Lecture 11. Landau Fermi liquid (IV)

§ Ward identities — Abriksov et al. P158

Consider an external small pertarbation $H'_{(t)} = \int \delta U(t)\, \psi^\dagger_\alpha(r)\psi_\alpha(r)\, d\mathbf{r}$.

we study the variation of green's function:

$$i G_{\alpha\beta}(x;x') = \int D\bar\psi D\psi\ \psi_\alpha(x)\bar\psi_\beta(x')\, e^{i\int dt\,(\mathcal{L}+\mathcal{L}')} \Big/ \int D\bar\psi D\psi\ e^{i\int dt\, \mathcal{L}+\mathcal{L}'}$$

where $\mathcal{L}' = -H'(t)$. correct to the linear order of $\delta U(t)$, we have

$$i G_{\alpha\beta}(x,x') = \frac{\int D\bar\psi D\psi\ \psi_\alpha(x)\bar\psi_\beta(x')\, e^{i\int dt\,\mathcal{L}}\,[1+i\int dt\,\mathcal{L}'] \Big/ \int D\bar\psi D\psi\ e^{i\int dt\,\mathcal{L}}}{\int D\bar\psi D\psi\ e^{i\int dt\,\mathcal{L}}\,[1+i\int dt\,\mathcal{L}'] \Big/ \int D\bar\psi D\psi\ e^{i\int dt\,\mathcal{L}}}$$

$$= \int D\bar\psi D\psi\ \psi_\alpha(x)\bar\psi_\beta(x')\, e^{i\int dt\,\mathcal{L}} \Big/ \int D\bar\psi D\psi\ e^{i\int dt\,\mathcal{L}}$$

$$+ \int D\bar\psi D\psi\ \left[\psi_\alpha(x)\bar\psi_\beta(x')\ i\int dt\,\mathcal{L}'\right] e^{i\int dt\,\mathcal{L}} \Big/ \int D\bar\psi D\psi\ e^{i\int dt\,\mathcal{L}}$$

$$- \frac{\int D\bar\psi D\psi\ [\psi_\alpha(x)\bar\psi_\beta(x')\, e^{i\int dt\,\mathcal{L}}]}{\int D\bar\psi D\psi\ e^{i\int dt\,\mathcal{L}}} \cdot \frac{\int D\bar\psi D\psi\ (i\int dt\,\mathcal{L}')\, e^{i\int dt\,\mathcal{L}}}{\int D\bar\psi D\psi\ e^{i\int dt\,\mathcal{L}}}$$

$$\rightarrow i\,\delta G_{\alpha\beta}(x,x') = \langle \psi_\alpha(x)\bar\psi_\beta(x')\,(-i)\int dt_y\ \delta U(t_y)\,\bar\psi_\gamma(y)\psi_\gamma(y)\rangle$$

$$- \langle \psi_\alpha(x)\bar\psi_\beta(x')\rangle\,\langle(-i)\int dy\ \delta U(t_y)\,\bar\psi_\gamma(y)\psi_\gamma(y)\rangle$$

Path integral average, or time-ordered in the operator language

According to the definition of the vertex function

$$\langle \psi(x_1)\psi(x_2)\psi^\dagger(x_3)\psi^\dagger(x_4)\rangle = -\langle\psi(x_1)\psi^\dagger(x_3)\rangle\langle\psi(x_2)\psi^\dagger(x_4)\rangle + \langle\psi(x_1)\psi^\dagger(x_4)\rangle\langle\psi(x_2)\psi^\dagger(x_3)\rangle$$

$$+\ i\int dx_1'\,dx_2'\,dx_3'\,dx_4'\ \langle\psi(x_1)\psi^\dagger(x_1')\rangle\langle\psi(x_2)\psi^\dagger(x_2')\rangle\langle\psi(x_3')\psi^\dagger(x_3)\rangle\langle\psi(x_4')\psi^\dagger(x_4)\rangle$$

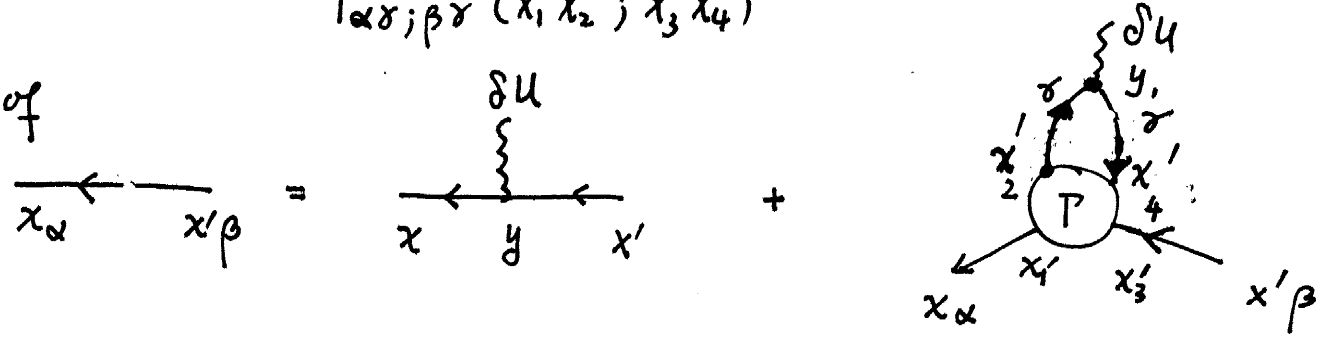
$$\Gamma(x_1'x_2';x_3'x_4') \quad \leftarrow \quad x_1 = x \quad x_3 = x' \quad x_2 = y \quad x_4 = y$$

$$\Rightarrow \delta G_{\alpha\beta}(x,x') = \delta_{\alpha\beta}\int d^4y\ \delta u(t_y)\ G(x-y)\,G(y-x')$$

$$-i\int d^4y\ dx_1'dx_2'dx_3'dx_4'\ \delta u(t_y)\ G(x-x_1')\,G(y-x_2')\ G(x_3'-x')\ G(x_4'-y)$$

