Lect 7 Boltzmann transport (I)

§ Wigner distribution

We need to consider a spatially inhomogenous (slowly varing) system. We will talk about the particle with momentum $\vec{p}$ at point $\vec{r}$, and the distribution function $n_{\alpha\beta}(p;r,t)$ in the phase space. A more rigours definition is as follows

define $f(p;k;t) = \langle C^{+}_{p+k/2} C_{p-k/2} \rangle (t)$, and perform the Fourier transform over small variable k, and arrive at

$$n_{\alpha\beta}(p;r,t) = \sum_k f(p;k;t)\, e^{ikr}$$

another way to express $n_{\alpha\beta}(p;r,t) = \int dr' e^{ip\cdot r'} \langle \psi^{+}_{\alpha}(r+\frac{r'}{2})\, \psi_{\beta}(r-\frac{r'}{2}) \rangle$

where "r" is the center of mass coordinate, r' is the ralative coordinate.

$n_{\alpha\beta}(p;r,t)$ is a semiclassical distribution. The resolution of Δp and Δr need to satisfy $\Delta p \cdot \Delta r \geq \hbar/2$.

§ Boltzmann equation:

Let us study the equation of motion of $n(p;r;t)$. It has three contributions: ① Flow in the momentum space

② Flow in the real space

③ Collisions

$$\frac{\partial n}{\partial t}(p,r,t) + \nabla_r \Big[\overbrace{v_p(r,t)}^{\dot{r}}\, n(p,r,t) \Big] + \nabla_p \Big[\overbrace{f_p(r,t)}^{\dot{p}}\, n(p,r,t) \Big] = I_{coll}$$

where $v_p(r,t) = \nabla_p \mathcal{E}_p(r,t)$, $\quad f_p(r,t) = -\nabla_r \mathcal{E}_p(r,t)$

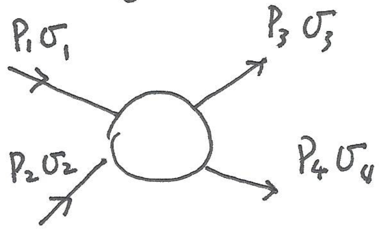
$$\Rightarrow \boxed{\frac{\partial}{\partial t} n(p,r,t) + \nabla_p \mathcal{E}_p(r,t)\nabla_r n(p,r,t) - \nabla_r \mathcal{E}(p,r,t)\nabla_p \cdot n(p,r,t) = I_{coll}}$$

Question: how to divide the collision events from flow in momentum space?

Actually, the answer depends. Typically, the momentum flow is due to force $f_p(r,t)$, which is assumed to be slow varing, an averaged smooth interaction And collision is due an abrupt inteaction. ~~which~~

The dividing between them is kind of arbitary.

Collision:



$$\frac{2\pi}{\hbar}|\langle 34|t|12\rangle|^2 n_1 n_2 (1-n_3)(1-n_4)\,\delta(\mathcal{E}_1+\mathcal{E}_2-\mathcal{E}_3-\mathcal{E}_4)$$
$$\Big\{ n_3 n_4 (1-n_1)(1-n_2) \cdots$$

$$\frac{2\pi}{\hbar}|\langle 34|t|12\rangle|^2 = \frac{1}{V^2}\,\omega(12;34)\,\delta_{p_1+p_2,\,p_3+p_4}\,\delta_{\sigma_1+\sigma_2=\sigma_3+\sigma_4}$$

$$\Rightarrow I_{coll}[n_p] = \frac{1}{V^2}\sum_{p_2\sigma_2 p_3\sigma_3 p_4\sigma_4} W(12;34)\,\delta_{p_1+p_2,\,p_3+p_4}\,\delta_{\sigma_1+\sigma_2=\sigma_3+\sigma_4}\,\delta(\mathcal{E}_1+\mathcal{E}_2-\mathcal{E}_3-\mathcal{E}_4)$$
$$\big[n_3 n_4 (1-n_1)(1-n_2) - n_1 n_2 (1-n_3)(1-n_4) \big]$$

In most situation, we adopt the relaxation time approximation

$$\Big(I[n_p]\Big)_{RA} = -\,\delta n / \tau \,, \qquad \text{if } \omega\tau \gg 1,$$

the collision integral can be neglected!

