta X_i - \delta X_{i-1})\frac{X_b - X_a}{t_b - t_a}$$

Add together, the linear term of δX_i vanishes,

$$\Rightarrow iG(x_b t_b ; x_a t_a) = \left(\frac{m}{2\pi i \Delta t}\right)^{N/2} e^{iS_c} \int dx_1 \cdots dx_{N-1} \prod_{i=2}^{N-1} e^{i\frac{m}{2}\frac{(\delta x_i - \delta x_{i-1})^2}{\Delta t}} \cdot \underbrace{e^{i\frac{m}{2}\left[(\delta x_1)^2 + (\delta x_{N-1})^2\right]}}$$

$$\exp\left[i \sum_{j,k} \delta x_j \, M_{jk} \, \delta x_k \right]$$

with
$$M_{jk} = \frac{m}{2\Delta t} \begin{bmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & & \ddots & \\ & & -1 & 2 \end{bmatrix}$$

$$\int dx_1 \cdots dx_{N-1} \exp\Big[i \sum_{\substack{j,k \\ 1,\cdots N-1}} \delta x_j \, M_{jk} \, \delta x_k \Big] = \frac{(\sqrt{\pi})^{N-1}}{\sqrt{\mathrm{Det}(-iM)}} \quad \text{(gaussian fluctation)}$$

tricks to calculate determinaint of $M_N = \begin{pmatrix} 2\cosh(u) & -1 & & \\ -1 & 2\cosh u & -1 & \\ & & \ddots & \\ & & -1 & 2\cosh u \end{pmatrix}$

$$\begin{cases} \mathrm{Det}\, M_N = 2\cosh u \, \mathrm{Det}\, M_{N-1} - \mathrm{Det}\, M_{N-2} \\ \mathrm{Det}\, M_1 = 2\cosh u, \quad \mathrm{Det}\, M_2 = 4\cosh^2 u - 1 \end{cases}$$

can be solved by the ansatz $\mathrm{Det}\, M_N = a e^{nu} + b e^{-nu}$

$$\Rightarrow \mathrm{Det}\, M_N = \frac{\sinh (N+1) u}{\sinh u} \longrightarrow N+1 \text{ as } u \to 0$$

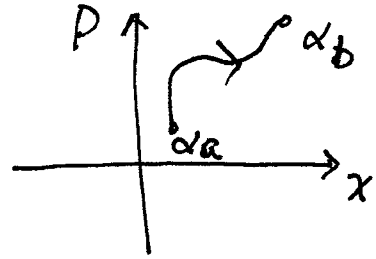
$$\Rightarrow \mathrm{Det}\,(-i M_{jk}) = \left(\frac{m}{2\Delta t i}\right)^{N-1} \times N$$

$$\Rightarrow iG(x_b t_b, x_a t_a) = \left(\frac{m}{2\pi i \Delta t}\right)^{N/2} \left(\frac{m}{2\pi i \Delta t}\right)^{-\left(\frac{N-1}{2}\right)} N^{-1/2} e^{iS_{cl}}$$

$$\boxed{iG(x_b t_b, x_a t_a) = \left(\frac{m}{2\pi i (t_b - t_a)}\right)^{1/2} e^{iS_{cl}}}$$

§ Coherent state path integral

Consider a harmonic oscilator $H = \omega a^\dagger a$, and we define the coherent state $|\alpha\rangle$ satisfying $a|\alpha\rangle = \alpha|\alpha\rangle$. The propogator in the coherent state Rep is even simpler.



$$iG(\alpha_b t_b ; \alpha_a t_a) = \langle \alpha_b | \mathcal{U}(t_b t_a) | \alpha_a \rangle$$

Resolution identity

$$\boxed{\int \frac{d\,\mathrm{Re}\,\alpha \; d\,\mathrm{Im}\,\alpha}{\pi} |\alpha\rangle\langle\alpha| = 1}$$

Plug in
$$\boxed{\begin{aligned} |\alpha\rangle &= e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \\ &= N_\alpha e^{\alpha a^\dagger} |0\rangle \end{aligned}}$$