$$\Gamma_{\alpha\gamma;\beta\gamma}(x_1'x_2';x_3'x_4')$$

Change of

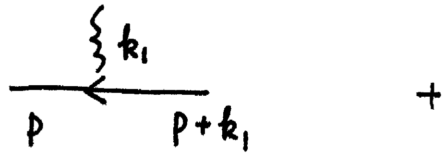


$\rightarrow$ Fourier

$$\delta G_{\alpha\beta}^{(p;p+k_1)} = \delta_{\alpha\beta}\,G(p)\ \delta u(\omega_1)\ G(p+k_1) - i\,G(p)\,G(p+k_1)$$

$$\times\int \Gamma_{\alpha\gamma;\beta\gamma}(p\,q;k_1)\ G(q)\,\delta u(\omega_1)\,G(q-k_1)\ \frac{d^4q}{(2\pi)^4}$$

where $k_1 = (0,\omega_1)$



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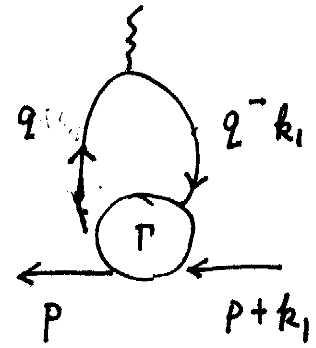


taking 1/2 of the trace

$$\delta G = G(p)\, \delta u(\omega_1)\, G(p+k_1) - i\, G(p)\, G(p+k_1)\, \frac{1}{2} \int \Gamma_{\alpha\beta;\alpha\beta}(pq;k_1)$$

$$\otimes\, G(q)\, \delta u(\omega_1)\, G(q-k_1)\, \frac{d^4q}{(2\pi)^4}$$

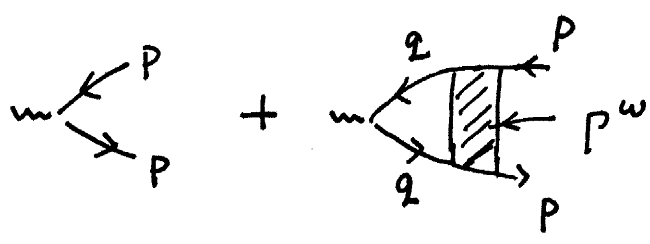
Set $\omega_1 \to 0$, which means we add $-\delta u$ into the frequency of $G(p)$

$$\Rightarrow \frac{\delta G}{\delta u} = - \frac{\partial G}{\partial \omega}$$

$$\Rightarrow \frac{\partial G}{\partial \omega} = -[G^2(p)]_\omega \left[1 - \frac{i}{2} \int \Gamma^\omega_{\alpha\beta;\alpha\beta}(pq)\, [G^2(q)]_\omega \frac{d^4q}{(2\pi)^4}\right.$$

We are considering the limit of $\lim_{\omega\to 0} \lim_{k\to 0}$

$$\Rightarrow \boxed{\frac{\partial G^{-1}(p)}{\partial \omega} = \frac{1}{Z} = 1 - \frac{i}{2} \int \frac{d^4q}{(2\pi)^4}\, \Gamma^\omega_{\alpha\beta;\alpha\beta}(pq)\, \left([G^2(q)\right]_\omega}$$



② Now let's consider the particle has charge

$$\frac{(\vec{P} - \frac{e}{c}\vec{A})^2}{2m}$$

$\rightarrow H' = -\frac{e}{mc} \int \psi^\dagger_\alpha(r)\, \hat{p}\, \psi_\alpha(r)\, d^3r$. Let's consider the static but slowly varying limit $\lim_{k\to 0} \lim_{\omega\to 0}$ ← Static small B field.

$\Rightarrow$ Similarly, we get the variation of

$$\delta G = -G(p)\ \frac{e}{mc}(\vec{p}\cdot\vec{A})\,G(p+k_2) + \frac{i}{2}G(p)G(p+k_2)\times$$

$$\int\frac{d^4q}{(2\pi)^4}\ \Gamma_{\alpha\beta;\alpha\beta}(p\,q\,;k_2)\,\frac{e}{mc}(\vec{q}\cdot\vec{A})\,G(q)\,G(q+k_2)$$

where $k_2 = (\vec{k},0)$. Set $\vec{A}$ to be small, this means

$$p \to p - \frac{e}{c}A$$

$$\Rightarrow\quad \delta G = \frac{\partial G}{\partial \vec{p}}\left(-\frac{e}{c}\vec{A}\right) \Rightarrow$$

$$-\frac{\partial G}{\partial \vec{p}} = -\left(G^2(p)\right)_k\frac{\vec{p}}{m} + \frac{i}{2}\left\{G^2(p)\right\}_k\int\frac{d^4q}{(2\pi)^4}\ \Gamma^k_{\alpha\beta;\alpha\beta}(pq)\,G^2(q)\frac{\vec{q}}{m}$$

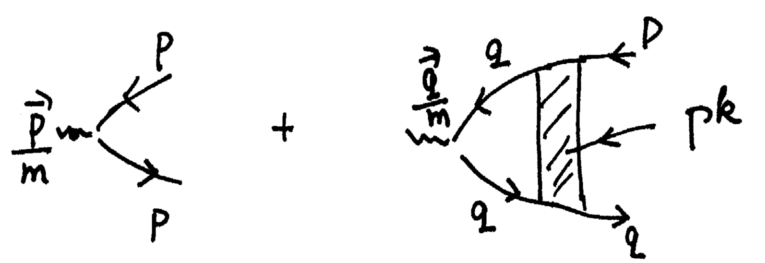
$$\|$$

$$\Rightarrow\ \frac{\partial G^{-1}(p)}{\partial \vec{p}} = -\frac{\vec{p}}{m} + \frac{i}{2}\int\frac{d^4q}{(2\pi)^4}\ \Gamma^k_{\alpha\beta;\alpha\beta}(pq)\left[G^2(q)\right]_k\frac{\vec{q}}{m}$$

$$\uparrow$$

$$-\frac{\vec{v}}{z} = -\frac{\vec{p}}{m^* z}\ ,\qquad \text{i.e}$$

$$\boxed{\frac{\vec{p}}{m^* z} = \frac{\vec{p}}{m} - \frac{i}{2}\int\frac{d^4q}{(2\pi)^4}\ \Gamma^k_{\alpha\beta;\alpha\beta}(pq)\left[G^2(q)\right]_k\frac{\vec{q}}{m}}$$