$\boxed{\text{Warm up.}}$ — the Boltzman Eq for the density channel, for ordinary FL. — only density mode

$$\frac{\partial}{\partial t} n(rpt) + \nabla_p \mathcal{E}(rpt)\, \nabla_r n(rpt) - \nabla_r \mathcal{E}(rpt)\, \nabla_p n(rpt) = I_{coll}$$

linearing the equation

$$\mathcal{E}(rpt) = \mathcal{E}_0(p) + \frac{1}{V}\sum_{p'} f^s(p,p')\, \delta n_{p'}(r,t)$$

$$n(rpt) = n_0(p) + \delta n(r,p,t)$$

$\nabla_r n(r,p,t)$ and $\nabla_r \mathcal{E}(r,p,t)$ are already at the order of δn

thus we keep $\nabla_p \mathcal{E}(r,pt) = \nabla_p \mathcal{E}_0(p)$ and $\nabla_p n(rpt) = \nabla_p n_0(p)$

at the zeroth order.

$$\Rightarrow \frac{\partial}{\partial t}\delta n(r,p,t) + \vec{v}_p \cdot \nabla_r \delta n(r,p,t) - \nabla_p n_0(p) \int \frac{d\vec{p}'}{(2\pi)^3} f^s_{\vec{p}\vec{p}'}\, \nabla_r \delta n_{p'}(r,t) = 0$$

$$\frac{\partial}{\partial t}\delta n(r,p,t) + \vec{v}_p \cdot \nabla_r \left[\delta n(r,p,t) - \frac{\partial n_0(p)}{\partial \mathcal{E}} \int \frac{d\vec{p}'}{(2\pi)^3} f^s_{\vec{p}\vec{p}'}\, \delta n(r,p',t) \right] = 0$$

do Fourier transform

$$\delta n(rpt) = \sum_q \delta n(\vec{p})\, e^{i(qr-\omega t)} \quad \leftarrow \boxed{\begin{array}{l}\text{Collision neglected}\\ \text{at } \omega\tau \gg 1\end{array}}$$

$$\Rightarrow (-i\omega + i\vec{v}_p\cdot\vec{q})\,\delta n(\vec{p}) - \frac{\partial n_0(p)}{\partial \mathcal{E}}\, \vec{v}_p \cdot i\vec{q} \left[\int \frac{d\vec{p}'}{(2\pi)^3} f^s_{\vec{p}\vec{p}'}\, \delta n(p' \ldots) \right] = 0$$

$$(\omega - v_f q\cos\theta_p)\,\delta n(\vec{p}) + \frac{\partial n_0(p)}{\partial \mathcal{E}}\, v_f q \cos\theta_p \left[\int \frac{d\vec{p}'}{(2\pi)^3} f^s(\vec{p}\vec{p}')\, \delta n(p') \right] = 0$$

$$(s-\cos\theta_p)\,\delta n(\vec p) - \left(-\frac{\partial n_0(p)}{\partial \varepsilon_p}\right)\cos\theta_p \int \frac{d^3p'}{(2\pi)^3}\, f^s(\vec p,\vec p')\,\delta n(\vec p') = 0$$

define $\delta n(\hat p) = \int \frac{p^2dp}{(2\pi)^3}\,\delta n(\vec p)$ — integrate over radial

$$\Rightarrow (s-\cos\theta_p)\int\frac{p^2dp}{(2\pi)^3}\,\delta n(\vec p) - \cos\theta_p\int\frac{p^2dp}{(2\pi)^3}\int\frac{p'^2dp'\,d\Omega_{p'}}{(2\pi)^3}\left(-\frac{\partial n_0(p)}{\partial\varepsilon_p}\right) f^s(\hat p,\hat p')\,\delta n(\vec p')=0$$

$$(s-\cos\theta_p)\,\delta n(\hat p) - \cos\theta_p \underbrace{\int\frac{p^2dp}{(2\pi)^3}\left(-\frac{\partial n_0(p)}{\partial\varepsilon_p}\right)}\int d\Omega_{p'}\, f^s(\hat p,\hat p')\,\delta n(\hat p')=0$$

