Lecture 17

Functional field integrals

§ Coherent states for fermions

operators $\{\hat{\psi}^\dagger, \hat{\psi}\} = 1$, $\{\hat{\psi}, \hat{\psi}\} = \{\hat{\psi}^\dagger, \hat{\psi}^\dagger\} = 0$

$N = \hat{\psi}^\dagger \hat{\psi}$, $\begin{cases} N|0\rangle = 0 \\ N|1\rangle = |1\rangle \end{cases}$ $\begin{cases} \hat{\psi}^\dagger |0\rangle = |1\rangle \\ \hat{\psi}^\dagger |1\rangle = 0 \end{cases}$. $\begin{cases} \hat{\psi}|1\rangle = |0\rangle \\ \hat{\psi}|0\rangle = 0 \end{cases}$

define fermion coherent state

① $\boxed{|\psi\rangle = |0\rangle - \psi|1\rangle}$, where ψ is a grassman number satisfying $\psi^2 = 0$.

$\Rightarrow \hat{\psi}|\psi\rangle = -\hat{\psi}\psi|1\rangle = \psi\hat{\psi}|1\rangle = \psi|0\rangle = \psi(|0\rangle - \psi|1\rangle)$

$\Rightarrow \boxed{\hat{\psi}|\psi\rangle = \psi|\psi\rangle}$

② $\boxed{\langle\bar{\psi}| = \langle 0| - \langle 1|\bar{\psi} = \langle 0| + \bar{\psi}\langle 1|}$

similarly, we can prove $\boxed{\langle\bar{\psi}|\hat{\psi}^\dagger = \langle\bar{\psi}|\bar{\psi}}$

define inner product

$$\langle\bar{\psi}|\psi\rangle = (\langle 0| - \langle 1|\bar{\psi})(|0\rangle - \psi|1\rangle) = 1 + \bar{\psi}\psi = e^{\bar{\psi}\psi}$$

- Functions & integrals of Grassman numbers

$F(\psi) = F_0 + F_1\psi$. no higher power terms

integrals $\boxed{\int 1\, d\psi = 0 ,\ \int \psi\, d\psi = 1 ,\ \text{and} \int d\psi\, \psi = -1}$

just rules. which yield good results later.

- $\bar{\psi}$ is a different Grassman number

$\int \bar{\psi}\psi\, d\psi\, d\bar{\psi} = 1$, but $\int \bar{\psi}\psi\, d\bar{\psi}\, d\psi = -1$

- Gaussian integral

① $$\langle \bar{\psi}\psi \rangle = \frac{\int \bar{\psi}\psi\, e^{a\bar{\psi}\psi}\, d\bar{\psi}\, d\psi}{\int e^{a\bar{\psi}\psi}\, d\bar{\psi}\, d\psi} = \frac{\int \bar{\psi}\psi(1+a\bar{\psi}\psi)\, d\bar{\psi}\, d\psi}{\int (1+a\bar{\psi}\psi)\, d\bar{\psi}\, d\psi}$$

$$= \frac{-1}{-a} = \frac{1}{a}$$

② generalize to a set of Grassman numbers

$$\psi = \begin{bmatrix} \psi_1 \\ \vdots \\ \vdots \\ \psi_n \end{bmatrix} \qquad \bar{\psi} = [\bar{\psi}_1, \cdots \bar{\psi}_n]$$

$$\boxed{\int e^{-\bar{\psi}_i M_{ij} \psi_j} \, d\bar{\psi} d\psi = \det M ,} \quad \text{where } d\bar{\psi} d\psi = \prod_{i=1}^{n} d\bar{\psi}_i \, d\psi_i$$

Proof: $e^{-\bar{\psi}_i M_{ij} \psi} = 1 - \bar{\psi} M \psi + \cdots + \frac{(-)^n}{n!} (\bar{\psi}_i M_{ij} \psi_j)^n$

only the last term contributes to the integral, and let's organize

$$\underbrace{\bar{\psi}_{i_1} M_{i_1 j_1} \psi_{j_1}}_{1} \; \underbrace{\bar{\psi}_{i_2} M_{i_2 j_2} \psi_{j_2}}_{2} \cdots \underbrace{\bar{\psi}_{i_n} M_{i_n j_n} \psi_{j_n}}_{n}$$

$$= \bar{\psi}_{i_1} \psi_{j_1} \bar{\psi}_{i_2} \psi_{j_2} \cdots \bar{\psi}_{i_n} \psi_{j_n} \; M_{i_1 j_1} M_{i_2 j_2} \cdots M_{i_n j_n}$$

$$= \bar{\psi}_{i_1} \bar{\psi}_{i_2} \cdots \bar{\psi}_{i_n} \psi_{j_n} \psi_{j_{n-1}} \cdots \psi_{j_2} \psi_{j_1} \; M_{i_1 j_1} M_{i_2 j_2} \cdots M_{i_n j_n}$$

$$= (-)^{P_{i_1 \cdots i_n}} (-)^{P_{j_1 \cdots j_n}} \; \bar{\psi}_1 \bar{\psi}_2 \cdots \bar{\psi}_n \psi_n \cdots \psi_1 \; M_{i_1 j_1} M_{i_2 j_2} \cdots M_{i_n j_n}$$

where we define $i_1, i_2, \cdots i_n$ as a permutation of $1, 2, \cdots n$

and $P_{i_1 \cdots i_n}$ as it's even or oddness.