Current vertex

③ Let us consider a boust $H' = -\vec{v}\cdot\vec{P} = -\vec{v}\cdot\int \psi^{\dagger}_{\alpha}(r)\,(-i\hbar\nabla)\,\psi_{\alpha}(r)\,d\vec{r}$

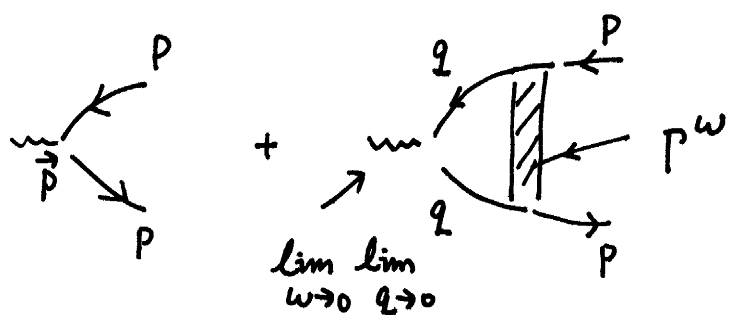
$\Rightarrow$ the variation of Green's function. We consider the $\lim\limits_{\omega\to 0}\lim\limits_{k\to 0}$ $\rightarrow$ small E-field

$$\delta G = -G(p)(\vec{P}\cdot\vec{v})G(p+k_1) + \frac{i}{2}G(p)G(p+k_1)\int \Gamma_{\alpha\beta;\alpha\beta}(pq;k_1)(\vec{q}\cdot\vec{v})G(q)G(q-k_2)\frac{d^4q}{(2\pi)^4}$$

where $k_1 = (0,\omega_1)$. The doppler shift the spectra $\varepsilon_p \to \varepsilon_p - \vec{p}\cdot\vec{v}$, which is equalivllent to $\omega \to \omega + \vec{p}\cdot\vec{v}$. Set $\lim\limits_{\omega_1\to 0}\lim\limits_{k_1\to 0}$

$$\delta G = \frac{\partial G}{\partial \omega}\vec{p}\cdot\vec{v} \Rightarrow -G^{-2}(p)\frac{\partial G}{\partial\omega}\vec{p} = \vec{p} - \frac{i}{2}\int \Gamma^{\omega}_{\alpha\beta}(pq)\,\vec{q}\,\left[G(q)\right]^2_{\omega}$$

$$\Rightarrow \underbrace{\frac{\partial G^{-1}}{\partial\omega}\vec{p}}_{\frac{\vec{p}}{Z}} = \boxed{\vec{p} - \frac{i}{2}\int\frac{d^4q}{(2\pi)^4}\,\Gamma^{\omega}_{\alpha\beta}(pq)\,\vec{q}\,\left[G(q)\right]^2_{\omega} = \frac{\vec{P}}{Z}}$$



④ Let us consider a variation of $\mathcal{U}(r)$ in the limit of $\lim\limits_{k\to 0}\lim\limits_{\omega\to 0}$ (static limit). $\Rightarrow$ $\underline{H' = \int dr\,\delta\mathcal{U}(r)\,\psi^{\dagger}(r)\psi(r)}$

$$\delta G = G(p)\,\delta\mathcal{U}(\vec{k})\,G(p+k_2) - \frac{i}{2}G(p)G(p+k_2)\int\frac{d^4q}{(2\pi)^4}\Gamma_{\alpha\beta;\alpha\beta}(pq;k_2)\,G(q)\,\delta\mathcal{U}(\vec{k})\,G(q+k_2)$$

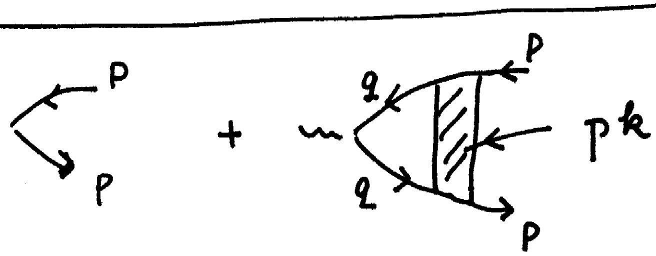
where $k_2 = (\vec{k}, 0)$

In the limit of $k \to 0$, this corresponds to $\mu + \delta U = \text{const}$

thus $\delta\mu = -\delta U$. $\Rightarrow$ $\delta G = \frac{\partial G}{\partial \mu}(-\delta U)$

$$\Rightarrow -\frac{\partial G}{\partial \mu} = G^2(p) \mp \frac{i}{2} G^2(p) \int \frac{d^4 q}{(2\pi)^4} \Gamma^k_{\alpha\beta;\alpha\beta}(p\, q) \left[G^2(q)\right]_k$$

$$\Rightarrow \boxed{\frac{\partial G^{-1}}{\partial \mu} = 1 - \frac{i}{2} \int \frac{d^4 q}{(2\pi)^4} \Gamma^k_{\alpha\beta;\alpha\beta}(p\, q) \left[G^2(q)\right]_k}$$

