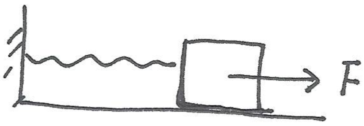
Fermi Liquid theory (II) — renormalizations to physical quantities

⊛ Simple examples

Elasticity



$$\begin{cases} E_{el} = \frac{1}{2} K x^2, \\ dE_{el} = F\,dX \end{cases} \qquad F = kX \quad \Rightarrow \quad \chi = -\frac{dx}{dF} = K^{-1}$$

susceptibility

$$E = E_{el} - FX$$

$$= \frac{1}{2} K x^2 - FX \quad \Rightarrow \quad E(F) = \frac{K}{2}\left(\frac{F}{K}\right)^2 - F\left(F/K\right)$$

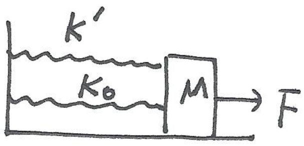
change variable from x to F $\qquad = -F^2/2K$

$$M/\chi = H$$

→ Magnetism $\quad E_M = \frac{M^2}{2\chi} \qquad dE_M = H\,dM$

$$E(H) = E_M - MH = \frac{(\chi H)^2}{2\chi} - \chi H^2 = -\frac{1}{2}\chi H^2$$

Now Consider two springs in parallel



$$E = \frac{1}{2}(K_0 + K')\, x^2$$

$$1/\chi = K_0 + K' = 1/\chi_0 + K'$$

$$\Rightarrow \chi = \frac{\chi_0}{1 + \chi_0 K'}$$

Then consider the situation of magnetism

$$E = \frac{M^2}{2\chi_0} + \frac{1}{2} f_0 M^2$$

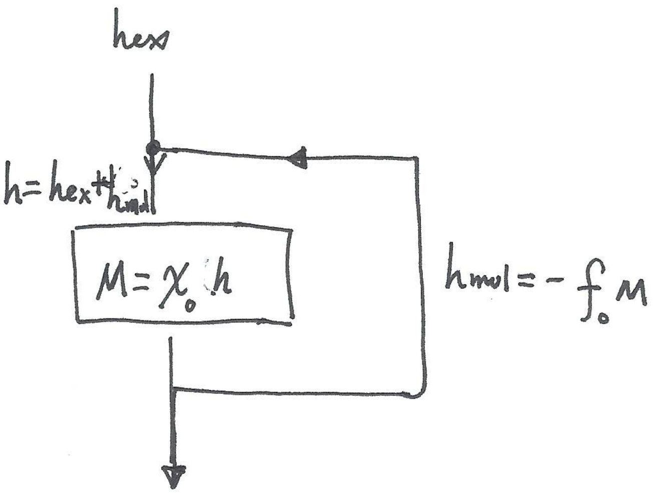
$$\frac{1}{\chi} = \frac{1}{\chi_0} + f_0 \Rightarrow \boxed{\chi = \frac{\chi_0}{1 + f_0\chi_0}}$$

(*) Molecular field method — feedback effect

$$E = \frac{M^2}{2\chi_0} + \frac{1}{2} f_0 M^2$$

← view this term as an internal molecule field effect

$$\begin{cases} \Delta E = -\int h_{mol} \cdot dM = \frac{f_0}{2} M^2 \\ h_{mol} = -f_0 M \end{cases}$$



$$\chi = M/h_{ex}$$

$$M = \chi_0 h = \chi_0 (h_{ex} + h_{mol})$$

$$= \chi_0 h_{ex} - f_0 M \chi_0$$

$$\Rightarrow (1 + f_0\chi_0) M = \chi_0 h_{ex}$$

$$\chi = \frac{M}{h_{ex}} = \frac{\chi_0}{1 + f\chi_0}$$

RPA: $M = M_0 + M_1 + M_2 + \cdots$

$$= \chi_0 h_{ex} + \chi_0 (\underbrace{-f_0 M_0}_{h^{(1)}_{mol}}) + \chi_0 (\underbrace{-f_0 M_1}_{h^{(2)}_{mol}}) + \cdots$$

$$= \chi_0 h_{ex} (1 - f_0 \chi_0 + (-f_0\chi_0)^2 + \cdots) = \frac{\chi_0}{1 + f_0\chi_0} h_{ex}$$

