Lecture 9

Phenomenology of Superconductivity (I)

Contents

In early days, solid state physics was not considered as fundamental physics but applied. Although various classic and quantum methods were applied to solids, such as the Drude theory of transport, Sommerfeld theory of electrons, Debye theory of phonons, and the Bloch theory of band structure, there were no original principles arising from this field. Superconductivity, or, superfluidity, was an outstanding problem that puzzled the entire physics community not just that of condensed matter. The study of superconductivity distinguished condensed matter physics from "applied physics" to some extent.

1. Wolfgang Pauli: Solid-state physics is "Schmutzphysik", i.e., the physics of dirt.
2. P. W. Anderson – More is different (Science 1972). (I highly recommend young students to read this wonderful philosophical paper.)

A variety of amazing and puzzling facts for superconductivity, which cannot be understood in the framework of the single-electron theory, or, the band theory of electrons.

1. What protects the zero resistivity? We know that in the macroscopic world, there always exist dissipations due to friction.
2. Why does a superconductor exhibit the complete diamagnetism? Diamagnetism is very common but typically very weak. (A frog can be magnetically levitated alive in a strong B-field – Ig Nobel prize by Geim.)
3. The best metals are poor superconductors or non-superconducting.

Indeed, superconductivity is not a property of one or two electrons, but emerges as a collective behavior of a society of electrons, *i.e.*, it is a collective behavior. New principles emerge such as the off-diagonal long-range order, phase coherence, and the Anderson-Higgs mechanism, which have big impacts in the entire field of physics including both condensed matter and high energy.

1 A brief history of superconductivity

1. 1911 Onnes' discovery of superconductivity of Hg
2. 1933 Meissner-Ochsenfeld effect
3. 1935 London equation
4. 1937 Kapitsa, Allen, Misener's discovery of superfluid ^{4}He
5. 1950 Ginzburg-Landau theory
6. 1957 Abrikosov's vortex state
7. 1957 The microscopic theory – BCS
8. 1962 Josephson effect
9. 1962 Anderson-Higgs mechanism
10. 1971 ^{3}He superfluidity
11. 1986 Bednorz and Mueller's discovery of high T_c superconductivity
12. 2006 Iron-based superconductivity
13. 2010's Topological superconductivity, Majorana fermion
14. 2010's Superconductivity of hydrides, H_2S, LaH_{10}

2 Zero resistivity

Superconductivity was first discovered by H. K. Onnes in 1911. Before that he achieved the liquification of ^{4}He, which opened a new era of low temperature physics. It was natural for him to apply liquid He to cool materials down to an unprecedented low temperature, and reexamine their properties. He found that a jump of resistance of Mercury at 4.2K: Within a change of temperature of 0.01K, the resistance drops from 0.1Ω to below $10^{-6}\Omega$. Not just Mercury, many metals and alloys were found become superconducting at transition temperatures at the order of a few Kelvins.

In addition to the critical temperature T_c, there are also other evidence for a transition

1. Superconductivity can also be suppressed by carrying currents and being applied magnetic fields, i.e, there also exist critical current I_c and critical field H_c.

2. Specific heat discontinuity at $T = T_c$.

3. Superconductivity (conventional) is not sensitive to weak disorder. Perfect superconductivity is not a consequence of very long mean-free path, but a consequence of a new state.

3 Diamagnetism

The zero resistance below T_c is certainly remarkable, but by itself it is not sufficient to justify a superconductor, but rather an "ideal conductor". Superconductors have another property that the magnetic field cannot enter the bulk of superconductors (for type I superconductors), which is called the Meissner effect.

Let us first consider the process of applying a magnetic field to a normal metal. The Lentz law, or, the Faraday's law, says that eddy currents are induced to resist the flux to enter the metal. Due to resistance, eddy currents dissipate into heat and decay, and then the magnetic field finally enters the bulk, and the system reaches the equilibrium of a normal metal in a magnetic field. For both a superconductor and an ideal conductor, due to the zero resistance, the eddy currents do not decay and keep the magnetic field outside.

Figure 1: Apply a B-field to a superconductor and a perfect conductor.

Nevertheless, there exists a crucial difference between an ideal conductor and a superconductor. Say, we begin with

1. $T < T_c$ and zero field, and then apply the B-field. As pointed out, screening currents develop in both cases to prevent the B-field to penetrate inside.

2. Then we increase $T > T_c$ to the normal state, then the screening current decays and the B-field enters the bulk.

3. Then, we cool the sample again below T_c, and we will see a sharp contrast between an ideal conductor and a superconductor: For an ideal conductor, the B-field still stays inside the system, while in a superconductor, the B-field is expelled to outside.

Hence, whether the B-field is maintained or expelled depending on its history, while in a superconductor, the Meissner state is a stable thermodynamic phase which is reversible and independent on its history.

Actually, as pointed by J. Hirsch, the direction of the screening current in a superconductor during step (II) is actually opposite with the direction you would expect from Faraday's law. The dynamic process of how the screening current is generated I think is still not completely understood.