Remember $\det M = \sum_{P_{j_1 \cdots j_n}} (-)^{P_{j_1 \cdots j_n}} M_{1 j_1} M_{2 j_2} \cdots M_{n j_n}$

but for each $i_1 \cdots i_n$, we also have

$$\det M = (-)^{P_{i_1 \cdots i_n}} \sum_{P_{j_1 \cdots j_n}} (-)^{P_{j_1 \cdots j_n}} M_{i_1 j_1} M_{i_2 j_2} \cdots M_{i_n j_n}$$

$$\Rightarrow (\bar{\psi} M \psi)^n = n! \det M \; \bar{\psi}_1 \bar{\psi}_2 \cdots \bar{\psi}_n \psi_n \cdots \psi_1$$

$$\Rightarrow \int (\bar\psi_1 \bar\psi_2 \cdots \bar\psi_n)(\psi_n \cdots \psi_1)\, d\bar\psi_1 d\psi_1 \cdots d\bar\psi_n d\psi_n$$

$$= \int (\bar\psi_1 \psi_1)(\bar\psi_2 \psi_2) \cdots (\bar\psi_n \psi_n)\, d\bar\psi_1 d\psi_1 \cdots d\bar\psi_n d\psi_n = (-)^n$$

$$\Rightarrow \int e^{-\bar\psi M \psi}\, d\bar\psi d\psi = \frac{(-)^n}{n!} (-)^n\, n!\, \det M = \det M .$$

- Resolution identity

$$\boxed{\int |\psi\rangle\langle\bar\psi|\, e^{-\bar\psi\psi}\, d\bar\psi d\psi = I}$$

Proof: $\int |\psi\rangle\langle\bar\psi|\, e^{-\bar\psi\psi}\, d\bar\psi d\psi = \int |\psi\rangle\langle\bar\psi|\, (1-\bar\psi\psi)\, d\bar\psi d\psi$

$$= \int \big(|0\rangle - \psi|1\rangle\big)\big(\langle 0| - \langle 1|\bar\psi\big)(1-\bar\psi\psi)\, d\bar\psi d\psi$$

$$= \int |0\rangle\langle 0|\, (-\bar\psi\psi)\, d\bar\psi d\psi + |1\rangle\langle 1| \int \psi\bar\psi\, d\bar\psi d\psi = |0\rangle\langle 0| + |1\rangle\langle 1|$$

- Trace identity: for any bosonic operater $\hat{O}$

$$\boxed{\mathrm{tr}[\hat{O}] = \int \langle -\bar\psi|\, \hat{O}\, |\psi\rangle\, e^{-\bar\psi\psi}\, d\bar\psi d\psi}$$

Proof: $\int \langle -\bar\psi|\, \hat{O}\, |\psi\rangle\, e^{-\bar\psi\psi}\, d\bar\psi d\psi = \int \{\langle 0| + \bar\psi\langle 1|\}\, \hat{O}\, \{|0\rangle - \psi|1\rangle\}$

$$(1-\bar\psi\psi)\, d\bar\psi d\psi$$

$$= \int \langle 0|\hat{O}|0\rangle (-)\bar\psi\psi\, d\bar\psi d\psi + \int \langle 1|\hat{O}|1\rangle (-)\bar\psi\psi\, d\bar\psi d\psi$$

$$= \langle 0|\hat{O}|0\rangle + \langle 1|\hat{O}|1\rangle = \mathrm{tr}\,\hat{O} .$$

§ The fermion path integral (single particle)

$$Z = Tr(e^{-\beta H}) = tr[(1-\epsilon H)\cdots(1-\epsilon H)], \quad \epsilon = \beta/N$$

$$\Rightarrow \quad Z = \int \langle -\bar{\psi}_1 | (1-\epsilon H) \underbrace{|\psi_{N-1}\rangle \, e^{-\bar{\psi}_{N-1}\psi_{N-1}} \langle \bar{\psi}_{N-1}|}_{\text{resolution Identity}} (1-\epsilon H) |\psi_{N-2}\rangle e^{-\bar{\psi}_{N-2}\psi_{N-2}}$$

anti-peroidical boundary condition

$$\cdots \underbrace{|\psi_2\rangle e^{-\bar{\psi}_2\psi_2} \langle \bar{\psi}_2|}_{\text{resolution}} (1-\epsilon H) \underbrace{|\psi_1\rangle e^{-\bar{\psi}_1\psi_1}}_{\text{trace identity}} \prod_{i=1}^{N-1} d\bar{\psi}_i d\psi_i$$

$$\langle \bar{\psi}_{i+1} | (1-\epsilon H) | \psi_i \rangle \approx e^{\bar{\psi}_{i+1}\psi_i} (1 - \epsilon H(\bar{\psi}_i, \psi_i))$$

$$\Rightarrow Z = \int \prod_{i=1}^{N-1} e^{\bar{\psi}_{i+1}\psi_i} e^{-\epsilon H(\bar{\psi}_i, \psi_i)} e^{-\bar{\psi}_i\psi_i} \, d\bar{\psi}_i d\psi_i$$

$$\Rightarrow Z = \int \prod_{i=1}^{N-1} \exp\left\{\left[\frac{1}{\epsilon}(\bar{\psi}_{i+1} - \bar{\psi}_i)\psi_i - H(\bar{\psi}_i, \psi_i)\right]\epsilon\right\} d\bar{\psi}_i d\psi_i$$

$$\rightarrow \boxed{Z = \int e^{-\int_0^\beta d\tau \, \bar{\psi}(\tau)\left[\frac{\partial}{\partial\tau} + H\right]\psi(\tau)} [d\bar{\psi} d\psi]}$$

partial derivative $\partial_\tau \bar{\psi}\,\psi \to -\bar{\psi}\,\partial_\tau \psi$