④ d Dimensionless Landau interaction function

$$f_{s,a}(\cos\theta) = \sum_l f_{l,sa} P_l(\cos\theta)$$

$F_{s,a} = N_0 f_{l;s,a}$ N_0: density of state

The interaction effects are summarized in two sets of Landau parameters.

S-wave channel: molecular method

spin-susceptibilities: $f_0^a \sigma\sigma' = N_0^{-1} \cdot F_0^a \sigma\sigma'$

$$f_a(\vec{p},\vec{p}')\,\vec{\sigma}_{\alpha\beta}\cdot\vec{\sigma}_{\gamma\delta}\,\delta n_{p\beta\alpha}\cdot\delta n_{p'\delta\gamma} \longrightarrow f_0^a\,\sigma\sigma'\delta n_{p\sigma}\delta n_{p'\sigma'}$$

in the case that $n_p, n_{p'}$ are diagonal i.e. σ_z eigenstates

Hence, $$\delta\mathcal{E}^{(2)} = \frac{1}{2N_0V}F_0^a \sum_{pp',\sigma\sigma'} \sigma\sigma'\,\delta n_{p\sigma}\,\delta n_{p\sigma'}.$$

$$= \frac{V}{2N_0} F_0^a (S_z)^2 \quad \text{where } S_z = \frac{1}{V}\sum_{p\sigma} \sigma\,\delta n_{p\sigma}$$

define molecule field $\Delta E = -V\int h_{mol}\cdot dS_z$

$$h_{mol} = -\frac{\partial E}{\partial S} = -\frac{1}{N_0} F_0^a S_z$$

$$S_z(1+\chi_0 F_0^a N_0^{-1}) = \chi_0 h_{ex}$$

$$h_{tot} = h_{ex} + h_{mol} = h_{ex} - N_0^{-1} F_0^a S_z$$

$$S_z = \chi_0 h_{tot} = \chi_0(h_{ex} - N_0^{-1} S_z F_0^a)$$

$$\boxed{\chi = \frac{\chi_0}{1+\chi_0 F_0^a N_0^{-1}}}$$

Since $\chi_0 = N_0$, as will be seen later.

(check dimension $E \propto \frac{1}{\chi_0 V}$ (extensive dimensionless quantity)^2

hence $[\chi_0] = [VE]^{-1}$ which is the dimension of DOS.

Then

$$\boxed{\chi = \frac{N_0}{1 + F_0^a}}$$

Please note that N_0 is the density of state of the interacting system, but not that of the free Fermi gas.

(*) Compressibility

$$\chi_{comp} = -\frac{1}{V}\frac{\partial V}{\partial P} \qquad V = N/n$$

$$\Delta(PV) = N d\mu$$

$$\chi_{comp} = -\frac{1}{N}\frac{\partial(N/n)}{d\mu} = \frac{1}{n^2}\frac{dn}{d\mu}$$

we will consider the Fermi liquid correction of $dn/d\mu$

$$\delta \mathcal{E} = \frac{1}{2N_0} F_0^s \sum_p \delta n_p \delta n_{p'} = \frac{V}{2}\frac{1}{N_0} F_0^s (\delta n)^2$$

where $\delta n = \frac{1}{V}\sum_p \delta n_p \Rightarrow h_{mol} = -\frac{F_0^s}{N_0}\delta n \Rightarrow \boxed{\frac{dn}{d\mu} = \frac{N_0}{1+F_0^s}}$

④ p-wave channel — effective mass

define $n(r,t) = \frac{1}{V}\sum_\sigma \sum_p n_{p,\sigma}(r,t) = \sum_\sigma \int \frac{d^3p}{(2\pi)^3} n_{p,\sigma}(r,t)$