$$\int\frac{p^2dp}{(2\pi)^3}\,\delta(\varepsilon_p-\varepsilon_f) = \frac{N_0}{4\pi} \leftarrow \text{single component} \quad \text{DOS}$$

$$\Rightarrow \boxed{(s-\cos\theta_p)\,\delta n(\hat p) - N_0\cos\theta_p\int\frac{d\Omega_{p'}}{4\pi}\,f^s(\hat p,\hat p')\,\delta n(\hat p')=0}$$

next step is another Fourier transform

$$\delta n(\hat p) = \sum_l Y_{l0}(\hat p)\, u_l \quad (\text{set the direction } q \text{ as } \hat z\text{-axis})$$

$$\sum_l (s-\cos\theta_p)\,Y_{l0}(\hat p)\,u_l - N_0\cos\theta_p\int\frac{d\Omega_{p'}}{4\pi}\sum_{l'm'} f^s_{l'}\,\frac{4\pi}{2l'+1}\,Y_{l'm'}(\hat p)\,Y^*_{l'm'}(\hat p')\sum_l Y_{l0}(\hat p')\,u_l=0$$

$$\boxed{\sum_l Y_{l0}(\hat p)\,u_l - \frac{\cos\theta_p}{s-\cos\theta_p}\sum_l\frac{F^s_l}{2l+1}\,Y_{l0}(\hat p)\,u_l=0}$$

$$\int d\Omega_p \, Y^*_{l'm'}(\hat p) \sum_l Y_{l0}(\hat p)\, u_l - \sum_{\substack{ll'\\m'}} \int d\Omega_p \frac{\cos\theta_p}{s-\cos\theta_p} Y^*_{l'm'}(\hat p) \frac{F_l^s}{2l+1} Y_{l0}(\hat p)\, u_l = 0$$

exchange $l \leftrightarrow l'$

$$\Rightarrow \quad u_l - \sum_{l'} \int d\Omega_p \, Y^*_{l0}(\hat p) \frac{F^s_{l'}}{2l'+1} Y_{l'0}(\hat p) \frac{\cos\theta_p}{s-\cos\theta_p} u_{l'} = 0$$

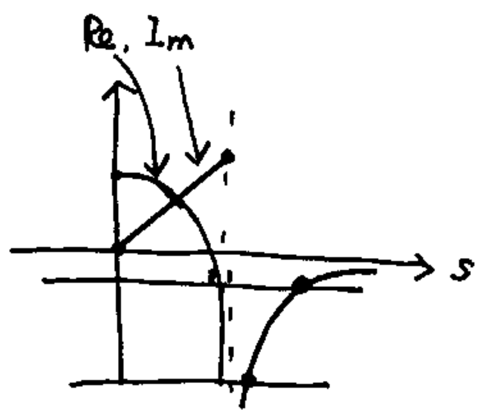
$$\boxed{\begin{aligned} &\frac{u_l}{\sqrt{2l+1}} + \sum_{l'} F^s_{l'} \Omega_{ll'}(s) \frac{u_{l'}}{\sqrt{2l'+1}} = 0 \\ &\text{where} \quad \Omega_{ll'}(s) = \frac{-1}{\sqrt{(2l'+1)(2l+1)}} \int d\Omega_p \, Y^*_{l0}(\hat p)\, Y_{l'0}(\hat p) \frac{\cos\theta_p}{s-\cos\theta_p} \end{aligned}}$$

① truncate at $l=0$ $\Rightarrow$

$$u_0 + F_0^s\, \Omega_{00}(s)\, u_0 = 0 \quad \Rightarrow \quad \frac{-1}{F_0^s} = \Omega_{00}(s)$$

$$\Omega_{00}(s) = \int \frac{d\Omega_p}{4\pi} \frac{-\cos\theta_p}{s-\cos\theta_p} = \int \frac{d\Omega_p}{4\pi} \left[\frac{-s}{s-\cos\theta_p + i\eta} + 1 \right] = + \left[1 + \frac{s}{2} \ln\left| \frac{s-1}{s+1} \right| \right] + i\frac{\pi}{2} s\, \Theta(1-s)$$

$$\Rightarrow \quad \boxed{\frac{1}{-F_0^s} = 1 + \frac{s}{2} \ln\left| \frac{s-1}{s+1} \right|}$$



for $F_0^s \to 0^+$, $s \to 1^+$

$$1 + \frac{1}{2} \ln \frac{s-1}{2} = -\frac{1}{F_0^s} \quad \Rightarrow \quad s \approx 1 + 2e^{-\frac{2}{F_0^s}}$$

for $F_0^s \gg 1$ $\Rightarrow$ $1 + \frac{s}{2} \ln\left| \frac{s-1}{s+1} \right| \sim -\frac{1}{3s^2}$ $\Rightarrow$ $s = \sqrt{\frac{F_0}{3}}$ (s is the sound velocity)