$\psi(\beta) = -\psi(0)$ for fermion: anti-periodical boundary condition

Fourier transform:

$$\psi(\tau) = \sum_n \frac{e^{-i\omega_n \tau}}{\sqrt{\beta}} \psi(\omega_n), \quad \text{where } \omega_n = \frac{(2n+1)\pi}{\beta}$$

$$\psi(\omega_n) = \frac{1}{\sqrt{\beta}} \int_0^\beta e^{i\omega_n \tau} \psi(\tau)\, d\tau$$

→ generalize to fermiunic field theory

$$\psi(\tau) \to \psi(x,\tau)$$

$$\boxed{\begin{aligned} &Z = \int D[\bar{\psi}(x,\tau)\, \psi(x,\tau)]\, e^{-S[\bar{\psi},\psi]} \\ &\text{where } S[\bar{\psi},\psi] = \int_0^\beta d\tau \left[\bar{\psi}(x,\tau)\, \partial_\tau \psi(x,\tau) + H(\bar{\psi},\psi) - \mu \bar{\psi}\psi \right] \end{aligned}}$$

§ Bosonic system:

Coherent state $\hat{a}|a\rangle = a|a\rangle$, where $\hat{a}$ is boson annilation operator & a is a complex number.

we have $|a\rangle = e^{-\frac{|a|^2}{2} + a\hat{a}^\dagger} |0\rangle = e^{-\frac{|a|^2}{2}} \sum_{n=0}^{\infty} \frac{a^n}{\sqrt{n!}} |0\rangle$

and

$$\boxed{\begin{aligned} &\int \frac{da\,da^*}{\pi} |a\rangle\langle a| = 1 \quad \text{where } da\,da^* = d\mathrm{Re}\,a\, d\mathrm{Im}\,a \\ &\langle a|a'\rangle = e^{-\frac{1}{2}(|a|^2+|a'|^2) + a^* a'} \end{aligned}}$$

trace identity: for a single mode:

$$\boxed{tr\, \hat{O} = \int \langle a | \hat{O} | a \rangle \frac{da^* da}{\pi}}$$

Proof: $\int \frac{da\, da^*}{\pi} \langle a | \hat{O} | a \rangle = \int \frac{d\theta}{\pi} |a| d|a| \; e^{-|a|^2} \sum_n \frac{|a|^{2n}}{n!} \langle n | \hat{O} | n \rangle$

$$= \sum_n \underbrace{\int_0^{+\infty} dx \; \frac{e^{-x^2}}{n!} (x^n)}_{=1} \; \langle n | \hat{O} | n \rangle = \sum_n \langle n | \hat{O} | n \rangle = tr[\hat{O}]$$

partition function for a single mode:

$$Z = tr[e^{-\beta H}] = \int \langle a_N | (1 - \epsilon H) | a_{N-1} \rangle \langle a_{N-1} | (1 - \epsilon H) | a_{N-2} \rangle \cdots$$

$$\langle a_2 | (1 - \epsilon H) | a_1 \rangle \quad D[a_i]$$

$$\langle a_{i+1} | (1 - \epsilon H) | a_i \rangle = e^{-\frac{1}{2}[a_{i+1}^* a_{i+1} + a_i^* a_i - 2 a_{i+1}^* a_i]} (1 - \epsilon H(a^* a$$

$$= e^{-\frac{\epsilon}{2} a_{i+1}^* \frac{a_{i+1} - a_i}{\epsilon} + \frac{a_i^* - a_{i+1}^*}{\epsilon} a_i} \quad e^{-\epsilon H(a^* a)}$$

$$= e^{-\frac{\epsilon}{2}[a_{i+1}^* \partial_\tau a_{i+1} - \partial_\tau a_{i+1}^* a_i] - \epsilon H(a_i^*, a_i)}$$

$$\xrightarrow[\text{partial derivative}]{} e^{-\epsilon [a_i^* \partial_\tau a_* + H(a^*, a)]}$$

$$\Rightarrow \boxed{Z = \int D a(\tau) \; e^{-S(a^*, a)}, \quad \text{where } S = \int_0^\beta d\tau \left(a^* \frac{\partial}{\partial \tau} a + H(a^*, a)\right)}$$

Many-body field theory for bosons, $a(\tau) \longrightarrow a(x,\tau)$

$$\boxed{\begin{aligned} &\mathcal{Z} = \int \mathcal{D}a(x,\tau)\ \exp\left[-S(a^*(x,\tau), a(x,\tau))\right] \\ &\text{where } S = \int_0^\beta d\tau \int dx\ a^*(x,\tau)\frac{\partial}{\partial\tau}a(x,\tau) + H\left[a^*(x,\tau), a(x,\tau)\right] \end{aligned}}$$

peridical boundary condition $\quad a(x\ \beta) = a(x, 0)$

$$\Rightarrow \quad a(\tau) = \frac{1}{\sqrt{\beta}} \sum_n e^{-i\omega_n \tau}\, a(\omega_n)$$

where $\omega_n = \dfrac{2n\pi}{\beta}$.

$$a(\omega_n) = \frac{1}{\sqrt{\beta}} \int_0^\beta e^{i\omega_n \tau}\, a(\tau)$$

* Gaussian integrals for real variables

1 $$\boxed{\int dx_1 \cdots dx_n \; e^{-\frac{1}{2} x_i A_{ij} x_j} = (2\pi)^{n/2} (\det A)^{-1/2}}$$

real variable

2 adding source field

$$\boxed{\int dx_1 \cdots dx_n \; e^{-\frac{1}{2} x_i A_{ij} x_j + J_i x_i} = (2\pi)^{n/2} (\det A)^{-1/2} \exp\left[\tfrac{1}{2} J_i A^{-1}_{ij} J_j\right]}$$