$\vec{j}(r,t) = \sum_\sigma \int \frac{d^3p}{(2\pi)^3} \vec{\nabla}_p \varepsilon_{p\sigma}(r,t)\, n_{p\sigma}(r,t)$

Allow a slow spatial variation $n_{p,\sigma}(r,t)$, $\varepsilon_{p,\sigma}$

linearize $\vec{j}(r,t)$ by using

$$\varepsilon_{p\sigma}(r,t) = \varepsilon_p^0 + \int \frac{d^3p'}{(2\pi)^3} f_{\sigma\sigma'}(p,p')\,\delta n_{p'\sigma'}(r,t)$$

$$n_{p\sigma}(r,t) = n_p^0 + \delta n_{p\sigma}(r,t)$$

$$\Rightarrow \vec{j}(r,t) = \sum_\sigma \int \frac{d^3p}{(2\pi)^3} \nabla_p \varepsilon_{p\sigma}^0\, \delta n_{p\sigma}(r,t) + \nabla_p \delta\varepsilon_p \cdot n_p^0$$

$$= \sum_\sigma \int \frac{d^3p}{(2\pi)^3} \nabla_p \varepsilon_{p\sigma}^0\, \delta n_{p\sigma}(r,t) - \nabla_p n_p^0\, \delta\varepsilon_{p\sigma}(r,t)$$ ← partial derivative

$$= \sum_\sigma \int \frac{d^3p}{(2\pi)^3} v_p \left[\delta n_{p\sigma}(r,t) - \frac{\partial n_{p\sigma}^0}{\partial \varepsilon_p} \int \frac{d^3p'}{(2\pi)^3} f_{\sigma\sigma'}(p,p')\,\delta n_{p'\sigma'}(r,t)\right]$$

$$= \int \frac{d^3p}{(2\pi)^3} v_p\, \delta n_p(r,t) + \int \frac{d^3p}{(2\pi)^3} v_p \left(-\frac{\partial n_p^0}{\partial \varepsilon_p}\right) \int \frac{d^3p'}{(2\pi)^3} f^s(p,p')\,\delta n_{p'}(r,t)$$

$$\int \frac{d^3p}{(2\pi)^3} \vec{v}_p \left(-\frac{\partial n^0_{p\sigma}}{\partial \varepsilon_p}\right) f^s(p,p') = N(0) \int \frac{d\Omega}{4\pi} \sum_l f^s_l P_l(\cos\theta)\, v_F \cos\theta\ \hat{z}$$

↳ set p' along z axis in the p-space

$$= \frac{N(0)}{3} f_1^s v_F \hat{z},$$

{ other two direction average to zero

$$\Rightarrow \int \frac{d^3p}{(2\pi)^3} \vec{v}_p \left(-\frac{\partial n^0_{p\sigma}}{\partial \varepsilon_p}\right) f^s(\vec{p},\vec{p}') = \frac{N(0)}{3} f_1^s \vec{v}_{p'} = \frac{F_1^s}{3} \vec{v}_{p'}$$

$$\vec{j}(r,t) = \int \frac{d^3p}{(2\pi)^3} \vec{v}_p \delta n_p(r,t) + \frac{F_1^s}{3} \int \frac{d^3p'}{(2\pi)^3} \vec{v}_{p'} \delta n_{p'}(r,t)$$

$$= \int \frac{d^3p}{(2\pi)^3} \vec{v}_p \left(1+\frac{F_1^s}{3}\right) \delta n_p(r,t) = \int \frac{d^3p}{(2\pi)^3} \frac{\vec{p}}{m^*} \left(1+\frac{F_1^s}{3}\right) \delta n_p(r,t)$$

on other hand, by adiabatic continuity

$$\vec{j}(r,t) = \int \frac{d^3p}{(2\pi)^3} \frac{\vec{p}}{m} \delta n_p(r,t) \Rightarrow \boxed{\frac{1}{m} = \frac{1}{m^*}\left(1+\frac{F_1^s}{3}\right)}$$