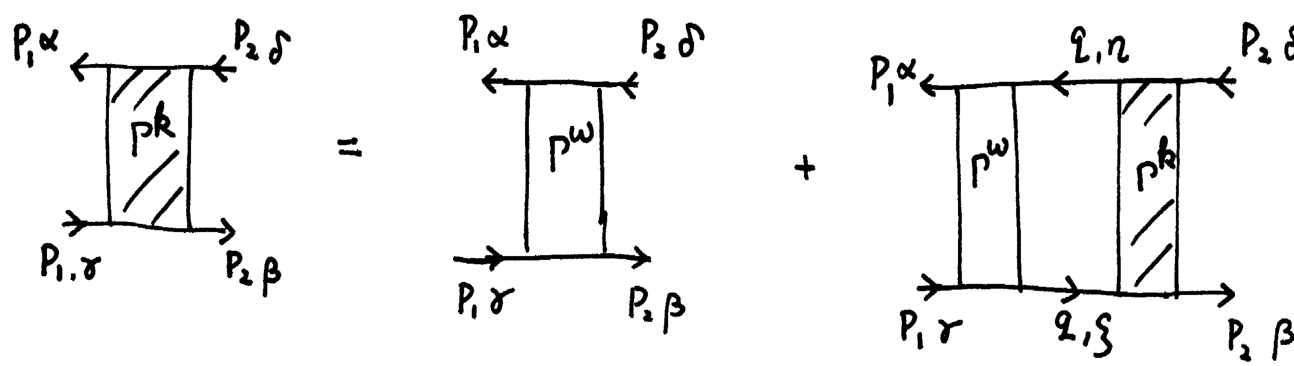


scalar vertex

§ Fermi surface volume

correct close to Fermi Surface

we need to use $\Gamma^k_{\alpha\beta;\gamma\delta}(P_1, P_2) = \Gamma^\omega_{\alpha\beta;\gamma\delta}(P_1; P_2) - \frac{k_F^2 Z^2}{(2\pi)^3 v_F} \int \Gamma^\omega_{\alpha\xi;\gamma\eta}(P_1 q)\, \Gamma^k_{\eta\beta;\xi\delta}(q, P_2)$



Plug in this Eq into

$$\frac{\vec{p}}{m^* Z} = \frac{\vec{p}}{m} - \frac{i}{2} \int \frac{d^4 q}{(2\pi)^4} \Gamma^k_{\alpha\beta;\alpha\beta}(p\, q) \left[G^2(q)\right]_k \frac{\vec{q}}{m}$$

$$\Rightarrow \quad \frac{\vec{p}}{m^* Z} - \frac{\vec{p}}{m} = -\frac{i}{2}\int \frac{d^4q}{(2\pi)^4}\, \Gamma^{\omega}_{\alpha\beta;\alpha\beta}(p\,q)\, \frac{\vec{q}}{m}\, \left[G^2(q)\right]_k$$

$$+ \frac{1}{2}\, \frac{k_F^2 Z^2}{(2\pi)^3 v_F} \int d\Omega\; \Gamma^{\omega}_{\alpha\beta;\alpha\beta}(p\,q) \left(-\frac{\vec{q}}{m^* Z} + \frac{\vec{q}}{m}\right)$$

by using the previous Eq again.

plug in $\left[G^2(p)\right]_k = \left[G^2(p)\right]_\omega - \dfrac{2\pi i Z^2}{v_F}\, \delta(\omega)\, \delta(|p| - k_F)$

$$\Rightarrow \quad \frac{\vec{p}}{m^* Z} = \frac{\vec{p}}{m} - \frac{i}{2}\int \frac{d^4q}{(2\pi)^4}\, \Gamma^{\omega}_{\alpha\beta;\alpha\beta}(pq)\, \frac{\vec{q}}{m}\, \left[G^2(p)\right]_\omega$$

$$- \frac{\pi Z^2}{v_F} \int \frac{d^4q}{(2\pi)^4}\, \Gamma^{\omega}_{\alpha\beta;\alpha\beta}(p,q)\, \delta(\omega)\, \delta(|q| - k_F)\, \frac{\vec{q}}{m}$$

$$+ \frac{1}{2}\, \frac{k_F^2 Z^2}{(2\pi)^3 v_F} \int d\Omega\; \Gamma^{\omega}_{\alpha\beta;\alpha\beta}(p,q) \left(\frac{-\vec{q}}{m^* Z} + \frac{\vec{q}}{m}\right)$$

The first line $= \dfrac{\vec{p}}{Zm}$

The second line $= -\dfrac{\pi Z^2}{v_F}\, \dfrac{1}{(2\pi)^4} \cdot k_F^2 \displaystyle\int d\Omega\; \Gamma^{\omega}_{\alpha\beta\,\alpha\beta}(p,q)\, \frac{\vec{q}}{m}$

$$\Rightarrow \quad \frac{\vec{p}}{m^* Z} = \frac{\vec{p}}{Zm} - \frac{1}{2}\, \frac{k_F^2 Z}{(2\pi)^3 v_F} \int d\Omega\; \Gamma^{\omega}_{\alpha\beta;\alpha\beta}(p,q)\, \frac{\vec{q}}{m^*}$$

$\Rightarrow$

$$\boxed{\frac{1}{m^*} = \frac{1}{m} - \frac{k_F^2}{2(2\pi)^3} \int d\Omega\; Z^2\, \Gamma^{\omega}_{\alpha\beta;\alpha\beta}(p.q)\, \frac{\vec{q}}{k_F}}$$

$$\frac{Z^2}{2}\Gamma^{\omega}_{\alpha\beta;\alpha\beta}(p,q) = 2f^{S}(p,q)$$

$$\Rightarrow \frac{1}{m^*} = \frac{1}{m} - \frac{\tilde{N}_0}{m}\frac{1}{3}f_1^S$$

where $f_1^S = \frac{1}{2}\int_{-1}^{1} d\cos\theta \; f^S(\cos\theta)\cos\theta$

and $\tilde{N}_0 = \frac{2\cdot k_f^2\cdot 4\pi}{(2\pi)^3 v_f^0} = \frac{2k_f m\cdot 4\pi}{(2\pi)^3}$

where $\tilde{N}_0$ is the DOS of free Fermi gas.

or $\frac{1}{m^*} = \frac{1}{m} - \frac{N_0}{m^*}\frac{1}{3}f_1^S \Rightarrow \frac{1+\frac{F_1^S}{3}}{m^*} = \frac{1}{m}$ or $\boxed{\frac{m^*}{m} = 1+\frac{F_1^S}{3}}$

N_0 is the DOS of Fermi liquid.

§ Luttinger theorem:

we prove the Fermi wave vector does not change in Fermi liquid.