Proof: change of variable

$$-\frac{1}{2} x_i A_{ij} x_j \quad = -\frac{1}{2}[x_i - A^{-1}_{ii'} J_{i'}] A_{ij} [x_j - A^{-1}_{jj'} J_{j'}]$$

$$-\frac{1}{2} x_i (A A^{-1})_{ij'} J_{j'} - \frac{1}{2} A^{-1}_{ii'} J_{i'} A_{ij} x_j + \frac{1}{2} \; A^{-1}_{ii'} J_{i'} (AA^{-1})_{ij'} J_{j'}$$

$$= -\frac{1}{2} y_i A_{ij} y_j - \frac{1}{2}\left(x_i J_i + J_{i'} x_{i'}\right) + \frac{1}{2} \; J_{i'} A^{-1}_{i'i} J_i$$

$$= -\frac{1}{2} y_i A_{ij} y_j - x_i J_i + \frac{1}{2} J_{i'} A^{-1}_{i'i} J_i$$

(remember $A = A^T$, $A^{-1} = (A^{-1})^T$

where $y_i = x_i - (A^{-1} J)_i$

$$\Rightarrow e^{-\frac{1}{2} x_i A_{ij} x_j + J_i x_i} = e^{-\frac{1}{2} y_i A_{ij} y_j} \; e^{\frac{1}{2} J_i A^{-1}_{ij} J_j}$$

$$\Rightarrow \int dx_1 \cdots dx_n \; e^{-\frac{1}{2} x_i A_{ij} x_j + J_i x_i} = (2\pi)^{n/2} (\det A)^{-1/2} \exp\left[\tfrac{1}{2} J_i A^{-1}_{ij} J_j\right]$$

3 take derivatives

$$\partial_{j_h} \partial_{j_\ell} \left[\int dx_1 \cdots dx_n \; e^{-\frac{1}{2} x_i A_{ij} x_j + J_i x_i}\right] \Bigg|_{j=0}$$

$$= \int dx_1 \cdots dx_n \; x_h x_\ell \; e^{-\frac{1}{2} x_i A_{ij} x_j}$$

$$= (2\pi)^{n/2} (\det A)^{-1/2} \; \partial_{j_h} \partial_{j_\ell} \; e^{\frac{1}{2} j_i (A^{-1})_{ij} j_j} \Big|_{J=0}$$

$$= (2\pi)^{n/2} (\det A)^{-1/2} \; e^{\frac{1}{2} j_i (A^{-1})_{ij} j_j} \; (A^{-1})_{h\ell}$$

$$\Rightarrow \boxed{\frac{\int dx_1 \cdots dx_n \; x_h x_\ell \; e^{-\frac{1}{2} x^T A x}}{\int dx_1 \cdots dx_n \; e^{-\frac{1}{2} x^T A x}} = (A^{-1})_{h\ell} = \langle x_h x_\ell \rangle}$$

Similarly, we can antinue to do derivatives

$$\partial_{J_{i_1}} \partial_{J_{i_2}} \partial_{J_{i_3}} \partial_{J_{i_4}} \left[\int dx_1 \cdots dx_n \; e^{-\frac{1}{2} x_i A_{ij} x_j + J_i x_i} \right] \Big|_{J=0} = \int dx_1 \cdots dx_n \; x_{i_1} x_{i_2} x_{i_3} x_{i_4} \; e^{-\frac{1}{2} x_i A_{ij} x_j}$$

$$\partial_{J_{i_1}} e^{\frac{1}{2} J_i A^{-1}_{ij} J_j} = e^{\frac{1}{2} J A^{-1} J} (A^{-1})_{i_1 j_1} J_{j_1}$$

$$\partial_{J_{i_1}} \partial_{J_{i_2}} e^{\frac{1}{2} J_i A^{-1}_{ij} J_j} = e^{\frac{1}{2} J A^{-1} J} \left[(A^{-1})_{i_2 j_2} (A^{-1})_{i_1 j_1} J_{j_1} J_{j_2} + (A^{-1})_{i_1 i_2} \right]$$

$$\partial_{J_{i_1}} \partial_{J_{i_2}} \partial_{J_{i_3}} \partial_{J_{i_4}} \; e^{\frac{1}{2} J_i A^{-1}_{ij} J_j} \Big|_{J=0} = (A^{-1})_{i_2 j_2} (A^{-1})_{i_1 j_1} \left[\delta_{j_1 i_3} \delta_{j_2 i_4} + \delta_{j_1 i_4} \delta_{j_2 i_3} \right]$$

$$+ (A^{-1})_{i_1 i_2} (A^{-1})_{i_3 i_4}$$

$$= (A^{-1})_{i_1 i_2} (A^{-1})_{i_3 i_4} + (A^{-1})_{i_1 i_3} (A^{-1})_{i_2 i_4} + (A^{-1})_{i_1 i_4} (A^{-1})_{i_2 i_3}$$

$$\Rightarrow \langle x_{i_1} x_{i_2} x_{i_3} x_{i_4} \rangle = (A^{-1})_{i_1 i_2} (A^{-1})_{i_3 i_4} + (A^{-1})_{i_1 i_3} (A^{-1})_{i_2 i_4} + (A^{-1})_{i_1 i_4} (A^{-1})_{i_2 i_3}$$

In general, we have Wick theorem for Gaussian integrals

$$\boxed{\begin{aligned} \langle x_{i_1} x_{i_2} \cdots x_{i_{2n-1}} x_{i_{2n}} \rangle &= \sum_{\text{all pairs}} \langle x_{i_{k_1}} x_{i_{k_2}} \rangle \cdots \langle x_{i_{k_{2n-1}}} x_{i_{k_{2n}}} \rangle \\ &= \sum_{\text{all pairs}} (A^{-1})_{i_{k_1}, i_{k_2}} \cdots (A^{-1})_{i_{k_{2n-1}} i_{k_{2n}}} \end{aligned}}$$

* Gaussian integrals for complex variables

$$\boxed{\int dz_1^* dz_1 \cdots dz_N^* dz_n \, e^{-z_i^* A_{ij} z_j} = \pi^n (\det A)^{-1}}, \text{ where } dz^* dz = d\mathrm{Re}z \, d\mathrm{Im}z$$

This is valid even where A is non-Hermitian.