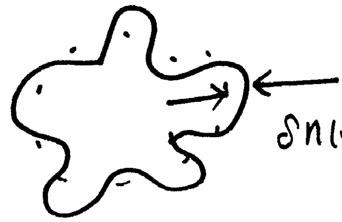
Similarly, we can derive spin current

$$j_i^\mu = 2 \int \frac{d^3p}{(2\pi)^3} \left(1+\frac{F_1^a}{3}\right) \frac{p_i}{m^*} \sigma_p^\mu(r,t)$$

we can define spin-effective mass $\frac{1}{m_s^*} = \frac{1}{m^*}\left(1+\frac{F_1^a}{3}\right)$

$$\boxed{\frac{m_s^*}{m} = \frac{1+\frac{1}{3}F_1^s}{1+\frac{1}{3}F_1^a}}$$

§ For general channels $F_l^{a,s}$



$$\delta n = V \int \frac{p^2 dp}{(2\pi)^3} \int d\Omega_p \, \delta n(p, \Omega_p) = V \int d\Omega \, \delta n(\hat{\Omega}_p)$$

where $\delta n(\hat{\Omega})$ is defined as $\int \frac{p^2 dp}{(2\pi)^3} \delta n(p, \Omega_p)$, i.e. integrate over radius direction.

we expand the angular distribution in terms of harmonic oscillators

$$\delta n(\hat{\Omega}_p) = \sum_{lm} \delta n_{lm} Y_{lm}(\hat{\Omega}_p)$$

$$E^{(2)} = \frac{1}{2V} \sum_{pp'} f_{\sigma\sigma'}(\hat{p}\,\hat{p}') \, \delta n_{p\sigma} \, \delta n_{p'\sigma'} = \frac{V}{2} \int d\Omega_p \, d\Omega_{p'} \, f_{\sigma\sigma'}(p\,p') \, \delta n_\sigma(\Omega_p) \, \delta n_{\sigma}(\Omega_p')$$

$$= \frac{V}{2} N^{-1}(0) \int d\Omega_p \, d\Omega_{p'} \underline{\sum_{lm} F_l^s \frac{4\pi}{2l+1} Y_{lm}^*(\Omega_p) Y_{lm}(\Omega_{p'})} \leftarrow \text{addition theorem}$$

$$\left[\left(\sum_{l_1 m_1} Y_{l_1 m_1}(\Omega_p) \, \delta n_{l_1 m_1}^s \right) \left(\sum_{l_2 m_2} Y_{l_2 m_2}(\Omega_p') \, \delta n_{l_2 m_2}^s \right) + (s \to a) \right]$$

where $F_{\sigma\sigma'} = F^s + F^a \sigma\sigma'$, $\delta n_{s,a} = \delta n_\uparrow \pm \delta n_\downarrow$

$$E^{(2)} = \frac{V}{2} N^{-1}(0) \left[\sum_{lm} F_l^s \frac{4\pi}{2l+1} \delta n_{lm}^{*(s)} \delta n_{lm}^{(s)} + (s \to a) \right]$$

The kinetic energy increase

$$\delta E^{(1)} = \sum \varepsilon_p \delta n_p = V \int d\Omega \int \frac{p^2 dp}{(2\pi)^3} \varepsilon_p \, \delta n(p, \Omega_p)$$

$$\int \frac{p^2 dp}{(2\pi)^3} \varepsilon_p \, \delta n(p, \hat{\Omega}_p) = \frac{p_F^2}{(2\pi)^3} v_F \cdot \frac{1}{2} (\delta p_F)^2 \leftarrow \quad \begin{array}{l} \varepsilon_p = v_F \cdot \delta p \\ p^2 \to p_F^2 \end{array}$$