Proof: $$\text{LHS} = \int \frac{d\,\mathrm{Re}\,\alpha \; d\,\mathrm{Im}\,\alpha}{\pi} e^{-|\alpha|^2} \sum_{nn'} \frac{\alpha^n \alpha^{*n'}}{\sqrt{n! n'!}} |n\rangle\langle n'|$$

$$= \int \frac{d\theta}{\pi} |\alpha| \, d|\alpha| \; e^{-|\alpha|^2} \sum_n \frac{|\alpha|^{2n}}{n!} |n\rangle\langle n|$$

For those $n \neq n'$, they vanish after $\int d\theta$

$$= \sum_n \underbrace{\int_0^\infty d|\alpha|^2 \, \frac{e^{-|\alpha|^2} (|\alpha^2|)^n}{n!}}_{\Gamma\text{-function}} |n\rangle\langle n| = \sum_n |n\rangle\langle n| = 1$$

Inner product

$$\langle \alpha | \alpha' \rangle = e^{-\frac{|\alpha|^2}{2}} e^{-\frac{|\alpha'|^2}{2}} \langle 0 | e^{\alpha^* \hat{a}} e^{\alpha' \hat{a}^\dagger} | 0 \rangle$$

$$e^{\alpha^* \hat{a}} e^{\alpha' \hat{a}^\dagger} = e^{\alpha' \hat{a}^\dagger} e^{\alpha^* \hat{a}} \; e^{\alpha^* \alpha' [\hat{a}, \hat{a}^\dagger]}$$

$$\Rightarrow \boxed{\langle \alpha | \alpha' \rangle = e^{-\frac{1}{2}(|\alpha|^2 + |\alpha'|^2) + \alpha^* \alpha'}}$$

$$iG(\alpha_b t_b ; \alpha_a t_a) = \int \frac{d^2a_1 \cdots d^2a_{N-1}}{\pi^{N-1}} \; iG(\alpha_b t_b , \alpha_{N-1} t_{N-1}) \cdots iG(\alpha_1 t_1 ; \alpha_a t_a)$$

$$iG(a_i t_i , a_{i-1} t_{i-1}) = \langle \alpha_i | \, e^{-i\Delta t \, \omega \hat{a}^{+} a} \, | \alpha_{i-1} \rangle = e^{-i\Delta t \, \alpha_i^* \alpha_{i-1} \omega} \langle \alpha_i | \alpha_{i-1} \rangle$$

$$= e^{-i\Delta t \, \omega \, \alpha_i^* \alpha_{i-1} - \frac{1}{2}|\alpha_i|^2 - \frac{1}{2}|\alpha_{i-1}|^2 + \alpha_i^* \alpha_{i-1}}$$

$$= e^{-i\Delta t \, \omega \, \alpha_i^* \alpha_{i-1} - \frac{1}{2}\alpha_i^*(\alpha_i - \alpha_{i-1}) + \frac{1}{2}\alpha_{i-1}(-\alpha_{i-1}^* + \alpha_i^*)}$$

$$= e^{i\Delta t \left[-\omega \, \alpha_i^* \alpha_i + \frac{i}{2}(\alpha_i^* \dot{\alpha}_i - \alpha_i \dot{\alpha}_i^*) \right]}$$

$$\Rightarrow iG(\alpha_b t_b ; \alpha_a t_a) = \int \frac{\prod_{i=1}^{N-1} d^2\alpha_i}{\pi^{N-1}} \; e^{i\int_{t_a}^{t_b} dt \left[\frac{i}{2}(\alpha^* \dot{\alpha} - \alpha \dot{\alpha}^*) - \omega \alpha^* \alpha \right]}$$

$$\longrightarrow \int D[\alpha(t)] \; e^{i\int_{t_a}^{t_b} dt \; \mathcal{L}}$$

where $\mathcal{L} = \frac{i}{2}(\alpha^* \dot{\alpha} - \alpha \dot{\alpha}^*) - \omega \alpha^* \alpha = p\dot{x} - H.$

