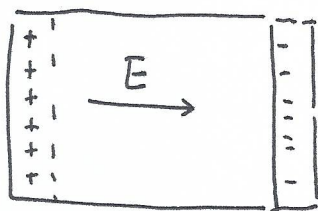


Interacting electron gas

§ Plasmon

Consider the fluctuation of electron density in the background of the positive charge background.


 Δx

$$E = 4\pi\sigma = -4\pi en_0 \Delta x$$

$$\Delta \ddot{x} = \frac{eE}{m} = -\frac{4\pi e^2 n}{m} \Delta x \Rightarrow \omega_p^2 = \frac{4\pi n e^2}{m}$$

$$\Delta \ddot{x} = -\omega_p^2 \Delta x$$

$$\frac{\partial}{\partial t} n + \nabla \cdot \vec{j} = 0, \text{ with } \vec{j} = n\vec{v} \text{ and } n = n_0 + \delta n$$

where n_0 is the

average electron density.

It cancels the positive background charge.

$$\frac{\partial^2}{\partial t^2} \delta n + \nabla \cdot \frac{\partial}{\partial t} (n\vec{v}) = \frac{\partial^2}{\partial t^2} \delta n + \nabla \cdot \left[\frac{d}{dt} (n\vec{v}) - \vec{v} \cdot \nabla (n\vec{v}) \right] = 0$$

$$\frac{d}{dt} (n\vec{v}) = \frac{en\vec{E}}{m}$$

$$\nabla \cdot \frac{d}{dt} (n\vec{v}) = \frac{e}{m} (n \nabla \cdot \vec{E} + \nabla \delta n \cdot \vec{E}) = \frac{4\pi e^2}{m} n_0 \delta n + \frac{e}{m} \nabla \delta n \cdot \vec{E}$$

$$\Rightarrow \frac{\partial^2}{\partial t^2} \delta n + \omega_p^2 \delta n + \underbrace{\frac{e}{m} \nabla \delta n \cdot \vec{E} - \nabla (\vec{v} \cdot \nabla (n\vec{v}))}_{\text{at long-wave length limit, negligible}} = 0$$

at long-wave length limit, negligible
 $\delta n = \tilde{n} e^{i\vec{q} \cdot \vec{r}}, \vec{v} = \vec{v}_0 e^{i\vec{q} \cdot \vec{r}}$

$$\Rightarrow \frac{\partial^2}{\partial t^2} \delta n + \omega_p^2 \delta n = 0$$

estimation of ω_p : Compare with Rydberg energy $E_R = \frac{e^4 m}{\hbar^2}$

$$\frac{\hbar \omega_p}{E_R} = \sqrt{4\pi} (n a_0^3)^{1/2}$$

$$r_s = \frac{\bar{n}^{1/3}}{a_0}$$

$$= \sqrt{4\pi} r_s^{-3/2}$$

In metal $r_s \approx 1 \sim 10$, $\hbar \omega_p$ is a very large energy scale
5 ~ 30 eV

• estimate SrTiO_3 $\hbar \omega_p = ?$

1 eV / $\hbar = 1.5 \times 10^{15}$ Hz, ω_p can often reach 10^{16} Hz in most metals.
which is comparable to Bohr frequency in atom.

* the relaxation time τ , typically 10^{-15} secs.
longer than
the condition $\omega_p \tau \gg 1$ is satisfied.

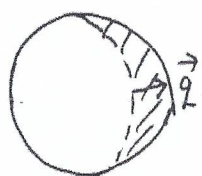
§2. particle-hole continuum:

$$C_{\vec{k}+\vec{q}}^\dagger \text{ or } C_{\vec{k}}^\dagger$$

Let's consider an excitation of moving an electron at \vec{k} to $\vec{k} + \vec{q}$
(inside the FS) (outside the FS)

$$\text{the excitation energy is } \hbar \omega_{\vec{k}\vec{q}} = \frac{\hbar^2}{2m} (q^2 + 2\vec{k} \cdot \vec{q})$$