Adding source field

$$\boxed{\int dz_1^* dz_1 \cdots dz_N^* dz_N \, e^{-z_i^* A_{ij} z_j + w_i^* z_i + z_i^* w_i'} = \pi^N (\det A)^{-1} e^{w_i^* A^{-1}_{ij} w_j'}}$$

where w_i and w_i' may be different.

Again $\boxed{\langle z_{i_1}^* z_{j_1} \rangle = (A^{-1})_{j_1 i_1}}$ ← note the sequence of indices.

$$\boxed{\langle z_{i_1}^* z_{i_2}^* \cdots z_{i_n}^* z_{j_n} \cdots z_{j_1} \rangle = \sum_P (A^{-1})_{j_1 i_{P_1}} (A^{-1})_{j_2 P_2} \cdots (A^{-1})_{j_n P_n}}$$

* Gaussian integral for fermions

$$\int e^{-\bar{\psi}_i M_{ij} \psi_j} \, d\bar{\psi}_1 d\psi_1 \cdots d\bar{\psi}_n d\psi_n = \det M$$

$$\int e^{-\bar{\psi} M \psi + \bar{\psi}_i \nu_i + \bar{\nu}_i \psi_i} \, d\bar{\psi}_1 d\psi_1 \cdots d\bar{\psi}_n d\psi_n = \det M \, e^{\bar{\nu}_i (M^{-1})_{ij} \nu_j}$$

$$\langle \psi_{j_1} \cdots \psi_{j_n} \bar{\psi}_{i_n} \cdots \bar{\psi}_{i_1} \rangle = \sum_P (\operatorname{sgn} P) \, A^{-1}_{j_1 i_{P_1}} \cdots A^{-1}_{j_n i_{P_n}}$$

* Gaussian functional integration (boson)

$$\int Da(x,\tau)\, Da^*(x,\tau)\, \exp\Big[-\Big\{\int dx d\tau dx' d\tau' \; a^*(x,\tau)\, M(x\tau; x'\tau')\, a(x'\tau')$$

$$+\int dx d\tau \; j(x\tau)\, a^*(x\tau) + j^*(x,\tau)\, a(x,\tau)\Big\}\Big]$$

$$\propto (\det M)^{-1} \exp\Big[\tfrac{1}{2}\int dx d\tau dx' d\tau' \; j^*(x,\tau)\; M^{-1}(x\tau, x'\tau')\, j(x',\tau')\Big]$$

where M^{-1} is defined as

$$\int dx' d\tau' \; M(x\tau, x'\tau')\, M^{-1}(x'\tau', x''\tau'') = \delta(x-x'')\,\delta(\tau-\tau'')$$

$$\langle a(x_1\tau_1)\, a^*(x_2\tau_2)\rangle = M^{-1}(x_1\tau_1; x_2\tau_2)$$

Similar results hold for fermions

$$\int D\bar{\psi}(x\tau)\, D\psi(x,\tau)\, \exp\Big[-\Big\{\int dx d\tau dx' d\tau' \; \bar{\psi}(x\tau)\, M(x\tau; x'\tau')\, \psi(x',\tau')$$

$$+\int dx d\tau \; \bar{\psi}(x\tau)\, j(x\tau) + \bar{j}(x\tau)\, \psi(x\tau)\Big\}\Big]$$

$$\propto (\det M)\cdot \exp\Big[\tfrac{1}{2}\int dx d\tau dx' d\tau' \; \bar{j}(x\tau)\; M^{-1}(x\tau, x'\tau')\, j(x',\tau')\Big]$$

$$\langle \psi(x_1\tau_1)\, \bar{\psi}(x_2,\tau_2)\rangle = M^{-1}(x_1\tau_1, x_2\tau_2).$$

Partition function / Green's function for free field

$$H = \sum_k \varepsilon_k a_k^\dagger a_k \quad \text{or} \quad \sum_k \varepsilon_k c_k^\dagger c_k$$

$$Z = \int D\bar{\phi} D\phi \; e^{-\int_0^\beta d\tau \, \bar{\phi}(x,\tau)\left[\frac{\partial}{\partial\tau} - H_0\right]\phi(x,\tau)},$$ ϕ can be either "c" or "a".

Fourier ~~transform~~: action

$$S = \sum_{k,\omega_n} \bar{\phi}(k, i\omega_n)\left[-i\omega_n + \underbrace{(\varepsilon_k - \mu)}_{\text{define as } \xi_k}\right]\phi(k, i\omega_n)$$

$$i\omega_n = \begin{cases} \dfrac{2\pi n}{\beta} & \text{for "a"} \\ \dfrac{(2n+1)\pi}{\beta} & \text{for "c"} \end{cases}$$

$\Rightarrow$

$$\langle \phi(k\, i\omega_n)\,\bar{\phi}(k, i\omega_n)\rangle = \frac{-1}{i\omega_n - \xi_k}$$

where $$\phi(x,\tau) = \frac{1}{\sqrt{V\beta}} \sum_{k,i\omega_n} e^{i(kx - \omega_n \tau)}\,\phi(k, i\omega_n)$$