§ Path integral Rep of partition function

$$Z(\beta) = \mathrm{tr}\, e^{-\beta H} = \int dx \; \mathcal{G}(x, x, \beta) \quad \text{where } \beta = \frac{1}{k_B T}$$

$$\mathcal{G}(x_b \, x_a ; \beta) \equiv \langle x_b | \, e^{-\beta H} \, | x_a \rangle$$

$$= \left(\frac{m}{2\pi \Delta\tau} \right)^{N/2} \int D x(\tau) \; e^{-\int_0^\beta d\tau \; \frac{m}{2}\left(\frac{dx}{d\tau}\right)^2 + V(x)}$$

$$\Rightarrow Z(\beta) = \int D x(\tau) \; e^{-\oint d\tau \left[\frac{m}{2}\left(\frac{dx}{d\tau}\right)^2 + V(x) \right]} \quad \text{for closed path } x(0) = x(\beta).$$

$$\mathcal{G}(x_b, x_a, \tau)\Big|_{\tau = it} = i\, G(x_b, x_a; t)$$

imaginary time path integral

path integral for dissipative system

Consider a QM system $H_p = \frac{p^2}{2m} + V(q)$. which furthur couples to an environment. The environment is described by a bounch of harmonic oscilators $H_B[q_\alpha] = \sum_{\alpha=1}^{N} \left(\frac{\hat{P}_\alpha^2}{2m_\alpha} + \frac{m_\alpha}{2} \omega_\alpha^2 q_\alpha^2 \right)$, and the coupling between them is linear to q_α, as $H_C = - \sum_{\alpha=1}^{N} f_\alpha(q) q_\alpha$.

Then consider the survival probability of a particle confined to a metastable minimum at $q = a$

$$\langle a | e^{-i\hat{H}t} | a \rangle = \int_{q(0)=q(t)=a} Dq \; e^{i S_p[q]} \int Dq_\alpha \, e^{i S_B[q_\alpha] + i S_C[q, q_\alpha]}$$

where $S_p[q] = \int_0^t dt \left(\frac{m}{2} \dot{q}^2 - V(q) \right)$, $S_B = \int_0^t dt \sum_\alpha \frac{m_\alpha}{2} \left(\dot{q}_\alpha^2 - \omega_\alpha^2 q_\alpha^2 \right)$

$$S_C[q] = - \int_0^t dt \sum_\alpha f_\alpha(q) q_\alpha$$

Now we change to Eucledian Rep, and integrate out environment degrees of freedom

$$\int Dq_\alpha \, e^{- \int_0^\beta d\tau \left[\left(\frac{m}{2} \frac{dq_\alpha}{d\tau} \right)^2 + \omega_\alpha^2 q_\alpha^2 + f_\alpha(q) q_\alpha \right]}$$

Peform Fourier transform $q(\tau) = \sum_{i\omega_n} q(i\omega_n) e^{-i\omega_n \tau}$

$$f_\alpha(q) = \sum_{i\omega_n} f_{\alpha,\omega_n}(q) \, e^{i\omega_n \tau}$$

(2)

Then, the above Eq $= \int Dq_\alpha \, e^{-\beta \sum_{\omega_n} \left[\frac{m}{2}(\omega_n^2+\omega_\alpha^2) q_\alpha(i\omega_n) q_\alpha(-i\omega_n) + f_{\alpha,-\omega_n}(q) q_\alpha(i\omega_n)\right]}$

$$= \int Dq_\alpha \exp\left[-\beta \sum_{i\omega_n} \frac{m(\omega_n^2+\omega_\alpha^2)}{2} \left[q_\alpha(i\omega_n) + \frac{f_{\alpha,\omega_n}}{m(\omega_n^2+\omega_\alpha^2)}\right]\left[q_\alpha(-i\omega_n) + \frac{f_{\alpha,-\omega_n}}{m(\omega_n^2+\omega_\alpha^2)}\right]\right]$$

$$\cdot \exp\left[\beta \sum_{\omega_n} \frac{f_{\alpha,\omega_n}(q) f_{\alpha,\omega_n}(q)}{2 m_\alpha (\omega_n^2+\omega_\alpha^2)}\right]$$