① For $q < 2k_f$,



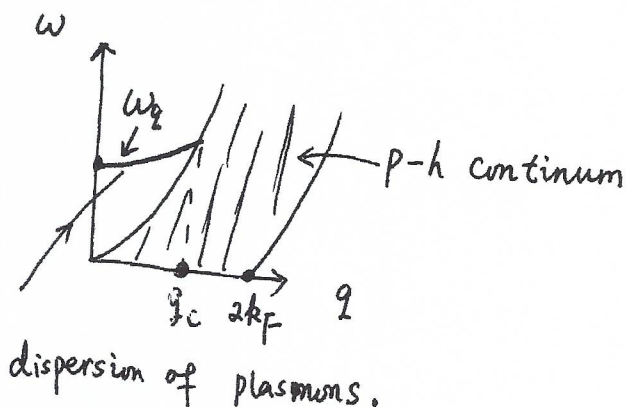
only particle inside the shaded area

$$\text{can be excited, } \omega_{\vec{k}\vec{q}, \text{max}} = \frac{\hbar^2}{2m} (q^2 + 2k_f q)$$

$$\omega_{\vec{k}\vec{q}, \text{min}} = 0$$

② For $q > 2k_F$, all the particle can be excited to create p-h excitation.

$$\hbar \omega_{k,q, \max} = \frac{\hbar^2}{2m} (q^2 + 2kq) \quad \hbar \omega_{k,q, \min} = \frac{\hbar^2}{2m} (q^2 - 2kq)$$



At $q < q_c$, plasmon excitation lies outside the p-h. continuum. it is not damped.

At $q > q_c$, plasmon can decay into p-h continuum.

§3: Random phase approximation:

$$H = - \sum_{i=1}^N \frac{\hbar^2 \nabla_i^2}{2m} + \frac{1}{2} \sum_{i,j}' \frac{e^2}{|r_i - r_j|} + H_{\text{positive charge background}}$$

Expand Coulomb interaction by using Fourier transform

$$\begin{aligned} \frac{1}{2} \sum_{i \neq j} \frac{e^2}{|r_i - r_j|} &= \frac{1}{2} \sum_q \sum_{i \neq j} V(q) e^{i \vec{q} \cdot (\vec{r}_i - \vec{r}_j)} = \frac{1}{2} \sum_q V(q) \left(\sum_i e^{i \vec{q} \cdot \vec{r}_i} \sum_j e^{-i \vec{q} \cdot \vec{r}_j} - N \right) \\ &= \frac{1}{2} \sum_q V(q) [P_q^\dagger P_q - N] \end{aligned}$$

$$\text{where } P_q = \sum_j e^{-i \vec{q} \cdot \vec{r}_j}, \quad \text{i.e. } \rho(\mathbf{r}) = \sum_{i=1}^N \delta(\mathbf{r} - \mathbf{r}_i)$$

the $q=0$ component $P_{q=0} = N$, $\Rightarrow \frac{V(0)}{2} [N^2 - N]$, this term

should just cancel the positive background charge. From now on, we should get rid of the $q=0$ component.

Change to ^{the} 2nd quantization

(4)

$$P_q = \sum_{k\sigma} C_{k-q,\sigma}^\dagger C_{k\sigma}$$

$$P_{-q} = \sum_{k,\sigma} C_{k\sigma}^\dagger C_{k+q,\sigma}, \quad \rho(r) = \frac{1}{V} \sum_q e^{-iqr} \rho(q)$$

$$H = \sum_{k\sigma} (\epsilon_{k\sigma} - \mu) C_{k\sigma}^\dagger C_{k\sigma} + \frac{1}{2V} \sum_{k,k'} \sum_{q \neq 0} \frac{4\pi e^2}{q^2} C_{k+q,\sigma}^\dagger C_{k'-q,\sigma'}^\dagger C_{k',\sigma'} C_{k,\sigma}$$