In order to make Z - dimensionless, when change to variable of $\phi(k, i\omega_n)$

then measure

$$Z = \int d\left(\bar{\phi}(k\, i\omega_n)/\sqrt{\beta}\right) d\left(\phi(k, i\omega_n)/\sqrt{\beta}\right) \cdot \exp\left[-\sum_{k, i\omega_n} \bar{\phi}(k, i\omega_n)\left[-i\omega_n + \xi_k\right] \phi(k, i\omega_n)\right]$$

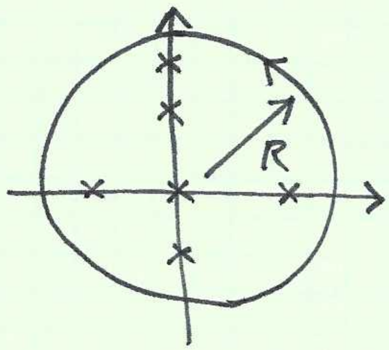
$$= \begin{cases} \displaystyle\prod_{k, i\omega_n} \frac{1}{\beta(-i\omega_n + \xi_k)} & \text{for bosons} \\ \displaystyle\prod_{k, i\omega_n} \beta(-i\omega_n + \xi_k) & \text{for fermions} \end{cases}$$

$$F = \frac{-1}{\beta}\ln Z = \begin{cases} \dfrac{-1}{\beta}\displaystyle\sum_{k,i\omega_n} \ln \dfrac{1}{\beta(-i\omega_n+\xi_k)} & \leftarrow \omega_n = \dfrac{2n\pi}{\beta} \text{ for bosons} \\ \dfrac{-1}{\beta}\displaystyle\sum_{k,i\omega_n} \ln \beta(-i\omega_n+\xi_k) & \leftarrow \omega_n = \dfrac{(2n+1)\pi}{\beta} \text{ for fermions} \end{cases}$$

§ Frequency summation

1. For boson frequency $\omega_n = \frac{2n\pi}{\beta}$. we evaluate $S = \frac{1}{\beta}\sum_n f(i\omega_n)$

Define $I = \lim_{R\to\infty} \oint \frac{dz}{2\pi i} f(z) \frac{1}{e^{\beta z}-1}$



If $\lim_{z\to\infty} |z f(z)| \to 0$ uniformly, we have

$I \to 0$ as $R\to\infty$. (If $f(z) \sim \frac{1}{z}$, we need to be careful, because $\left|\frac{1}{e^{\beta z}-1}\right| \longrightarrow 1$ one the left half plane).

Under this condition, we can emulate all the poles enclosed by the loop:

$$\Rightarrow \underbrace{\frac{1}{\beta}\sum_n f(i\omega_n)}_{\text{pole from } n_B(z)} + \underbrace{\sum_i \text{Res}\left[\frac{1}{e^{\beta z_i}-1} f(z)\right]\Big|_{z=z_i}}_{\text{poles of } f(z)} = 0$$

$$\Rightarrow \boxed{S = \frac{1}{\beta}\sum_n f(i\omega_n) = -\sum_i \text{Res}\left(\frac{1}{e^{\beta z_i}-1} f(z)\right)\Big|_{z=z_i}}$$

$$\boxed{= -\sum_i n_B(z_i)\, \text{Res}\, f(z)\Big|_{z=z_i}}$$

Example: $$S = \frac{1}{\beta} \sum_{i\omega_n} \frac{2\omega_q}{\omega_n^2 + \omega_q^2} \frac{1}{ip_n + i\omega_n - \xi_p}$$

Solution: $$f(z) = \frac{2\omega_q}{-z^2 + \omega_q^2} \frac{1}{z + ip_n - \xi_p}$$

$f(z)$ has 3 poles:

$$z_1 = \omega_q \qquad \mathrm{Res}\, f(z)\Big|_{z_1} = \frac{-1}{\omega_q - \xi_p + ip_n}$$

$$z_2 = -\omega_q \qquad \mathrm{Res}\, f(z)\Big|_{z_2} = \frac{1}{-\omega_q - \xi_p + ip_n}$$

$$z_3 = \xi_p - ip_n \qquad \mathrm{Res}\, f(z)\Big|_{z_3} = \frac{2\omega_q}{\omega_q^2 - (\xi_p - ip_n)^2}$$

$$\Rightarrow S = -n_B(\omega_q) \frac{(-1)}{\omega_q - \xi_p + ip_n} - n_B(-\omega_q) \frac{1}{-\omega_q - \xi_p + ip_n}$$

$$- n_B(\xi_p - ip_n) \left[\frac{1}{\omega_q - \xi_p + ip_n} - \frac{1}{-\omega_q - \xi_p + ip_n} \right]$$

$$n_B(-\omega_q) = -1 - n_B(\omega_q)$$

$$n_B(\xi_p - ip_n) = \frac{1}{e^{\beta\xi_p} e^{-\beta i p_n} - 1} = -n_F(\xi_p),$$ ← p_n is fermion frequency.

$$\Rightarrow \boxed{S = \frac{n_B(\omega_q) + n_F(\xi_p)}{\omega_q - \xi_p + ip_n} + \frac{1 + n_B(\omega_q) - n_F(\xi_p)}{-\omega_q - \xi_p + ip_n}}$$

$$\boxed{= \frac{1}{\beta} \sum_{i\omega_n} \frac{2\omega_q}{\omega_n^2 + \omega_q^2} \frac{1}{ip_n + i\omega_n - \xi_p}}$$

2: For fermion frequency $\omega_n = \frac{(2n+1)\pi}{\beta}$, we evaluate

$$S = \frac{1}{\beta} \sum_n f(i\omega_n)$$

Define $I = \lim_{R\to\infty} \oint \frac{dz}{2\pi i} f(z) \frac{1}{e^{\beta z}+1}$, again we need