§ Spin-related transport

Pauli matrix

Define $n_p(r,t) = \frac{1}{2} tr[n_{p,\alpha\beta}]$, $\vec{\sigma}_p(r,t) = \frac{1}{2} tr[n_{p,\alpha\beta} \ \vec{\tau}_{\beta\alpha}]$

$$\rightarrow \boxed{n_p(r,t) = n_p(r,t)\delta_{\alpha\alpha'} + \vec{\sigma}_p(r,t)\cdot\vec{\tau}_{\alpha\alpha'}}$$

Then the quasiparticle energy can be expressed as

$$\mathcal{E}(p,r,t) = \mathcal{E}_p(r,t)\delta_{\alpha\alpha'} + \vec{h}_p(r,t)\cdot\vec{\tau}_{\alpha\alpha'}$$

plug in the Boltzman Eq.

$$\frac{\partial}{\partial t}n(r,p,t) + \frac{1}{2}\frac{\partial}{\partial r}\left[\frac{\partial \mathcal{E}}{\partial p}n(r,p,t) + n(r,p,t)\frac{\partial\mathcal{E}}{\partial p}\right] + \frac{\partial}{2\partial p}\left[\left\{-\frac{\partial\mathcal{E}}{\partial r}\right\} n(r,p,t) + n(r,p,t)\left(-\frac{\partial\mathcal{E}}{\partial r}\right)\right]$$

$$\underbrace{-\frac{1}{i\hbar}[n(r,p,t), \mathcal{E}(r,p,t)]}_{\text{Larmor precession}} = I_{collision}$$

Separate variables

$$\boxed{\begin{aligned} &\frac{\partial n_p(r,t)}{\partial t} + \frac{\partial}{\partial r_i}\left[\frac{\partial\mathcal{E}_p}{\partial p_i}n_p + \frac{\partial\vec{h}_p}{\partial p_i}\vec{\sigma}_p\right] + \frac{\partial}{\partial p_i}\left[-\frac{\partial\mathcal{E}}{\partial r_i}n_p - \frac{\partial\vec{h}_p}{\partial r_i}\cdot\vec{\sigma}_p\right] = I_{coll} \\ &\frac{\partial\vec{\sigma}_p(r,t)}{\partial t} + \frac{\partial}{\partial r_i}\left[\frac{\partial\mathcal{E}_p}{\partial p_i}\vec{\sigma}_p + \frac{\partial\vec{h}_p}{\partial p_i}n_p\right] + \frac{\partial}{\partial p_i}\left[-\frac{\partial\mathcal{E}}{\partial r_i}\vec{\sigma}_p - \frac{\partial\vec{h}_p}{\partial r_i}n_p\right] \\ &= \frac{2}{\hbar}\vec{h}_p\times\vec{\sigma}_p + I_{coll}\end{aligned}}$$

Lamor precession: in an external magnet field $\mathcal{H}$, it couples to electron spin as $-\frac{\gamma}{2}\mathcal{H}\cdot\sigma \Rightarrow \frac{\partial \vec{\sigma}_p}{\partial t} = \vec{\sigma}_p \times \gamma \mathcal{H} \Rightarrow \omega_0 = \gamma \mathcal{H}$

In the Fermi liquid, $h_p = -\gamma \frac{\hbar}{2}\mathcal{H} + 2\int \frac{d^3p'}{(2\pi)^3} f^a(p,p')\,\sigma_{p'}$