Equation of motion analysis

$$P_q^\dagger = \sum_k P_{kq,\sigma}^\dagger = \sum_k C_{k+q,\sigma}^\dagger C_{k\sigma}, \quad P_{kq} = \sum_k C_{k-q}^\dagger C_k$$

$$[H, P_q^\dagger] = \sum_k [H, P_{kq}^\dagger] = \sum_k [H_0, P_{kq}^\dagger] + \sum_k [H_{int}, P_{kq}^\dagger]$$

$$[H_0, P_{kq,\sigma}^\dagger] = \left\{ \frac{\hbar^2}{2m} (k+q)^2 - \frac{\hbar^2}{2m} k^2 \right\} P_{kq,\sigma}^\dagger = \hbar \omega_{k+q} P_{kq,\sigma}^\dagger$$

$$[H_{int}, P_{kq,\sigma}^\dagger] = \sum_{q'} \frac{1}{2} V(q') \{ [P_{q'}^\dagger, P_{kq,\sigma}^\dagger] P_{q'} + P_{q'}^\dagger [P_{q'}, P_{k+q,\sigma}^\dagger] \}$$

$$= \sum_{q'} \frac{1}{2} V(q') \{ [P_{q'}^\dagger, P_{kq,\sigma}^\dagger] P_{q'}^\dagger + P_{q'}^\dagger [P_{q'}, P_{k+q,\sigma}^\dagger] \}$$

$$[P_{q'}^\dagger, P_{kq,\sigma}^\dagger] = \left[\sum_k C_{k'-q',\sigma'}^\dagger C_{k',\sigma'}, \sum_k C_{k+q,\sigma}^\dagger C_{k,\sigma} \right] = C_{k+q-q',\sigma}^\dagger C_{k,\sigma} - C_{k+q,\sigma}^\dagger C_{k+q',\sigma}$$

$$\Rightarrow [H, P_{kq,\sigma}^\dagger] = \hbar \omega_{kq} P_{kq,\sigma}^\dagger + \frac{1}{2} V(q) \{ (C_{k,\sigma}^\dagger C_{k,\sigma} - C_{k+q,\sigma}^\dagger C_{k+q,\sigma}) P_{q,\sigma}^\dagger + P_{q,\sigma}^\dagger (C_{k,\sigma}^\dagger C_{k,\sigma} - C_{k+q,\sigma}^\dagger C_{k+q,\sigma}) \}$$

$$+ \frac{1}{2} \sum_{q' \neq q} V(q') \{ (C_{k+q-q',\sigma}^\dagger C_{k,\sigma} - C_{k+q,\sigma}^\dagger C_{k+q',\sigma}) P_{q',\sigma}^\dagger + P_{q',\sigma}^\dagger (C_{k+q-q',\sigma}^\dagger C_{k,\sigma} - C_{k+q,\sigma}^\dagger C_{k+q',\sigma}) \}$$

the second summation represents the coupling between electron-hole excitations with different momentum. This coupling will be neglected basing on the following argument:

(5)

$\rho_q = \sum_j e^{-i q \cdot r_j}$ is a summation of phases, which are "random" at high densities.

thus $\rho_{q'} \rho_{q-q'}$ compared to $\rho_q \rho_{q-q} = \rho_q N$ is small. This approximation is called RPA.

We further replace $c_k^\dagger c_{k-\sigma} - c_{k+q, \sigma}^\dagger c_{k+q, \sigma}$ with their expectation value

$$\Rightarrow [H, \rho_{kq, \sigma}^\dagger] = \hbar \omega_{kq} \rho_{kq, \sigma}^\dagger + V(q) (n_k - n_{k+q}) \rho_q^\dagger$$