The integral over q_α is still Gaussian, which gives a q-independent result. Let us consider a simple case, that f_α is a linear function

$$f_\alpha[q(\tau)] = C_\alpha q(\tau) \longrightarrow f_{\alpha,\omega_n} = C_\alpha q(i\omega_n)$$

$$\Rightarrow S_{eff}[q] = S_p(q) - \frac{\beta}{2} \sum_{i\omega_n,\alpha} \frac{C_\alpha^2 q(i\omega_n) q(-i\omega_n)}{m_\alpha(\omega_n^2+\omega_\alpha^2)}$$

we would like to subtract a frequency independent term

$$\frac{1}{\omega_n^2+\omega_\alpha^2} \to \frac{1}{\omega_n^2+\omega_\alpha^2} - \frac{1}{\omega_\alpha^2} = \frac{-\omega_n^2}{\omega_\alpha^2(\omega_n^2+\omega_\alpha^2)}$$

The coupling shift the potential

$$V(q) \to V(q) + \left(\frac{1}{2}\sum_\alpha \frac{C_\alpha^2}{\omega_\alpha^2 m_\alpha}\right) q^2 ,$$

C carries the unit of $m\omega^2$.

This shift is non-interesting, we neglect it. then we ariveat

$$\boxed{\begin{aligned} &S_{eff}[q] = S_p[q] + \beta \sum_{i\omega_n} q(i\omega_n) K(i\omega_n) q(-i\omega_n), \text{ where} \\ &K(i\omega_n) = \frac{1}{2}\sum_\alpha \frac{C_\alpha^2 \omega_n^2}{m_\alpha \omega_\alpha^2(\omega_\alpha^2+\omega_n^2)} \end{aligned}}$$

Define spectra density $J(\omega) = \frac{\pi}{2} \sum_\alpha \frac{C_\alpha^2}{m_\alpha \omega_\alpha} \delta(\omega - \omega_\alpha)$

then $K(i\omega_n) = \int_0^{+\infty} \frac{d\omega}{\pi} J(\omega) \frac{\omega_n^2}{\omega(\omega^2+\omega_n^2)}$. Consider a weight of

Ohmic damping $J(\omega) = \eta\omega$. $\boxed{J\text{'s unite} = [m\omega^2]\ [\eta] = m\omega}$.

$$\boxed{K(i\omega_n) = \int_0^{+\infty} \frac{d\omega}{\pi} \eta \frac{\omega_n^2}{\omega^2+\omega_n^2} = \frac{|\omega_n|\eta}{\pi} \int_0^{\infty} dx \frac{1}{x^2+1} = \frac{\eta}{2}|\omega_n|},$$

then

go back to time-domain

$$q(i\omega_n) = \frac{1}{\beta}\int_0^\beta d\tau\, q(\tau) e^{i\omega_n\tau} \quad . \quad K(i\omega_n) = \frac{1}{\beta}\int_0^\beta d\tau'\, K(\tau') e^{-i\omega_n\tau'}$$

$$\to \beta\beta^{-3} \int_0^\beta d\tau \int_0^\beta d\tau' \int_0^\beta d\tau''\, q(\tau) q(\tau') \sum_{\omega_n} e^{i\omega_n(\tau-\tau'-\tau'')} K(\tau'')$$

$$\boxed{\overset{(*)}{\frac{1}{\beta}\sum_{\omega_n} e^{i\omega_n(\tau-\tau'-\tau'')}} = \sum_m \delta(\tau-\tau'-\tau''+m\beta)}$$

$$= \beta^{-1}\int_0^\beta d\tau \int_0^\beta d\tau'\, q(\tau) K(\tau-\tau') q(\tau')$$

$$\omega_n = \frac{2n\pi}{\beta}$$

(~~we neglect~~ ($K(\tau)$ is also a periodical function of τ

$\tau-\tau'$ may go outside $[0, \beta]$, which is equivalent

to put $\tau-\tau'$ inside $[0,\beta]$, but keep all possible $m \neq 0$ in $(*)$]

$$\Rightarrow \boxed{S_{eff}[q] = S_p[q] + \beta^{-1}\int_0^\beta d\tau \int_0^\beta d\tau'\, q(\tau) K(\tau-\tau') q(\tau')}$$

K's unit: $[K] = [m\omega^2]$.