↳ contribution from interaction

in the uniform system, we have

$$\frac{\partial \vec{\sigma}_p}{\partial t} = \gamma \vec{\sigma}_p \times \vec{\mathcal{H}} - \frac{4}{\hbar}\int \frac{d^3p'}{(2\pi)^3} (\vec{\sigma}_p \times \vec{\sigma}_{p'})$$

define $\delta\, \sigma(\hat{\Omega}) = 2\int \frac{dp}{(2\pi)^3} \cdot p^2 \sigma(p, \hat{\Omega})$, (we integrate out radius direction).

$$\Rightarrow 2\int \frac{p^2 dp}{(2\pi)^3} \frac{\partial \vec{\sigma}}{\partial t}(p,\hat{\Omega}) = 2\int \frac{p^2 dp}{(2\pi)^3} \vec{\sigma}(p,\hat{\Omega}) \times (\gamma \vec{\mathcal{H}}) - 8\int \frac{d\Omega'}{4\pi} \int \frac{p'^2 dp'}{(2\pi)^3} \int \frac{p^2 dp}{(2\pi)^3} \overset{f^a(pp')}{\sigma_p \times \sigma_{p'}}$$

$$\frac{\partial}{\partial t}\sigma(\hat{\Omega}_p) = \gamma \sigma(\hat{\Omega}_p) \times \vec{\mathcal{H}} - \frac{2}{\hbar}\int \frac{d\Omega_p'}{4\pi} f^a(p,p')\, \hat{\sigma}(\Omega_p)\, \hat{\sigma}(\Omega_{p'})$$

In the external field $\sigma(\hat{\Omega}_p) = \sigma^0 + \delta\sigma(\hat{p}) \Rightarrow$

$$\frac{\partial}{\partial t}\delta\vec{\sigma}(\Omega_p) = \gamma\, \delta\vec{\sigma}(\Omega_p) \times \vec{\mathcal{H}} - \frac{2}{\hbar}\int \frac{d\Omega_p'}{4\pi} f^a(pp')\left[\delta\sigma(\hat{p}) \times \sigma^0 + \sigma^0 \times \delta\sigma(\hat{p}')\right]$$

define $\delta\sigma_+(\hat{\Omega}_p) = \delta\sigma_x(\hat{\Omega}_p) + i\,\delta\sigma_y(\hat{\Omega}_p)$

$$\frac{\partial}{\partial t}\delta\sigma_+(\Omega_p) = -i\left[\omega_0\, \delta\sigma_+(\Omega_p) - \frac{2\sigma^0}{\hbar}\int \frac{d\Omega'}{4\pi} f^a(p,p')(\delta\sigma_+(\Omega_p) - \delta\sigma_+(\Omega_p'))\right]$$

$$= -i\left[(\omega_0 - \frac{2}{\hbar}N^{-1}(0)F_0^a \sigma^0)\,\delta\sigma_+(\Omega_p) + \frac{2}{\hbar}\sigma^0 \int \frac{d\Omega'}{4\pi} f^a(p,p')\,\delta\sigma_+(\Omega_p')\right]$$

expand $\delta\sigma_+(\Omega_p) = \sum_{lm} \delta\sigma_+(lm)\, Y_{lm}(\Omega_p)$

$$\int \frac{d\Omega'}{4\pi} f^a(p,p')\, \delta\sigma_+(\Omega_{p'}) = N^{-1}(0) \int \frac{d\Omega_p'}{4\pi} F_l^a \frac{4\pi}{2l+1} Y_{lm}(\Omega_p) Y^*_{lm}(\Omega_{p'}) \sum_{l'm'} Y_{l'm'}(\Omega_{p'})\, \delta\sigma_+(l'm')$$

$$= N^{-1}(0) \sum_{l,m} \frac{F_l^a}{2l+1} Y_{lm}(\Omega_p)\, \delta\sigma_+(lm)$$

$$\Rightarrow \frac{\partial}{\partial t}\delta\sigma_+(lm) = -i\left\{\omega_0 - \frac{2}{\hbar} N^{-1}(0) F_0^a \sigma^0 + N^{-1}(0)\frac{F_l^a}{2l+1}\right\}\delta\sigma_+(lm)$$

$$\Rightarrow \quad \omega_{l+} = \left\{\omega_0 - \frac{2}{\hbar}\sigma_0 N^{-1}(0)\left[F_0^a - \frac{F_l^a}{2l+1}\right]\right\}$$

$$\sigma^0 = \frac{\gamma\hbar}{2} \frac{N(0)}{1+F_0^a}\mathcal{H}, \quad \omega_0 = \gamma\mathcal{H}$$

$$\Rightarrow \boxed{\frac{\omega_{l+}}{\omega_0} = \frac{1+F_l^a/2l+1}{1+F_0^a}}$$

the $l=0$ channel Larmor frequency is not modified by interaction because Spin is conserved by interaction!