Let me assume the eigen operator is a linear superposition of ρ_{kq}^\dagger

$\sum_{k\sigma} a_{k\sigma} \rho_{kq, \sigma}^\dagger$, which satisfies $[H, \sum_{k\sigma} a_{k\sigma} \rho_{kq, \sigma}^\dagger] = \hbar \omega \sum_{k\sigma} a_{k\sigma} \rho_{kq, \sigma}^\dagger$

$$\Rightarrow \sum_{k\sigma} \hbar \omega_{kq} \rho_{kq, \sigma}^\dagger + \sum_{k\sigma} V(q) (n_k - n_{k+q}) a_{k\sigma} \rho_q^\dagger = \hbar \omega \sum_{k\sigma} a_{k\sigma} \rho_{kq, \sigma}^\dagger$$

$$\Rightarrow \sum_{k\sigma} \left\{ \hbar \omega_{kq} a_{k\sigma} + \sum_{k'\sigma'} V(q) (n_{k'} - n_{k'+q}) a_{k'\sigma'} \right\} \rho_{kq, \sigma}^\dagger = \hbar \omega \sum_{k\sigma} a_{k\sigma} \rho_{kq, \sigma}^\dagger$$

i.e. $\hbar \omega_{kq} a_{k, \sigma} + \sum_{k'\sigma'} V(q) (n_{k'} - n_{k'+q}) a_{k'\sigma'} = \hbar \omega a_{k\sigma}$

$$\frac{V(q)}{\hbar(\omega - \omega_{kq})} \sum_{k'\sigma'} (n_{k'} - n_{k'+q}) a_{k'\sigma'} = a_{k\sigma}$$

$$2V(q) \sum_k \frac{(n_k - n_{k+q})}{\hbar(\omega - \omega_{kq})} + \sum_{k'\sigma'} (n_{k'} - n_{k'+q}) a_{k'\sigma'} = \sum_{k\sigma} (n_k - n_{k+q}) a_{k\sigma}$$

$$\Rightarrow \boxed{1 + 2V(q) \sum_k \frac{n_{k+q} - n_k}{\hbar(\omega - \omega_{kq})} = 0}$$

§ dielectric function

Suppose we add an external potential

$$H_e(t) = \sum_i V_e(r_i) e^{-i\omega t + i\eta t} = \frac{1}{V} \sum_q \left[V_e(q) e^{-i\omega t + i\eta t} \right] \rho_q^\dagger$$

$\Rightarrow \rho_q$ and ρ_{kq} should have the same $e^{-i\omega t + i\eta t}$ dependence

$$\rho_{kq} = \sum_\sigma \rho_{kq,\sigma} \Rightarrow -i\hbar \dot{\rho}_{kq} = [H, \rho_{kq}] + [H_e(t), \rho_{kq}]$$

under RPA, $[H_e(t), \rho_{kq}] = \sum_{q'} V_e(q') e^{-i\omega t + i\eta t} [\rho_{q'}^\dagger, \rho_{kq}] = \underbrace{\frac{2}{V}}_{\text{sum over spin}} V_e(q) (n_k - n_{k-q}) e^{-i\omega t + i\eta t}$

$$-i\omega \langle \rho_{kq} \rangle_t = \frac{1}{V} V(q) 2(n_k - n_{k-q}) \langle \rho_q \rangle_t + \frac{2}{V} V_e(q) (n_k - n_{k-q}) e^{-i\omega t + i\eta t} + \hbar \omega_{k,-q} \langle \rho_{kq} \rangle_t$$

$$\Rightarrow (\hbar\omega + E_{k-q} - E_k) \langle \rho_{kq} \rangle_t + \frac{2}{V} (n_k - n_{k-q}) \left[V_e(q) e^{-i\omega t + i\eta t} + V(q) \langle \rho_q \rangle_t \right] = 0$$

$$\Rightarrow \langle \rho_q \rangle_t = \sum_k \langle \rho_{kq} \rangle_t = - \sum_k \frac{2}{V} \frac{(n_k - n_{k-q})}{[\hbar\omega - (E_k - E_{k-q})]} \left\{ V_e(q) e^{-i\omega t + i\eta t} + V(q) \langle \rho_q \rangle_t \right\}$$

$$-\nabla^2 V_i(r, t) = 4\pi(-e)^2 \rho(r, t) = 4\pi(-e)^2 \sum_{q \neq 0} \langle \rho_q \rangle e^{iqr - i\omega t + i\eta t}$$

$$V_i(q, t) = \frac{4\pi e^2}{q^2} \langle \rho_q \rangle_t$$

$$\Rightarrow \langle \rho_q \rangle_t = - \sum_k \frac{2}{V} \frac{(n_k - n_{k-q})}{[\hbar\omega - (E_k - E_{k-q})]} \underbrace{\left[V_e(q, t) + V_i(q, t) \right]}_{V_{tot}(q, t)}$$