Consider $G(p,\omega) = \frac{Z}{\omega - v_F(|p|-k_F) + i\,\mathrm{sgn}\,\omega\,\eta} + \cdots$

all of Z, k_F, v_F depends on μ. However, the contribution to $\frac{\partial G}{\partial \mu}$ mainly comes from $\partial k_F/\partial\mu$:

$$\frac{\partial G}{\partial \mu} = \frac{\partial Z/\partial\mu}{\omega - v_F(|p|-k_F) + i\,\mathrm{sgn}\,\omega} + \frac{-Z}{[\omega - v_F(|p|-k_F)+i\,\mathrm{sgn}\,\omega]^2}\left[-\frac{\partial v_F}{\partial\mu}(|p|-k_F) + v_F\frac{\partial k_F}{\partial\mu}\right]$$

as $|p|\to k_F$ and $\omega\to 0$, the last term dominates

$$\Rightarrow \frac{\partial G}{\partial \mu} \approx -G^2\frac{v_F}{Z}\frac{dk_f}{d\mu} \quad \text{or} \quad \boxed{v_F\frac{dk_f}{\partial\mu} = Z\left(\frac{\partial G^{-1}}{\partial\mu}\right)\Big|_{\substack{\omega=0\\|p|=k_F}}}$$

Plug in $\frac{\partial G^{-1}(p)}{\partial \mu} = 1 - \frac{i}{2}\int \Gamma^{k}_{\alpha\beta;\alpha\beta}(p\,q)\,\{G^2(q)\}_k \,\frac{d^4q}{(2\pi)^4}$

and $\Gamma^{k}_{\alpha\beta;\alpha\beta}(p\,q) = \Gamma^{\omega}_{\alpha\beta\alpha\beta}(p\,q) - \frac{k_F^2 z^2}{(2\pi)^3 v_F}\int d\Omega_{q'}\, \Gamma^{\omega}_{\alpha\xi;\alpha\eta}(p\,q')\, \Gamma^{k}_{\eta\beta;\xi\beta}(q',q)$

$$\Rightarrow \frac{v_F}{z}\frac{dk_F}{\partial\mu} = \left.\frac{\partial G^{-1}(p)}{\partial\mu}\right|_{\substack{\omega=0\\ |p|=k_F}} = 1 - \frac{i}{2}\int \Gamma^{\omega}_{\alpha\beta;\alpha\beta}(p\,q)\,\{G^2(q)\}_k\,\frac{d^4q}{(2\pi)^4}$$

$$+\frac{i}{2}\,\frac{k_F^2 z^2}{(2\pi)^3 v_F}\int d\Omega_{q'}\int\frac{d^4q}{(2\pi)^4}\,\Gamma^{\omega}_{\alpha\xi;\alpha\eta}(p\,q')\,\Gamma^{k}_{\eta\beta;\xi\beta}(q',q)\,\{G^2(q)\}_k$$

the second line: after integrate $\frac{d^4q}{(2\pi)^4}$

$$\rightarrow -\frac{k_F^2 z^2}{(2\pi)^3 v_F}\int d\Omega_{q'}\,\Gamma^{\omega}_{\alpha\xi;\alpha\eta}(p\,q')\,\frac{1}{2}\Big[\underbrace{\frac{\partial G^{-1}(q')}{\partial\mu} - 1}_{\left(\frac{v_F}{z}\frac{dk_F}{d\mu} - 1\right)}\Big]\,\delta_{\eta\xi}$$

(η has to be equal to ξ, otherwise, vertex $\Gamma_{\alpha\xi;\alpha\eta}$ and $\Gamma_{\eta\beta;\xi\beta}$ are not spin conserved

$$\Rightarrow \frac{v_F}{z}\frac{dk_F}{d\mu} = 1 - \frac{i}{2}\int \Gamma^{\omega}_{\alpha\beta;\alpha\beta}(p\,q)\,\{G^2(q)\}_k\,\frac{d^4q}{(2\pi)^4}$$

$$- \frac{1}{2}\,\frac{k_F^2 z^2}{(2\pi)^3 v_F}\int \Gamma^{\omega}_{\alpha\beta;\alpha\beta}(p\,q)\left(\frac{v_F}{z}\frac{dk_F}{d\mu} - 1\right) d\Omega_q$$

using $\{G^2(q)\}_k = \{G^2(q)\}_\omega - \frac{2\pi i z^2}{v_F}\,\delta(\varepsilon)\,\delta(|q| - k_F)$

The first line: $1 - \frac{i}{2}\int \Gamma^{\omega}_{\alpha\beta;\alpha\beta}(p.q)\,\{G^2(q)\}_\omega\,\frac{d^4q}{(2\pi)^4} + \frac{i}{2}\,\frac{2\pi i z^2}{v_F}$

$$\otimes\int \Gamma^{\omega}_{\alpha\beta;\alpha\beta}(p\,q)\,\delta(\omega_q)\,\delta(|q| - k_F)\,\frac{d^4q}{(2\pi)^4} = \frac{1}{z} - \frac{2\pi z^2}{2 v_F}\,\frac{k_F^2}{(2\pi)^4}\int \Gamma^{\omega}_{\alpha\beta}(p\,q)\, d\Omega_q$$

$$\frac{v_F}{z}\frac{dk_F}{d\mu} = \frac{1}{z} - \frac{k_F^2 z^2}{2(2\pi)^3 v_F}\,\frac{v_F}{z}\frac{dk_F}{d\mu}\int \Gamma^{\omega}_{\alpha\beta;\alpha\beta}(p,q)\,d\Omega_q$$

$$\Rightarrow \boxed{v_F\frac{dk_F}{d\mu} = \Big(1 + \underbrace{\frac{k_F^2}{2(2\pi)^3 v_F}}_{\frac{N_0}{4}}\int \underbrace{z^2\Gamma^{\omega}_{\alpha\beta;\alpha\beta}(p,q)}_{4 f_0^s}\,d\Omega_q\Big)^{-1}} = \frac{1}{1+F_0}$$

$$v_F = \frac{d\epsilon_F}{dk_F} \Rightarrow \frac{d\epsilon_F}{d\mu} = \frac{1}{1+F_0} \quad \text{i.e}\quad \boxed{\frac{dn}{d\mu} = \frac{dn}{d\epsilon}\frac{d\epsilon}{d\mu} = \frac{N_0}{1+F_0}}$$

From the definition of Green's function

$$n(k) = \langle \psi^\dagger(k)\psi(k)\rangle = -\langle T[\psi(k,t)\psi^\dagger(k,t+0^+)]\rangle = -i\,G(k,-0^+)$$

$$= -i\int\frac{d\omega}{2\pi}\,G(p,\omega)\,e^{i\omega\eta} \quad \text{where } \eta = 0^+$$