Compare with $\int \frac{p^2 dp}{(2\pi)^3} \delta n(p, \hat{\Omega}_p) = \frac{p_F^2}{(2\pi)^3} \delta p_F = \delta n(\Omega_p)$

$$\Rightarrow \int \frac{p^2 dp}{(2\pi)^3} \varepsilon_p \, \delta n(p, \hat{\Omega}_p) = \frac{v_F}{2} [\delta n(\Omega_p)]^2 \Big/ \frac{p_F^2}{(2\pi)^3} = 4\pi \, N^{-1}(0) \, [\delta n(\Omega_p)]^2$$

$$\delta E^{(1)} = V N^{-1}(0) \int d\Omega \, [\delta n(\Omega_p)]^2 = 2\pi \, V N^{-1}(0) \sum_{lm} |\delta n^s_{lm}|^2 + |\delta n^a_{lm}|^2$$

$$\Rightarrow \quad \Delta E = 2V N^{-1}(0) \sum_{lm} \left\{ \left(1 + \frac{F_l^s}{2l+1}\right) |\delta n^s_{lm}|^2 + (s \to a) \right\}$$

From thermodynamic properties. we know

$$\Delta E = \sum_{lm} \frac{1}{2\chi^s_l} |\delta n^s_{lm}|^2 + (s \to a)$$

$$\Rightarrow \quad \frac{1}{\chi_{l\,FL}^{s,a}} = \frac{1}{\chi^{s,a}_{l,0}} \left(1 + \frac{F_l^{s,a}}{2l+1}\right)$$

i.e. $$\boxed{\chi^{s,a}_{FL,l} = \frac{\chi^{s,a}_{l,}}{1 + \frac{F^{s,a}}{2l+1}}}$$

in ^3He $\quad F_0^s \simeq 10.8. \quad F_a^0 \approx -0.75$

Compressibility is greatly reduced

spin-susceptibility is greatly enhanced!

③

§: effective mass renormalization

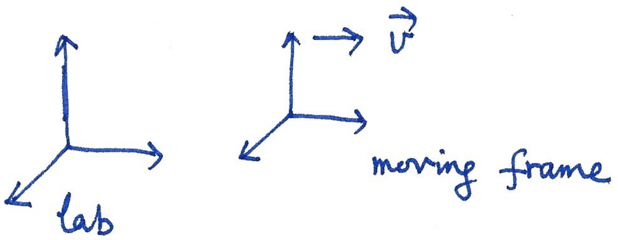
a moving frame with velocity $\vec{V}$

Consider that we do a Galilean transformation,

$$H = \sum_i \frac{P_i^2}{2m} \rightarrow \sum_i \frac{(\vec{P}_i + m\vec{V})^2}{2m}$$

where P_i is momentum in the moving frame

$P_i + m\vec{V}$ is the momentum in the lab frame.



P_i is cannonical momentum

$P_i + mV$ is mechanical momentum

In the lab frame, the current reads

$j(r,t) = \sum \delta(r - r_i)(\vec{P}_i + m\vec{V})$, which is zero because the system remains at rest.

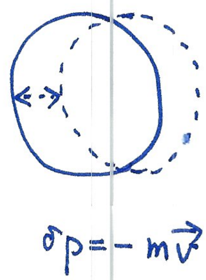
$$\langle j \rangle = \frac{1}{m}\langle p(t) \rangle + \vec{V} = 0 \Rightarrow \langle \vec{P}_i \rangle = -\vec{V} m$$

total mometum $\langle \vec{Q} \rangle = -N m \vec{V}$ in the moving frame.

Now let us calculate $\langle \vec{Q} \rangle$ in the moving frame by another method.

Let us conside $-\vec{v}$ as an external field, and $\vec{Q}$ as the response

$$\langle \vec{Q} \rangle = -\frac{\delta E}{\delta \vec{V}}$$



$\delta p = -m\vec{v}$

④

$$\langle \vec{Q} \rangle = \sum_{\sigma} p\, \delta n_{p\sigma} = V \cdot P_F \int d\Omega\, \Omega_p \int \frac{dp}{(2\pi)^3} P_F^2\, \delta n_{p\sigma} = V \cdot P_F \int d\Omega_p \cos\theta_p\, \delta n(\Omega_p)$$