(7)

define $\chi_0(q, \omega) = \frac{2}{V} \sum_{\mathbf{k}} \frac{n_{\mathbf{k}+\mathbf{q}} - n_{\mathbf{k}}}{\hbar\omega - (\epsilon_{\mathbf{k}+\mathbf{q}} - \epsilon_{\mathbf{k}}) + i\eta}$, (vacuum polarization)

$$\langle P_q \rangle_t = -\chi_0(q, \omega) V_{tot}(q, t)$$

$$V_{tot}(q, t) = V_e(q, t) + V_{induced}(q, t) = V_e(q, t) + \frac{4\pi e^2}{q^2} \langle P_q \rangle_t$$

$$= V_e(q, t) - v(q) \chi_0(q, \omega) V_{tot}(q, t)$$

$$V_{tot}(q, t) = V_e(q, t) / (1 + v(q) \chi_0(q, \omega))$$

Lindhard form of dielectric function (RPA approximation)

$$\epsilon(q, \omega) = 1 + v(q) \chi_0(q, \omega) = 1 + \frac{2}{V} v(q) \sum_{\mathbf{k}} \frac{n_{\mathbf{k}+\mathbf{q}} - n_{\mathbf{k}}}{(\hbar\omega + i\eta) - (\epsilon_{\mathbf{k}+\mathbf{q}} - \epsilon_{\mathbf{k}})}$$

$$\nabla \cdot \vec{D} = 4\pi \rho_e, \quad \vec{D} = \vec{E} + 4\pi \vec{P} = \vec{E} + 4\pi \chi \vec{E} = \epsilon \vec{E}$$

$$\epsilon = 1 + 4\pi \chi \leftarrow \text{polarizability}$$

$$\vec{j} = \partial_t \vec{P}, \quad \text{i.e. } \vec{j}(\omega) = -i\omega \vec{P}(\omega) = -i \omega \chi(\omega) \vec{E}(\omega)$$

$$= \frac{-i}{4\pi} (\epsilon - 1) \vec{E}(\omega)$$

or $\sigma(\omega) = \frac{-i}{4\pi} (\epsilon - 1) \omega$, i.e. $\epsilon(\omega) = 1 + \frac{4\pi \sigma(\omega)}{\omega} i$

$$\epsilon_1(q, \omega) = \text{Re } \epsilon(q, \omega) = 1 + \frac{2}{V} v(q) \sum_{\mathbf{k}} \mathcal{P} \left[\frac{n_{\mathbf{k}+\mathbf{q}} - n_{\mathbf{k}}}{\hbar\omega - (\epsilon_{\mathbf{k}+\mathbf{q}} - \epsilon_{\mathbf{k}})} \right] \leftarrow \text{Principle value}$$

$$\epsilon_2(q, \omega) = \text{Im } \epsilon(q, \omega) = \frac{2\pi v(q)}{\hbar} \sum_{\mathbf{k}} n_{\mathbf{k}} \left\{ \delta(\omega - (\omega_{\mathbf{k}+\mathbf{q}} - \omega_{\mathbf{k}})) - \delta(\omega + (\omega_{\mathbf{k}+\mathbf{q}} - \omega_{\mathbf{k}})) \right\}$$

* Plasma excitations in dielectric function at $q/k_F \rightarrow 0$

$$\epsilon(q, \omega) = 1 + v(q) \chi_0(q, \omega)$$

$$\chi_0(q, \omega) = \frac{2}{V} \sum_{\mathbf{k}} \frac{n_{\mathbf{k}+\mathbf{q}} - n_{\mathbf{k}}}{(\hbar\omega + i\eta) - (\epsilon_{\mathbf{k}+\mathbf{q}} - \epsilon_{\mathbf{k}})} \quad \text{— Lindhard}$$