Taking into account spin $\Rightarrow$

$$n = \frac{N}{V} = -2i\int\frac{d^3p}{(2\pi)^3}\frac{d\omega}{2\pi}\,G(p,\omega)\,e^{i\omega\eta} \leftarrow \eta\to 0^+$$

$$\frac{dn}{d\mu} = -2i\int\frac{\partial G(p)}{\partial\mu}\frac{d^4p}{(2\pi)^4} = 2i\int\frac{\partial G^{-1}(p)}{\partial\mu}\{G^2(p)\}_k\frac{d^4p}{(2\pi)^4}$$

Plug in $\displaystyle \frac{\partial G^{-1}(p)}{\partial\mu} = 1 - \frac{i}{2}\int\Gamma^{k}_{\alpha\beta;\alpha\beta}(p\,q)\{G^2(q)\}_k\frac{d^4q}{(2\pi)^4}$

$$\Rightarrow \frac{dn}{d\mu} = 2i\int\{G^2(p)\}_k\frac{d^4p}{(2\pi)^4} + \int\Gamma^{k}_{\alpha\beta;\alpha\beta}(p\,q)\{G^2(q)\}_k\frac{d^4q}{(2\pi)^4}\frac{d^4p}{(2\pi)^4}\{G^2(p)\}_k$$

Again, using

$$\Gamma^{k}_{\alpha\beta\,\alpha\beta}(p\,q) = \Gamma^{\omega}_{\alpha\beta\,\alpha\beta}(p\,q) - \frac{k_F^2 z^2}{(2\pi)^3 v_F}\int d\Omega_{q'}\,\Gamma^{\omega}_{\alpha\xi;\alpha\eta}(p\,q')\,\Gamma^{k}_{\eta\beta;\xi\beta}(q',q)$$

the second term goes

$$\int \{G^2(p)\}_k \, \Gamma^{\omega}_{\alpha\beta;\alpha\beta}(p\,q)\{G^2(q)\}_k \frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4} - \frac{k_F^2 Z^2}{(2\pi)^3 v_F}\int \frac{d^4p}{(2\pi)^4}\{G^2(p)\}_k$$

$$\otimes \int d\Omega_{q'}\, \Gamma^{\omega}_{\alpha\xi;\alpha\eta}(p\,q') \underbrace{\int \frac{d^4q}{(2\pi)^4}\, \Gamma^{k}_{\eta\beta;\xi\beta}(q',q)\{G^2(q)\}_k}_{\left[\frac{\partial G^{-1}(q')}{\partial\mu} - 1\right] 2i \frac{\delta_{\eta\xi}}{z}}$$

$$\Rightarrow \frac{dn}{d\mu} = 2i\int \{G^2(p)\}_k \frac{d^4p}{(2\pi)^4} + \int \{G^2(p)\}_k \,\Gamma^{\omega}_{\alpha\beta,\alpha\beta}(p\,q)\{G^2(q)\}_k \frac{d^4q}{(2\pi)^4}\frac{d^4p}{(2\pi)^4}$$

$$- \frac{i k_F^2 Z^2}{(2\pi)^3 v_F}\int \frac{d^4p}{(2\pi)^4}\{G^2(p)\}_k \, \Gamma^{\omega}_{\alpha\beta\,\alpha\beta}(p,q') \left[\frac{v}{z}\frac{dk_F}{d\mu} - 1\right]$$

$$\widehat{d\Omega_{q'}}$$

us $\{G^2(p)\}_k = \{G^2(p)\}_\omega - \frac{2\pi i Z^2}{v_F}\delta(\omega)\delta(|p| - k_F)$

The first term $\Rightarrow$

$$2i\int \{G^2(p)\}_\omega \frac{d^4p}{(2\pi)^4} + \frac{4\pi z^2}{v_F}\frac{4\pi k_F^2}{(2\pi)^4} = 2i\int [G^2(p)]_\omega \frac{d^4p}{(2\pi)^4} + \frac{8\pi Z^2 k_F^2}{(2\pi)^3 v_F}$$

the second term

$$\int \{G^2(p)\}_\omega \,\Gamma^{\omega}_{\alpha\beta\alpha\beta}(p\,q)\{G^2(q)\}_\omega \frac{d^4q}{(2\pi)^4}\frac{d^4p}{(2\pi)^4} - 2\times\frac{2\pi i Z^2}{v_F}\frac{k_F^2}{(2\pi)^4}\underbrace{\int \frac{d^4q}{(2\pi)^4}}_{\int d\Omega_p}\Gamma^{\omega}_{\alpha\beta,\alpha\beta}(p,q)\{G^2(q)\}_\omega \quad (*)$$

$$+ \left(\frac{2\pi i Z^2}{v_F}\right)^2 \frac{(k_F^2)^2}{(2\pi)^4(2\pi)^4}\int d\Omega_p\, d\Omega_q\, \Gamma^{\omega}_{\alpha\beta,\alpha\beta}(p,q) \quad (**)$$

the third one

$$-\frac{-ik_F^2 z^2}{(2\pi)^3 v_F}\int\frac{d^4p}{(2\pi)^4}\int d\Omega_{q'}\,\{G^2(p)\}_\omega\,\Gamma^\omega_{\alpha\beta\,\alpha\beta}(p\,q')\left[\frac{v}{z}\frac{dk_f}{d\mu}-1\right] \quad (*)$$

$$-\frac{2\pi\,k_F^2 z^4}{(2\pi)^3 v_F^2}\,\frac{k_F^2}{(2\pi)^4}\int d\Omega_p\int d\Omega_{q'}\;\Gamma^\omega_{\alpha\beta,\alpha\beta}(p\,q')\left[\frac{v}{z}\frac{dk_f}{d\mu}-1\right] \quad (**)$$

two (*) terms combine together

$$\frac{-ik_f^2 z^2}{(2\pi)^3 v_F}\int\frac{d^4p}{(2\pi)^4}\int d\Omega_{q'}\;\underbrace{\{G^2(p)\}_\omega\,\Gamma^\omega_{\alpha\beta\,\alpha\beta}(p\,q')}_{2i\left(\frac{1}{z}-1\right)}\left[\frac{v}{z}\frac{dk_f}{d\mu}+1\right]$$