Now we calculate $K(\tau-\tau')$:

$$K(\tau) = \sum_n K(\omega_n)\, e^{i\omega_n\tau} = \frac{\eta}{2}\sum_n |\omega_n|\, e^{i\omega_n\tau} = \eta \sum_{n=1}^{\infty} \frac{2n\pi}{\beta}\, e^{in\frac{2\pi}{\beta}(\tau+i0^+)}$$

set $x=\frac{2\pi}{\beta}$, $\quad \sum_{n=1}^{\infty} nx\, e^{inx(\tau+0^+)} = -i\frac{\partial}{\partial\tau}\sum_{n=1}^{\infty} e^{inx(\tau+i0^+)}$

$$= -i\frac{\partial}{\partial\tau}\sum_{n=0}^{\infty} e^{inx(\tau+i0^+)} = -i\frac{\partial}{\partial\tau}\frac{1}{1-e^{ix(\tau+0^+i)}}$$

$$= -i\,\frac{x(-)(-)ie^{ix\tau}}{(1-e^{ix\tau})^2} = \frac{x}{\left(e^{-i\frac{x\tau}{2}} - e^{\frac{i}{2}x\tau}\right)^2} = \frac{x}{-4\sin^2\frac{x\tau}{2}}$$

$$\Rightarrow K(\tau) = \eta\,\frac{2\pi}{\beta}\,\frac{1}{-4\sin^2\frac{\pi}{\beta}(\tau+i0^+)} = -\frac{\pi}{2}\frac{\eta}{\beta}\,\frac{1}{\sin^2\frac{\pi(\tau+i0^+)}{\beta}}$$

at $\tau \ll \beta \;\Rightarrow\; K(\tau) = \frac{-\eta\beta}{2\pi(\tau+i0^+)^2}$

Further $\quad q(\tau)K(\tau-\tau')q(\tau') = \left\{q^2(\tau) + q^2(\tau') - \frac{1}{2}\left[q(\tau)-q(\tau')\right]^2\right\}K(\tau-\tau')$

The first two terms again after integration only renormalize to $V(q)$, and will be droped. Then we arrive at

$$\boxed{S_{eff}[q] = \int_0^\beta d\tau\; \frac{m}{2}\left(\frac{dq}{d\tau}\right)^2 + V(q) + \frac{\eta}{4\pi}\int_0^\beta d\tau\int_0^\beta d\tau'\; \frac{\left(q(\tau)-q(\tau')\right)^2}{\left(\tau-\tau'+i0^+\right)^2}}$$

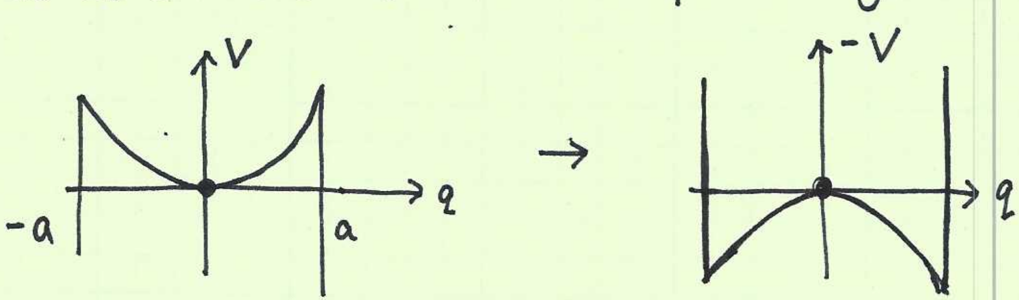
change back to Minkowski space $\tau \to it$

$$i S_{eff}[q(t)] = -S_{eff}(\tau \to it) = i\int dt \left[\frac{m}{2}\left(\frac{dq}{dt}\right)^2 - V(q)\right] - \int dt dt' \frac{\eta}{4\pi} \frac{(q(t)-q(t'))^2}{(t-t')^2}$$

damping

$$\boxed{S[q(t)] = \int dt \ \frac{m}{2}\left(\frac{dq}{dt}\right)^2 - V(q) + i\int dt dt' \frac{\eta}{4\pi} \frac{(q(t)-q(t'))^2}{(t-t')^2}}$$