$\lim_{|z|\to\infty} |z f(z)| \to 0$ uniformly, and using the same reasoning as in the boson case

$$-\frac{1}{\beta} \sum_n f(i\omega_n) + \sum_i n_F(z_i) \operatorname{Res} f(z)\Big|_{z=z_i} = 0$$

$$\Rightarrow \boxed{S = \frac{1}{\beta} \sum_n f(i\omega_n) = \sum_i n_F(z_i) \operatorname{Res} f(z)\Big|_{z=z_i}}$$

Example: $S = \frac{1}{\beta} \sum_{ip_n} \frac{1}{ip_n - \xi_p} \frac{1}{ip_n + i\omega_n - \xi_k}$

Solution: $f(z) = \frac{1}{z-\xi_p} \frac{1}{z+i\omega_n-\xi_k}$

$f(z)$ has two poles: $z_1 = \xi_p$, $\operatorname{Res} f(z)\Big|_{z_1} = \frac{1}{\xi_p - \xi_k + i\omega_n}$

$z_2 = \xi_k - i\omega_n$ $\operatorname{Res} f(z)\Big|_{z_2} = \frac{1}{\xi_k - \xi_p - i\omega_n}$

$$\Rightarrow S = \frac{n_F(\xi_p)}{\xi_p - \xi_k + i\omega_n} + \frac{n_F(\xi_k)}{\xi_k - \xi_p - i\omega_n} = \frac{n_F(\xi_p) - n_F(\xi_k)}{i\omega_n + \xi_p - \xi_k}$$

$$\Rightarrow \boxed{\frac{1}{\beta} \sum_{ip_n} \frac{1}{ip_n - \xi_p} \frac{1}{ip_n + i\omega_n - \xi_k} = \frac{n_F(\xi_p) - n_F(\xi_k)}{i\omega_n + \xi_p - \xi_k}}$$

§ Summation with convegence facter

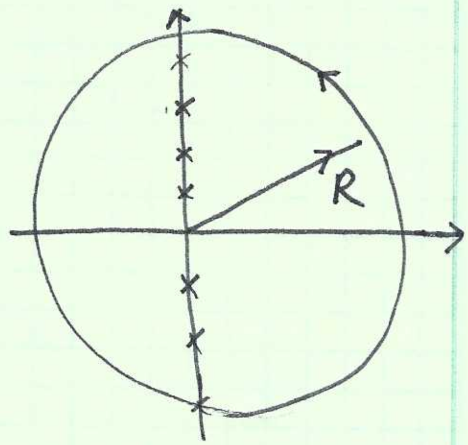
$$S = \begin{cases} \dfrac{-1}{\beta} \sum_n \dfrac{1}{i\omega_n - \xi_k} & \text{for } \omega_n = \dfrac{2n\pi}{\beta} \quad \text{boson} \\[2ex] \dfrac{1}{\beta} \sum_n \dfrac{1}{i\omega_n - \xi_k} & \text{for } \omega_n = \dfrac{2n+1}{\beta}\pi \quad \text{fermion} \end{cases}$$

Solution: In order to converge, we need to add a factor $e^{-i\omega_n \tau}$ and set $\tau \to 0^-$. This comes from the definition of Green's function.

$$n_k = \begin{cases} -\mathcal{G}(k, \tau = 0^-) = \lim\limits_{\tau \to 0^-} \langle T_\tau a_k(\tau) a^\dagger(0) \rangle \\[2ex] \mathcal{G}(k, \tau = 0^-) = -\lim\limits_{\tau \to 0^-} \langle T_\tau \psi_k(\tau) \psi^\dagger(0) \rangle \end{cases}$$

Then we still choose $I = \lim\limits_{R \to \infty} \oint \dfrac{dz}{2\pi i} f(z) \dfrac{1}{e^{\beta z} \mp 1}$

where $f(z) = \dfrac{e^{-z\tau}}{z - \xi_k}$.



set $z = R\cos\theta + iR\sin\theta$, the $n_{B,F}(z)$ suppresses the contribution for the right half circle.

and $e^{-z\tau}$ suppresses the contribution from the left half circle (remember τ is negative).

Then we have $I = 0$. This yields

$$\boxed{\lim_{\tau \to 0^-} \mp \frac{1}{\beta} \sum_{\omega_n} \frac{e^{-i\omega_n \tau}}{i\omega_n - \xi_k} = \begin{cases} n_B(z_1) \Big|_{z=\xi} \\ n_F(z) \Big|_{z=\xi} \end{cases} = \frac{1}{e^{\beta\xi} \mp 1}}$$

Then what happen if we take $\tau \to 0^+$

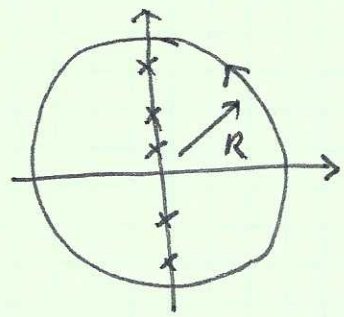
$$-\mathcal{G}(k,\tau=0^+) = \lim_{\tau\to 0^+} \langle T_\tau a_k(\tau) a^\dagger(0) \rangle = \frac{-1}{\beta} \sum_n \frac{e^{-i\omega_n \tau}}{i\omega_n - \xi_k} = 1 + n_B(\xi_k) \quad \text{boson}$$

$$\mathcal{G}(k,\tau=0^+) = \lim_{\tau\to 0^+} -\langle T_\tau \psi(\tau) \psi^\dagger(0) \rangle = \frac{1}{\beta} \sum_n \frac{e^{-i\omega_n \tau}}{i\omega_n - \xi_k} = -1 + n_F(\xi_k) \quad \text{fermion}$$

Check: we need to change to integrals

$$I = \lim_{R\to\infty} \oint \frac{dz}{2\pi i} f(z) \frac{1}{e^{-\beta z} \mp 1} \quad \text{with } f(z) = \frac{e^{-z\tau}}{z - \xi_k} \quad \text{as } \tau \to 0^+.$$

Again $z = R\cos\theta + iR\sin\theta$, at $R \to \infty$,



$\frac{1}{e^{-\beta z} \mp 1}$ protects convergence on the left half circle.