Consider the long-wave length limit $q/k_F \ll 1$, by shifting $\mathbf{k} \rightarrow \mathbf{k} - \mathbf{q}/2$

$$n_{\mathbf{k}+\mathbf{q}} - n_{\mathbf{k}} = n_{\mathbf{k}+\mathbf{q}/2} - n_{\mathbf{k}-\mathbf{q}/2} = \frac{\partial n}{\partial \epsilon} \cdot \frac{\partial \epsilon}{\partial \mathbf{q}} = -\delta(\epsilon - \epsilon_F) v_F q \cos \theta$$

$$\chi_0(q, \epsilon) = \frac{2}{(2\pi)^3} \int d\omega \cos \theta \int d\varphi \cdot k_F^2 dk \frac{-v_F q \cos \theta \delta(\epsilon - \epsilon_F)}{\omega - v_F q \cos \theta + i\eta}$$

$$= N_0 \int \frac{d\omega \cos \theta d\varphi}{4\pi} \frac{-v_F q \cos \theta}{\omega - v_F q \cos \theta + i\eta} \quad \leftarrow \text{define } S = \frac{\omega}{v_F q}$$

$$= \frac{N_0}{2} \int_{-1}^1 d\omega \cos \theta \frac{S - \cos \theta - S}{S - \cos \theta + i\eta} = N_0 \left[1 - \frac{1}{2} \int_{-1}^1 d\omega \cos \theta \frac{S}{S - \cos \theta + i\eta} \right]$$

$$\chi_0(q, \epsilon) = N_0 \left[1 - \frac{S}{2} \ln \left| \frac{1+S}{1-S} \right| \right] + i \frac{\pi}{2} N_0 S \Theta(|S| < 1)$$

two different limits:

$$\chi_0(q, \epsilon) = \begin{cases} N_0 (1 - S^2 + i \frac{\pi}{2} S \Theta(S < 1)) & (S \ll 1) \\ -\frac{1}{3S^2} - \frac{1}{5S^4} & (S \gg 1) \end{cases}$$

In the plasmon region, $s \gg 1$

$$\begin{aligned} \epsilon(q, \omega) &= 1 + \frac{4\pi e^2}{q^2} N_0 \left(\frac{-1}{3s^2} - \frac{1}{5s^4} \right) \\ &= 1 - 4\pi e^2 N_0 \left(\left(\frac{v_F}{\omega} \right)^2 + \frac{3}{5} \frac{(v_F q)^2}{\omega^2} \left(\frac{v_F}{\omega} \right)^2 \right) = 1 - \left(\frac{\omega_p^2}{\omega^2} + \frac{3}{5} \frac{\omega_p^2}{\omega^2} \left(\frac{v_F q}{\omega} \right)^2 \right) \end{aligned}$$

$$\boxed{\frac{\omega^2}{\omega_p^2} = 1 + \frac{3}{10} \left(\frac{v_F q}{\omega_p} \right)^2} \quad \text{--- no damping}$$

← plasmon.

(*) The static limit

$$\epsilon(q, \omega) = 1 + \frac{2v_F}{V} \sum_{\mathbf{k}} n_{\mathbf{k}} \left[\frac{1}{\hbar \omega_{\mathbf{k}q} - (\hbar \omega + i\eta)} + \frac{1}{\hbar \omega_{\mathbf{k}q} + (\hbar \omega + i\eta)} \right]$$