§2 Now let us come back to the survival probability



Consider the bounce solution $q(0) = q(\beta) \to 0$, at $\tau = \beta/2$, the particle bounce back. The classic equation of motion

$$-m\ddot{q} + m\omega_c^2 q + \frac{\eta}{\pi}\int_0^\beta d\tau' \frac{q(\tau)-q(\tau')}{(\tau-\tau')^2} = A\,\delta(\tau - \beta/2)$$

A is the coefficent to represent the discontinous change of the velocity

LHS: $$\sum_n m(\omega_n^2 + \omega_c^2) q(\omega_n) e^{-i\omega_n \tau} - \frac{2}{\beta}\int_0^\beta d\tau' \left[\sum_{n''} K(i\omega_{n''}) e^{i\omega_{n''}(\tau-\tau')}\right] \left\{\sum_n q(\omega_n)\left(e^{-i\omega_n\tau} - e^{-i\omega_n\tau'}\right)\right\}$$

$$-2\int_0^\beta d\tau' \frac{1}{\beta}\sum_{nn''} K(i\omega_{n''})\, q(\omega_n)\,(-)\,e^{i\omega_{n''}(\tau-\tau') - i\omega_n\tau'} = \eta \sum_n |\omega_n|\, q(\omega_n)\, e^{-i\omega_n\tau}$$

$$\int_0^\beta d\tau' \sum_{nn''} q(\omega_n)\, e^{-i\omega_n\tau}\, |\omega_{n''}|\, e^{-i\omega_{n''}(\tau-\tau')} = 0$$

$$\Rightarrow \text{LHS} = \sum_n \left\{ m(\omega_n^2+\omega_c^2) + \eta|\omega_n| \right\} q(\omega_n)\, e^{-i\omega_n\tau}$$

$$\text{RHS} = \frac{1}{\beta}\sum_n A\, e^{-i\omega_n(\tau-\beta/2)}$$

$$\boxed{\left(m(\omega_n^2+\omega_c^2)+\eta|\omega_n|\right) q(\omega_n) = \frac{A}{\beta}\, e^{i\omega_n\frac{\beta}{2}}}$$ ← Solution of classic equation.

$$\sum_n q(\omega_n)\, e^{-i\omega_n\frac{\beta}{2}} = \frac{A}{\beta}\sum_n \frac{1}{m(\omega_n^2+\omega_c^2)+\eta|\omega_n|}$$

$$a = q\left(\frac{\beta}{2}\right) = Af\,, \quad \text{where } f \triangleq \frac{1}{\beta}\sum_n \frac{1}{m(\omega_n^2+\omega_c^2)+\eta|\omega_n|}$$

↑ boundry condition of bouncing action.

or $A = \dfrac{a}{f}$, $\quad [f] = \left[\dfrac{1}{m\omega^2}\right]$

The action of the boucing solutions

$$S_0 = \beta\sum_n \frac{m}{2}|q(\omega_n)|^2(\omega_n^2+\omega_c^2) + \beta^{-1}\int_0^\beta d\tau\int_0^\beta d\tau' \sum_n q(\omega_n)\, e^{-i\omega_n\tau} \sum_{n'} K(\omega_{n'})\, e^{i\omega_{n'}(\tau-\tau')} \sum_{n''} q(\omega_{n''})\, e^{-i\omega_{n''}\tau'}$$

$$= \beta\sum_n \frac{m}{2}|q(\omega_n)|^2(\omega_n^2+\omega_c^2)$$

$$+\ \beta\sum |q(\omega_n)|^2 K(\omega_n) \leftarrow \frac{\eta}{2}|\omega_n|$$

$$\Rightarrow S_{bounce} = \frac{\beta}{2}\sum_n \left[m(\omega_n^2+\omega_c^2)+\eta|\omega_n|\right]|q_n|^2$$

plug in $q(\omega_n) = \frac{A}{\beta} e^{i\omega_n \frac{\beta}{2}} \Big/ \left(m(\omega_n^2 + \omega_c^2) + \eta|\omega_n|\right)$