$$=\frac{2k_f^2 z^2}{(2\pi)^3 v_F}\int d\Omega_{q'}\left(\frac{1}{z}-1\right)\left(\frac{v}{z}\frac{dk_f}{d\mu}+1\right)=\frac{8\pi k_f^2 z(1-z)}{(2\pi)^3 v_F}\left(\frac{v}{z}\frac{dk_f}{d\mu}+1\right)$$

two (**) terms combine together

$$\frac{-k_F^4 z^4}{(2\pi)^6 v_F^2}\int d\Omega_p\,d\Omega_{q'}\;\Gamma^\omega_{\alpha\beta\,\alpha\beta}(p\,q')\left[\frac{v}{z}\frac{dk_f}{d\mu}\right]=-4\pi\left[\frac{k_F^2 z^2}{(2\pi)^3 v_F}\right]^2$$

$$\otimes\int\Gamma^\omega_{\alpha\beta\,\alpha\beta}(p\,q')\,d\Omega_{q'}\left[\frac{v_F}{z}\frac{dk_f}{d\mu}\right]$$

$$\Rightarrow\ \frac{d(n)}{d\mu}=2i\int\{G^2(p)\}_\omega\frac{d^4p}{(2\pi)^4}+\int[G^2(p)]_\omega\,\Gamma^\omega_{\alpha\beta,\alpha\beta}(p\,q)\,[G^2(q)]_\omega\frac{d^4p\,d^4q}{(2\pi)^4(2\pi)^4}$$

$$+\frac{8\pi k_F^2 z}{(2\pi)^3 v_F}+\frac{8\pi k_f^2(1-z)}{(2\pi)^3 v_F}\,v_F\frac{dk_f}{d\mu}-4\pi\left(\frac{k_f^2 z^2}{(2\pi)^3 v_F}\right)^2\int\Gamma^\omega_{\alpha\beta\,\alpha\beta}(p\,q)\,d\Omega_q\left(\frac{v_F}{z}\frac{dk_f}{d\mu}\right)$$

only this term survives

The first two terms : $\Rightarrow 0$.

$$2i \int \frac{d^4p}{(2\pi)^4} \left[G^2(p)\right]_\omega \int \frac{d^4q}{(2\pi)^4} \left(1 - \frac{i}{2} \int \frac{d^4q}{(2\pi)^4} \Gamma^{\omega}_{\alpha\beta;\alpha\beta}(p\,q) \left[G^2(q)\right]_\omega \right]$$

$$= -2i \int \frac{d^4p}{(2\pi)^4} \frac{\partial G}{\partial \omega} = -2i \int \left[G(\vec{p}, \omega = +\infty) - G(\vec{p}, \omega = -\infty) \right] \frac{d^3p}{(2\pi)^3} = 0$$

this equals to the shift of total particle number due to shift of energy zero. $\left(\varepsilon \to \varepsilon + u \Rightarrow \frac{\partial G}{\partial \varepsilon} = -\frac{\partial G}{\partial u} \right)$.

The other 3-terms can be organized as

$$\frac{8\pi k_F^2 z}{(2\pi)^3 v_F} \left\{ 1 - \left(1 + \frac{k_F^2 z^2}{2(2\pi)^3 v_F} \int \Gamma^{\omega}_{\alpha\beta;\alpha\beta}(p\,q)\, d\Omega_q \right) \left[v_F \frac{dk_F}{d\mu} \right] \right\}$$

"= 0"

$$+ \frac{8\pi k_F^2}{(2\pi)^3 v_F} v_F \frac{dk_F}{d\mu} = \frac{8\pi k_F^2}{(2\pi)^3} \frac{dk_F}{d\mu}$$

$$\Rightarrow \frac{dn}{d\mu} = \frac{8\pi k_F^2}{(2\pi)^3} \frac{dk_F}{d\mu} \Rightarrow n = \frac{8\pi}{3} \frac{k_F^3}{(2\pi)^3}$$

$\Rightarrow k_F$ is the same as no-interacting system!

Lecture 11.5 Landau Fermi liquid (V)

§ More on Luttinger theorem — Luttinger & Ward

Another expression: $\frac{N}{V} = 2i\int\left[\frac{\partial}{\partial\omega}\ln G(p) - G(p)\frac{\partial}{\partial\omega}\Sigma(p)\right]e^{i\omega\eta}\frac{d^4p}{(2\pi)^4}$

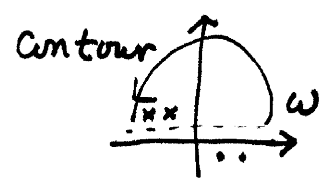
Proof:

$$\frac{\partial}{\partial\omega}\ln G(p) - G(p)\frac{\partial}{\partial\omega}\Sigma(p) = G^{-1}(p)\frac{\partial}{\partial\omega}G(p) - G\frac{\partial}{\partial\omega}\Sigma = (\omega-\xi-\Sigma)\frac{\partial}{\partial\omega}G$$

$$-G\frac{\partial}{\partial\omega}\Sigma = \omega\frac{\partial}{\partial\omega}G - \xi\frac{\partial}{\partial\omega}G - \frac{\partial}{\partial\omega}(\Sigma G) = \frac{\partial}{\partial\omega}(\omega G) - \frac{\partial}{\partial\omega}(\Sigma G) - \frac{\partial}{\partial\omega}(\xi G) - G$$

$$= \frac{\partial}{\partial\omega}[(\omega-\xi-\Sigma)G] - G \Rightarrow \boxed{\frac{N}{V} = -2i\int G\, e^{i\omega\eta}\frac{d^4p}{(2\pi)^4}}$$

— The second term of the integral can be proved to be zero. Let us postpone its proof for a while. $\frac{N}{V} = 2i\int\frac{\partial}{\partial\omega}\ln G(p)\, e^{i\omega\eta}\frac{d^4p}{(2\pi)^4}$



G is not analytic. $G_R = \begin{cases} G(\epsilon) & \epsilon>0 \\ G^*(\epsilon) & \epsilon<0, \end{cases}$ G_R is analytic for ω in the upper plan