Set $\omega = 0$

$$\epsilon(q, 0) = 1 + \frac{4\pi e^2}{q^2} \sum_{\mathbf{k} < k_F} \frac{2}{E_{\mathbf{k}+q} - E_{\mathbf{k}}}$$

$$= 1 + \frac{4\pi e^2}{q^2} \sum_{\mathbf{k} < k_F} \frac{4}{\frac{\hbar^2 k_F^2}{2m} \left[2 \frac{\vec{k}}{k_F} \cdot \frac{\vec{q}}{k_F} + \left(\frac{q}{k_F} \right)^2 \right]}$$

$$\swarrow \quad X = \frac{q}{2k_F}$$

$$= 1 + \frac{4\pi e^2}{q^2} \int \frac{k^2 dk}{(2\pi)^3} \int_{-1}^1 d\cos\theta \frac{2\pi \cdot 4}{\epsilon_F \left[2 \frac{k q \cos\theta}{k_F^2} + \left(\frac{q}{k_F} \right)^2 \right]}$$

$$= 1 + \frac{4\pi e^2}{q^2} \frac{k_F^3}{\epsilon_F} \frac{1}{4\pi^2} \int_0^1 d\left(\frac{k}{k_F}\right) \left(\frac{k}{k_F}\right)^2 \int_{-1}^1 d\cos\theta \frac{1}{\left(\frac{k}{k_F} X \cos\theta + X^2\right)}$$

$$= 1 + \frac{4\pi e^2}{q^2} N_0 \left[\frac{1}{2} + \frac{1-X^2}{4X} \ln \left| \frac{1+X}{1-X} \right| \right]$$

① $q \rightarrow 0$, Thomas - Fermi Screening

$$\epsilon(q) = 1 + 4\pi e^2 N_0 / q^2 = 1 + \frac{1}{\lambda^2 q^2}$$

$$V(q) = V_0(q) / \epsilon(q) = \frac{4\pi e^2}{q^2 + (1/\lambda)^2}, \quad \boxed{\frac{1}{\lambda^2} = 4\pi e^2 N_0}$$

$$\rightarrow V(r) = \int dq e^{iqr} V(q) = \frac{e^{-\lambda r}}{r}$$

T-F:
$$-\nabla^2 V(r) = 4\pi \frac{e^2}{\epsilon_0} (p_{ex} + p_{ind})$$

$$\left\{ \begin{array}{l} p_{ind} = - \left(\frac{\partial n}{\partial \mu} \right) \cdot V(r) \end{array} \right. \leftarrow n(\mu = \mu_0 - V) - n(\mu_0)$$

$$\Rightarrow \left[-\nabla^2 + \left(\frac{\partial n}{\partial \mu} \right) 4\pi e^2 \right] V(r) = 4\pi e^2 p_{ex} = - \frac{\partial n}{\partial \mu} \cdot V$$

$$\Rightarrow V(q) = \frac{4\pi e^2 p_{ex}}{q^2 + (1/\lambda)^2}$$

$$N_0 = 2 \int \frac{d^3k}{(2\pi)^3} \delta\left(\epsilon - \frac{\hbar^2 k^2}{2m}\right) = \frac{4\pi k_F^3}{\hbar^2 k_F / m} \frac{2}{8\pi^3} = \frac{m}{\pi^2 \hbar^2} k_F$$

$$\lambda = (4\pi e^2 N_0)^{-1/2}$$

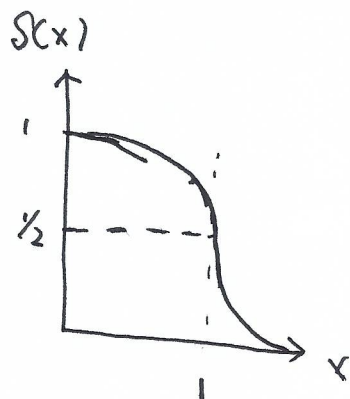
$$\Rightarrow \lambda \cdot k_F = \frac{1}{\left[4\pi e^2 \frac{m}{\pi^2 \hbar^2 k_F} \right]^{1/2}} = \frac{1}{\sqrt{\frac{4}{\pi}}} \left[\frac{e^2 \cdot k_F}{\frac{\hbar^2 k_F^2}{m} \cdot 2} \right]^{1/2} \sim \sqrt{\frac{E_K}{E_{int}}}$$

$\Rightarrow \boxed{\lambda \sim \sqrt{r_s}}$ Hence, T-F screening length is at the order of $1/k_F$.