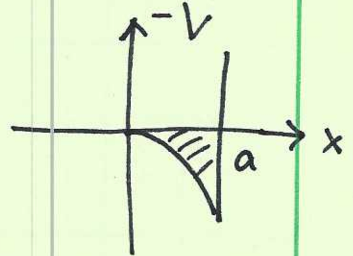
$$\Rightarrow S_{bounce} = \frac{A^2}{2\beta} \sum_n \frac{1}{(\cdots)} = \frac{A^2}{2} f = \boxed{\frac{a^2}{2f} = S_{bounce}}.$$

⊛ and the tunneling rate $\boxed{\Gamma \sim e^{-S_{bounce}}}$.

① Consider the limit $\eta \to 0$ and $\beta \to \infty$: zero ~~temperature~~, no dissipation

$$f = \frac{1}{\beta} \sum_n \frac{1}{m(\omega_n^2 + \omega_c^2)} = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \frac{1}{m(\omega^2 + \omega_c^2)} = \frac{1}{2\pi m \omega_c} \int_{-\infty}^{+\infty} dx \frac{1}{x^2+1}$$

$$= \frac{1}{2m\omega_c} \Rightarrow S_{bounce} = m\omega_c a^2$$



$$S = 2\int_0^a dx \left(\frac{dx}{d\tau}\right) m = 2\int_0^a dx\, \omega_c x\, m = m\omega_c a^2$$

bound back and forth

so $\Gamma \sim e^{-m\omega_c a^2/\hbar}$ is controlled by the barrier height $\frac{m\omega_c^2 a^2}{2}$

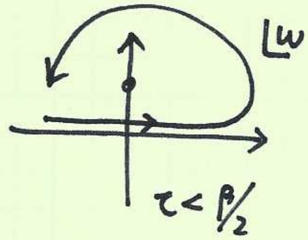
at the attempt frequecy ω_c.

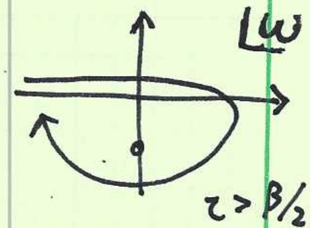
The classic solution

$$q(\tau) = \sum_{\omega_n} q(\omega_n) e^{-i\omega_n \tau} = \beta \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \frac{A}{\beta} \frac{e^{-i\omega (\tau - \beta/2)}}{m(\omega^2 + \omega_c^2) + \eta|\omega_n|} \quad \leftarrow \eta = 0$$

$$= \frac{a}{mf} \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \frac{e^{-i\omega(\tau - \beta/2)}}{\omega^2 + \omega_c^2} = \frac{-2\pi i}{2\pi} \frac{a}{mf} \left(\frac{1}{\omega - \omega_c i} e^{-i\omega(\tau - \beta/2)} \right) \Bigg|_{\omega \to -\omega_c i} \quad \boxed{\text{at } \tau > \beta/2}$$

$$q(\tau) = -\frac{2\pi i}{2\pi}\,\frac{a}{mf}\,\frac{1}{-2\omega_c i}\,e^{-\omega_c|\tau-\beta/2|}$$ ← combine results at $\tau<\beta/2$ from residual theory





$$= \frac{a}{2\cancel{\pi} m f\omega_c}\,e^{-\omega_c|\tau-\beta/2|}$$

$$= a\,e^{-\omega_c|\tau-\beta/2|}$$

② consider $\beta\to+\infty$, and overdamped reigon $\eta \gg m\omega_c$

$$f = \frac{1}{\beta}\sum_n \frac{1}{m(\omega_n^2+\omega_z^2)+\eta|\omega_n|} = \int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\,\frac{1}{m(\omega^2+\omega_c^2)+\eta|\omega|}$$

in the regin: $\left.\begin{array}{l}\eta|\omega| \gg m\omega^2\\ \eta|\omega| \gtrsim m\omega_c^2\end{array}\right\} \Rightarrow \frac{m\omega_c^2}{\eta} \ll |\omega| \ll \eta/m$

the contribution ~~the comes from~~ $\cancel{\frac{1}{\eta|\omega|}}$

$$\Rightarrow f_1 \simeq \frac{2}{2\pi\eta}\int_{\frac{m\omega_c^2}{\eta}}^{\eta/m} d\omega\,\frac{1}{\omega} = \frac{1}{\pi\eta}\ln\frac{\eta/m}{m\omega_c^2/\eta} = \frac{2}{\pi\eta}\ln\frac{\eta}{m\omega_c}$$