$e^{-z\tau}$ $(\tau \to 0^+)$ protects convergece on the right half circle. $\Rightarrow I = 0$.

For bosons $\quad \sum_{i\omega_n} -\frac{1}{\beta} f(i\omega_n) + \frac{1}{e^{-\beta \xi_k} - 1} = 0$

$$\boxed{\Rightarrow \lim_{\tau\to 0^+} -\frac{1}{\beta} \sum_{i\omega_n} \frac{e^{-i\omega_n \tau}}{i\omega_n - \xi_k} = -\frac{1}{e^{-\beta \varepsilon_k} - 1} = 1 + n_B(\xi_k) = -n_B(-\xi_k)}$$

For fermions $\quad \sum_{i\omega_n} \frac{1}{\beta} f(i\omega_n) + \frac{1}{e^{-\beta\varepsilon} + 1} = 0$

$$\boxed{\lim_{\tau\to 0^+} \frac{1}{\beta} \sum_{i\omega_n} \frac{e^{-i\omega_n \tau}}{i\omega_n - \xi_k} = -\frac{1}{e^{-\beta\varepsilon} + 1} = -1 + n_F(\xi_k) = -n_F(-\xi_k)}$$

To calculate free energy (fermions)

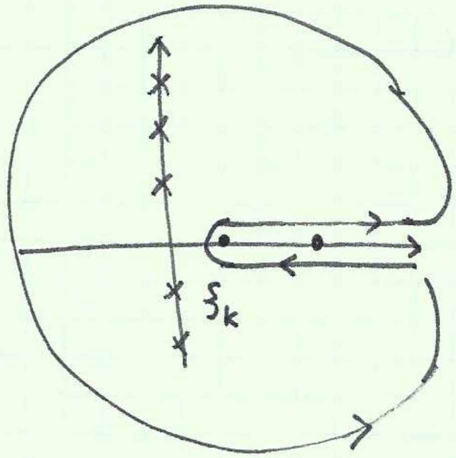
$$F = -\frac{1}{\beta} \sum_{i\omega_n} \ln \beta(-i\omega_n + \xi_k) \qquad \omega_n = \frac{(2n+1)\pi}{\beta}$$

$$= \lim_{\tau \to 0^-} -\frac{1}{\beta} \sum_{i\omega_n} e^{-i\omega_n \tau} \ln \beta(-i\omega_n + \xi_k)$$

consider $I = \oint \frac{dz}{2\pi i} f(z)\, n_F(z)$, where $f(z) = e^{-z\tau} \ln[\beta(z - \xi_k)]$

The branch cut of $\ln(-\beta(z-\xi_{k}))$

The convegence factor $e^{-z\tau}$ $(\tau \to 0^-)$ and $n_F(z)$, suppress the contribution on the big circle. But the contribution from the branch cut is not.



$$\Rightarrow -\frac{1}{\beta} \sum_{i\omega_n} e^{-i\omega_n \tau} \ln -\beta(i\omega_n - \xi_k) = \int_{\xi_k}^{+\infty} \frac{dx}{2\pi i} \left[\ln(-\beta(x + i\eta - \xi_k)) - \ln(-\beta(x - i\eta - \xi_k))\right] \frac{1}{e^{\beta x}+1}$$

$$\Rightarrow F = -\frac{1}{\beta} \sum_{i\omega_n} e^{-i\omega_n \tau} \ln[-\beta(i\omega_n - \xi_k)]$$

$$= \int_{\xi_k}^{+\infty} \frac{dx}{2\pi i} \ln\left(\frac{x + i\eta - \xi_k}{x - i\eta - \xi_k}\right) \frac{1}{e^{\beta x}+1}$$

we can extend the lower boundary to $-\infty$, because at $x < \xi_k$, there's no branch cut, $\ln \frac{x+i\eta-\xi_k}{x-i\eta-\xi_k} = 0$ at $x < \xi_k$

$$= \int_{-\infty}^{+\infty} \frac{dx}{2\pi i} \ln\left(\frac{x + i\eta - \xi_k}{x - i\eta - \xi_k}\right) \frac{1}{e^{\beta x}+1}$$

$$= -\frac{1}{\beta} \int_{-\infty}^{+\infty} \ln\left(\frac{x + i\eta - \xi_k}{x - i\eta - \xi_k}\right) \frac{d}{dx} \ln(1 + e^{-\beta x})\, \frac{dx}{2\pi i}$$

$$= \frac{1}{\beta}\int_{-\infty}^{+\infty} \ln(1+e^{-\beta x})\,\underbrace{\left(\frac{1}{x+i\eta-\xi_k} - \frac{1}{x-i\eta-\xi_k}\right)}_{-2\pi i\,\delta(x-\xi_k)}\frac{dx}{2\pi i}$$

$$= -\frac{1}{\beta}\ln(1+e^{-\beta\xi_k})$$

Similarly, for bosons, we have

$$F_k = \frac{1}{\beta}\ln(1-e^{-\beta\epsilon_k}).$$