② Friedel oscillation

(11)

$$\epsilon(q, 0) = 1 + \frac{1}{\lambda^2 q^2} S\left(\frac{q}{2k_F}\right), \quad S(x) = \frac{1}{2} \left[1 + \frac{1-x^2}{2x} \ln \left| \frac{1-x}{1+x} \right| \right]$$



at $x=1$, $S(x)$ has a sudden dip

$$\text{at } q > 2k_F, \quad \omega_{in} = \frac{\hbar}{2m} ((k-q)^2 - k^2) > 0$$

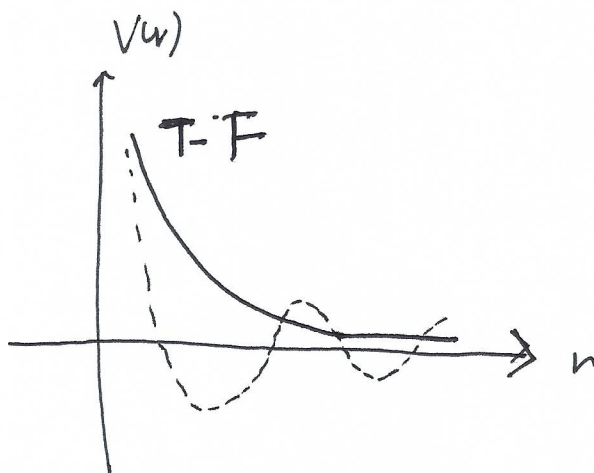
$$V(r) = \int d^3\vec{q} \, e^{i\vec{q} \cdot \vec{r}} \frac{4\pi Ze^2}{q^2 + \lambda^2 S(q/2k_F)}$$

Screen-potential
by a charged impurity
 Ze .

uncontinuity from the Fermi surface.

at $r \rightarrow +\infty$

$$V(r) \sim \text{const.} \frac{\cos 2k_F r}{r^3}$$



§: electron - electron interaction

Not only the external potential but also the interaction between electrons is renormalized into

$$V_{\text{eff}}(q, \omega) = \frac{4\pi e^2}{q^2 + 4\pi e^2 \chi_0(q, \omega)}, \text{ the HF difficulty}$$

$$\delta \mathcal{E}_{\text{HF}}(k) \rightarrow - \sum_{\mathbf{q}} n_{k+\mathbf{q}} \frac{4\pi e^2}{q^2 + 4\pi e^2 \chi_0(q, 0)}, \text{ then the exchange interaction is reduced.}$$

§: Wigner crystal:

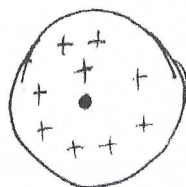
$$E_K \propto k_F^2, \quad E_{\text{int}} \propto \frac{e^2}{l} \propto k_F. \quad \text{define dimensionless parameter}$$

$$r_s = \frac{E_{\text{int}}}{E_K} = \frac{\frac{e^2}{l}}{\frac{\hbar^2 k_F^2}{m}} = \frac{l}{\frac{\hbar^2 e^2}{m}} \sim \frac{l}{a_0}$$

at low density region; $r_s \gg 1$, E_{int} is much stronger than E_K .

The above perturbative picture stop working. Electron starts to form regular crystal. In 3D, Electrons form fcc lattice. In 2D electrons form triangular lattice.

Vibration frequency.



$$E \cdot 4\pi \cdot r^2 = 4\pi \cdot \frac{4\pi}{3} \rho r^3$$

$$E = \frac{4\pi}{3} \rho r = \frac{4\pi}{3} \frac{e}{\frac{4\pi}{3} r_0^3} r = \frac{e}{r_0^3} r \Rightarrow$$

$$\boxed{\omega^2 = \frac{e^2}{m(r_s a_0)^3} = \frac{1}{3} \omega_p^2}$$

Fermi liquid

???

Wigner Crystal