At $|\omega| \gg \eta/m$, we have $f' = \frac{1}{\pi m}\int_{\eta/m}^{+\infty} d\omega\,\frac{1}{\omega^2} = \frac{1}{\pi m}\,\frac{m}{\eta} \simeq \frac{1}{\pi\eta}$

$|\omega| \ll \frac{m\omega_c^2}{\eta} \quad\rightarrow\quad f'' = \frac{1}{\pi m}\int_0^{\frac{m\omega_c^2}{\eta}} \frac{1}{\omega_c^2} \simeq \frac{1}{\pi\eta}$

combine together, the contribution mainly from $\frac{m\omega_c^2}{\eta} \ll |\omega| \ll \eta/m$

$$\Rightarrow f \simeq \frac{2}{\pi\eta}\left[\ln\frac{\eta}{m\omega_c}+1\right] \simeq \frac{2}{\pi\eta}\ln\frac{\eta}{m\omega_c}$$

then $\boxed{S_{bounce} = \frac{\pi \eta a^2}{4 \ln \eta/m\omega_c}}$, at $\eta \gg m\omega_c$, this soultion

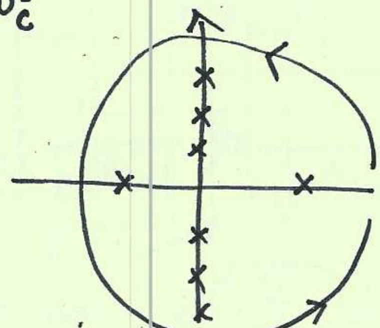
is much larger than the coherent case $S_{bounce} = m\omega_c a^2$.

$\Rightarrow$ tunneling probability is exponentially suppressed.

$\longrightarrow$ The particle becomes classic, - no tunnely.

③ $\eta \to 0$ but $\beta \neq 0$, thermal : $f = \frac{1}{m\beta} \sum_n \frac{1}{\omega_n^2 + \omega_c^2}$

define $\oint \frac{dz}{2\pi i} \frac{1}{e^{\beta z}-1} \frac{1}{m(-\omega^2+\omega_c^2)}$



$$0 = \frac{1}{m\beta} \sum_n \frac{1}{\omega_n^2+\omega_c^2} + \sum_i \mathrm{Res} \frac{1}{e^{\beta z}-1} \frac{1}{m(-\omega^2+\omega_c^2)}\Bigg|_{z=\pm\omega_c}$$

$$\Rightarrow f = +\left[\frac{1}{e^{\beta\omega_c}-1}\frac{1}{2\omega_c} + \frac{1}{e^{-\beta\omega_c}-1}\left(-\frac{1}{2\omega_c}\right)\right] = \frac{1}{2\omega_c m}\frac{e^{\beta\omega_c}+1}{e^{\beta\omega_c}-1} = \frac{\coth\frac{\beta\omega_c}{2}}{2m\omega_c}$$

$$\Rightarrow S = \frac{a^2}{\cancel{4}\left(\frac{\coth\frac{\beta\omega_c}{2}}{m\omega_c}\right)} = m\omega_c a^2 \cdot \tanh\frac{\beta\omega_c}{2} \rightarrow \begin{cases} m\omega_c a^2 \text{ at } \beta\to\infty \\ \frac{\beta}{2} m\omega_c^2 a^2 \text{ at } \beta \to 0 \end{cases}$$

at $\beta \to 0$ (high T): $P \sim e^{\frac{m\omega_c^2 a^2}{2 k_B T}}$ (classic thermal activated behavior!)