~~§ spin hydrodynamic equations~~

Integrate over momentum for the Boltzman transport equation $\Rightarrow$

$$\frac{\partial}{\partial t}\vec{\sigma}(r,t) + \frac{\partial}{\partial r_i}\vec{j}_i(\vec{r},t) = -\frac{2}{\hbar}\int\frac{d\vec{p}}{(2\pi)^3}\, \vec{\sigma}_p \times \left[-\frac{\gamma}{2}\mathcal{H}^{(r,t)} + \int\frac{d\vec{p}'}{(2\pi)^3} f^a(p,p')\vec{\sigma}_{p'}\right)$$

$$\boxed{\frac{\partial}{\partial t}\vec{\sigma}(r,t) + \frac{\partial}{\partial r_i}\vec{j}_i(\vec{r},t) = \gamma\,\vec{\sigma}(r,t)\times\mathcal{H}(r,t)}$$

(inteaction part cancels).

where
$$\vec{\sigma}(r,t) = 2\int\frac{d^3p}{(2\pi)^3}\,\vec{\sigma}(r,p,t)$$

$$\vec{j}_i(r,t) = 2\int\frac{d^3p}{(2\pi)^3}\left[\frac{\partial\varepsilon_p}{\partial p_i}\vec{\sigma}(r\,p\,t) + \frac{\partial \vec{h}_p}{\partial p_i} n_p(r,p,t)\right]$$

$$\vec{j}_i(r,t) = 2\int\frac{d^3p}{(2\pi)^3}\left[\frac{\partial\varepsilon_p^0}{\partial p_i}\,\vec{\sigma}(r,p,t) - \frac{\partial n_p^0}{\partial p}(r,p,t)\,\vec{h}_p\right] \quad (\text{linearize}).$$

$$= 2\int\frac{d^3p}{(2\pi)^3}\, v_{p_i}\left(\vec{\sigma}_p - \frac{\partial n_p^0}{\partial \varepsilon_p}\vec{h}_p\right)$$

v_pi is odd function, after integration, this term goes to zero

$$= 2\int\frac{d^3p}{(2\pi)^3}\, v_{p_i}\left(\vec{\sigma}_p - \frac{\partial n_p^0}{\partial \varepsilon_p}\left(-\frac{\gamma}{2}\frac{\hbar}{2}\mathcal{H} + 2\int\frac{d^3p'}{(2\pi)^3} f^a(p,p')\,\vec{\sigma}_{p'}\right)\right]$$

$$= 2\int\frac{d^3p}{(2\pi)^3}\, v_{p_i}\,\vec{\sigma}_p - 4\int\frac{d^3p}{(2\pi)^3}\, v_{p_i}\frac{\partial n_p^0}{\partial \varepsilon_p}\int\frac{d^3p'}{(2\pi)^3} f^a(p,p')\,\vec{\sigma}_{p'}$$

$$2\int\frac{d^3p}{(2\pi)^3}\, v_{p_i}\frac{\partial n_p^0}{\partial \varepsilon_p} f^a(p,p') = -N(0)\int\frac{d\Omega}{4\pi}\sum_l f_l^a \quad P_l(\cos\theta)\; v_F\cdot\cos\theta$$ (set p' along z axis).

$$= -\frac{F_1^a}{3}v_F$$

$$\Rightarrow \vec{j}_i(r,t) = 2\int\frac{d^3p}{(2\pi)^3}\, v_{p_i}\,\vec{\sigma}_p + 2\int\frac{d^3p'}{(2\pi)^3}\,\frac{F_1^a}{3}\, v_{p_i'}\,\vec{\sigma}_{p'}$$

$$\boxed{\vec{j}_i(r,t) = 2\int\frac{d^3p}{(2\pi)^3}\, v_{p_i}\,\vec{\sigma}_p\left(1+\frac{F_1^a}{3}\right)}$$