So

$$\frac{N}{V} = 2i\left[\int_0^{\infty}\frac{d\omega}{2\pi}\int\frac{d^3\vec{p}}{(2\pi)^3}\frac{\partial}{\partial\omega}\ln G_R(p) + \int_{-\infty}^{0}\frac{d\omega}{2\pi}\int\frac{d^3\vec{p}}{(2\pi)^3}\frac{\partial}{\partial\omega}\ln G(p)\right]$$

$$= 2i\left[\underbrace{\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\int\frac{d^3\vec{p}}{(2\pi)^3}\frac{\partial}{\partial\omega}\ln G_R(p)}_{=0} + \int_{-\infty}^{0}\frac{d\omega}{2\pi}\int\frac{d^3\vec{p}}{(2\pi)^3}\frac{\partial}{\partial\omega}\ln\frac{G(\vec{p},\omega)}{G^*(\vec{p},\omega)}\right]$$

$$= \frac{2i}{2\pi}\int\frac{d^3\vec{p}}{(2\pi)^3}\ln\frac{G(p,\omega)}{G^*(p,\omega)}\Bigg|_{-\infty}^{0} = -\frac{2}{\pi}\int\frac{d^3\vec{p}}{(2\pi)^3}[\varphi(0)-\varphi(-\infty)],$$

Where φ denotes the phase of G. We need to consider the variation of φ from $\omega=-\infty$ to $\omega=0$. From the Lehmann Rep. we have

$$G(\omega)= e^{\beta\Omega}\sum_{n,m}\langle n|\psi|m\rangle\langle m|\psi^{\dagger}|n\rangle\left\{\frac{e^{-\beta E_n}}{\omega+E_n-E_m+i\eta}+\frac{e^{-\beta E_m}}{\omega+E_n-E_m-i\eta}\right\}$$

$$\xrightarrow{T=0K} =\sum_m \frac{\langle 0|\psi|m\rangle\langle m|\psi^{\dagger}|0\rangle}{\omega-E_m+i\eta}+\sum_n\frac{\langle 0|\psi^{\dagger}|n\rangle\langle n|\psi|0\rangle}{\omega+E_n-i\eta}$$

$$=\int_0^{+\infty}dE\ \frac{A(E)}{\omega-E+i\eta}+\frac{B(E)}{\omega+E-i\eta},$$

$\boxed{A, B \text{ are positive}}$

where
$$A(E)=\sum_m\langle 0|\psi|m\rangle\langle m|\psi^{\dagger}|0\rangle\,\delta(E-E_m)$$
$$B(E)=\sum_n\langle 0|\psi^{\dagger}|n\rangle\langle n|\psi|0\rangle\,\delta(E-E_n).$$

As $\omega\to\infty \Rightarrow G(\omega)\to\frac{1}{\omega}\int_0^{+\infty}(A(E)+B(E))\,dE$

$$=\sum_{m,n}\frac{1}{\omega}\left\{\langle 0|\psi|m\rangle\langle m|\psi^{\dagger}|0\rangle+\langle 0|\psi^{\dagger}|n\rangle\langle n|\psi|0\rangle\right\}=\frac{1}{\omega}\langle 0|\psi\psi^{\dagger}+\psi^{\dagger}\psi|0\rangle$$

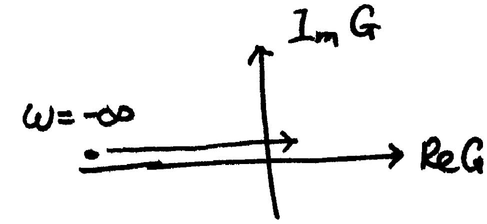
$$=\frac{1}{\omega}\quad \text{i.e.}\quad \boxed{G(\omega)\to\frac{1}{\omega},\ \text{as}\ \omega\to\infty}.$$

$$\mathrm{Re}\,G(\omega)=P\int_0^{+\infty}dE\ \frac{A(E)}{\omega-E}+\frac{B(E)}{\omega+E}$$

$$\mathrm{Im}\,G(\omega)=\begin{cases}-\pi A(\omega) & \text{for } \omega>0\\ \pi B(-\omega) & \text{for } \omega<0\end{cases}.$$

As $\omega\to-\infty$, $\mathrm{Re}\,G(\omega)\to-\frac{1}{\omega}<0$. $\mathrm{Im}\,G(\omega)$ at $\omega<0$, describe the hole excitation below the Fermi surface. If we assume that

the hole-excitation is bounded from below, then $\mathrm{Im}\, G(\omega) > 0$ decays faster. $\Rightarrow$ $\varphi(\omega \to -\infty) = \pi$.
towards 0



We set $\mathrm{Im}\, G(\omega) = 0$ at $\omega = 0$, because all the E_n, E_m in the Lenhmann Rep > 0. The φ of $G(\omega=0)$ is determined by the real part $\mathrm{Re}\, G(\omega=0)$. If $\mathrm{Re}\, G(\omega=0) < 0 \Rightarrow \varphi = \pi$,

$\mathrm{Re}\, G(\omega=0) > 0 \Rightarrow \varphi = 0$.

$$\Rightarrow \boxed{\frac{N}{V} = 2 \int_{\mathrm{Re}\, G(p,\omega=0) > 0} \frac{d^3 p}{(2\pi)^3}}$$

The region $G(p,\omega=0)$ is bound by a surface of either zero or diveregence.

- Vanishing of $G(p,\omega=0)$ corresponds to $\Sigma \to \infty$, which corresponds to superconductivity $G = \frac{u_k^2}{\omega - \sqrt{\varepsilon_k^2 + \Delta^2}} + \frac{v_k^2}{\omega + \sqrt{\varepsilon_k^2 + \Delta^2}}$.

$\Rightarrow G(0,k) = \frac{-u_k^2 + v_k^2}{E_k}$, (we do not consider this possibility here).

On the other hand, $G(p,\omega=0) = \frac{Z}{\omega - \xi_p + i\eta} + \cdots \to +\infty$

corresponds to location of $\xi_p = 0$, i.e. <u>the location of FS.</u>

$$G(p,\omega=0) \to -\frac{Z}{\xi_p} \to +\infty \quad \xi_p < 0$$

$$-\infty \quad \text{for } \xi_p > 0 .$$

> This proof does not assume the isotropy of FS. Interaction may deform the shape but not its volume!