$$= V \cdot P_F \sqrt{\frac{4\pi}{3}}\, \delta n_{10}^s$$

$$E(\vec{v}) = E(\vec{v}=0) + V P_F \left(\sqrt{\frac{4\pi}{3}}\, v\right) \delta n_{10}^s$$

$$\frac{E(\vec{v})}{V} = 2\pi N^{-1}(0)\left(1+\frac{F_1^s}{3}\right)(\delta n_{10}^s)^2 + P_F\left(\sqrt{\frac{4\pi}{3}}\, v\right)\delta n_{10}^s$$

$$\Rightarrow \langle \delta n_{10}^s \rangle = -\frac{N(0)\, P_F \left(\sqrt{\frac{4\pi}{3}}\right)}{4\pi\left(1+\frac{F_1^s}{3}\right)}\, v = -\frac{k_F^2\, P_F}{4\pi^3 \hbar v_F}\, \frac{\sqrt{\frac{4\pi}{3}}}{1+\frac{F_1^s}{3}}\, v$$

$$\langle Q \rangle = -V \frac{P_F}{\hbar}\, \frac{k_F^2\, m^* v}{3\pi^2\left(1+\frac{F_1^s}{3}\right)} = -V\, \frac{k_F^3}{3\pi^2}\, \frac{m^* v}{1+\frac{F_1^s}{3}} = -\frac{N m^* \boldsymbol{v}}{1+\frac{F_1^s}{3}}$$

$$\Rightarrow \quad m = \frac{m^*}{1+\frac{F_1^s}{3}} \qquad \text{i.e.} \quad \boxed{\frac{m^*}{m} = 1 + \frac{F_1^s}{3}}$$

§ Pomeranchuk instability

Consider the Fermi surface as an elastic membrane in momentum space. The deformation of the Fermi surface not only changes the Kinetic energy, but also changes the interaction energy.

As we showed before,

$$\delta E \propto \left(1+\frac{F_l^{s,a}}{2l+1}\right)|\delta n_{lm}^{s,a}|^2 + O(\delta n_{lm}^{s})^4 + \cdots$$

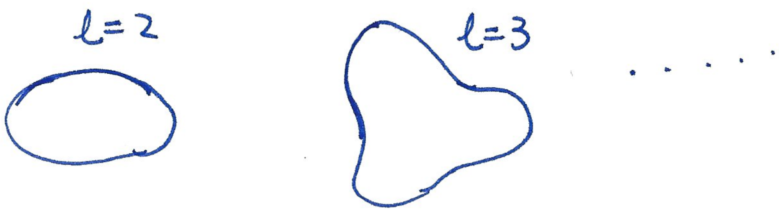
if $F_l^{s,a} < -(2l+1)$, then the Fermi surface will not be spheric stable, but develop distortions.

$F_0^s \rightarrow$ phase seperation : diveragence of Compressibility

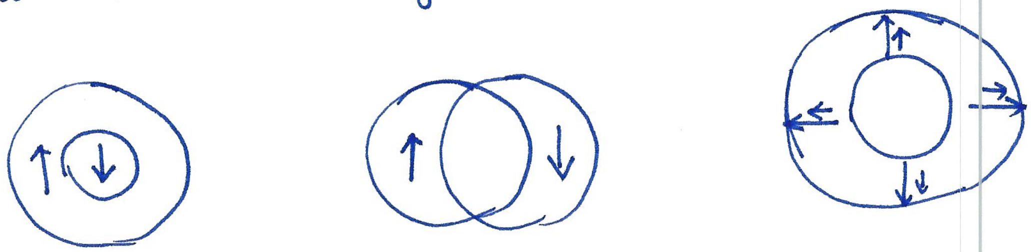
$F_0^a \rightarrow$ Ferromagnetism : divergence of spin-susceptibility

for $l \geq 1$, Fermi surface anisotropic distortions.

electronic liquid crystal phase



* ~~no~~ Unconventional magnetism



F_0^a F_1^a

p-wave magnetism

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