3.1 London theory

Below $T < T_c$, we divide electrons into "normal fraction" and "superconducting fraction". For the superconducting fraction, from Newton's 2nd law, we have

$$\begin{aligned}
&\frac{d\mathbf{v_s}}{dt} = \frac{\partial \mathbf{v_s}}{\partial t} + (\mathbf{v_s}\cdot\nabla)\mathbf{v_s} = \frac{e}{m}(\mathbf{E} + \frac{1}{c}\mathbf{v_s}\times\mathbf{B}) \Rightarrow \\
&\frac{\partial \mathbf{v_s}}{\partial t} - \frac{e}{m}\mathbf{E} + \frac{1}{2}\nabla v_s^2 = \mathbf{v_s}\times\Big(\nabla\times\mathbf{v_s} + \frac{e}{mc}\mathbf{B}\Big) \Rightarrow \\
&\frac{\partial}{\partial t}(\nabla\times\mathbf{v_s} + \frac{e}{mc}\mathbf{B}) = \nabla\times\Big(\mathbf{v_s}\times(\nabla\times\mathbf{v_s} + \frac{e}{mc}\mathbf{B})\Big).
\end{aligned} \tag{1}$$

Please note that we have use $\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v_s}\cdot\nabla$, which is well-known in the context of fluid mechanics.

Now let us use a stronger assumption that if at an initial time $t = 0$, we have

$$\nabla\times\mathbf{v_s} + \frac{e\mathbf{B}}{mc} = \nabla\times(\mathbf{v_s} + \frac{e}{mc}\mathbf{A}) = 0 \tag{2}$$

it will remains zero for all the time. This is the London equation proposed by Fritz and Hentz London brothers. We often write the London equation in the Coulomb gauge, i.e.,

$$\mathbf{v}_s = -\frac{e^2}{mc}\mathbf{A}, \quad \text{with} \quad \nabla\cdot\mathbf{A} = 0 \Longrightarrow \mathbf{J}_s = -\frac{e^2 n_s}{mc}\mathbf{A} = -\frac{c}{4\pi\lambda_L^2}\mathbf{A}. \tag{3}$$

where λ_L is a length scale satisfying

$$\frac{1}{\lambda_L^2} = \frac{4\pi n_s e^2}{mc^2} = \frac{\omega_p^2}{c^2}, \tag{4}$$

with ω_p is the plasma frequency of the superfluid component of electrons. λ_L is called the London penetration depth, whose physical meaning will be clarified later.

To see why this property is remarkable. Let us first consider the E&M response in a normal metal. Consider the Fourier transform of the vector potential $\mathbf{A}(\mathbf{q})$. We can decompose $\mathbf{A}(\mathbf{q})$ into the longitudinal part $\mathbf{A}_L$ and the transverse part $\mathbf{A}_T$, such that $\mathbf{q}\perp\mathbf{A}_T$ and $\mathbf{q}\parallel\mathbf{A}_L$ whose configurations are plotted in Fig. 2. In other words,

$$\begin{aligned}
\mathbf{A}_{L,i}(\mathbf{q}) &= \frac{\hat{q}_i\hat{q}_j}{q^2}\mathbf{A_j}, \\
\mathbf{A}_{T,i}(\mathbf{q}) &= \Big(\delta_{ij} - \frac{\hat{q}_i\hat{q}_j}{q^2}\Big)\mathbf{A_j}.
\end{aligned} \tag{5}$$

Since the longitudinal $\mathbf{A_L}$ is a pure gauge, if we apply $\mathbf{A}_T$ to any systems, it should not have physical observable effects, hence the response current is zero, i.e, $J^L(\mathbf{q}) =$

$\chi^L_{JJ}(\mathbf{q},0)\mathbf{A}_L(\mathbf{q}) = 0$ where χ_{JJ} is the current-current susceptibility. In other words, gauge invariance requires that

$$\chi^L_{JJ}(\mathbf{q},0) = 0. \tag{6}$$

On the other hand, in the long-wave length limit $q \to 0$, longitudinal and transverse vector fields are not easy to be distinguished. (The direction of $\mathbf{q}$ needs a large spacial size of $1/q$ to be determined.) Hence, we would expect that the difference of the responses between the longitudinal and transverse vector fields should vanish as q. Indeed, in the normal state, we have

$$\lim_{q\to 0} \chi^T_{JJ}(\mathbf{q},\omega = 0) = \chi^L_{JJ}(\mathbf{q},0) = 0. \tag{7}$$

However, the superconducting state does see the difference! We should still have $\chi^{\parallel}_{JJ} = 0$ as required by the gauge invariance. Let us transform the London equation into the Fourier space

$$\mathbf{q} \times \mathbf{J_s}(\mathbf{q}) = -\frac{n_s e^2}{mc}\mathbf{q} \times \mathbf{A}(\mathbf{q}). \tag{8}$$

The above equation shows that for the transverse vector field, we have the transverse current-current susceptibility

Figure 2: Configurations of a transverse and a longitudinal $\mathbf{A}$ field.

$$\chi^T_{JJ}(\mathbf{q},0) = -\frac{n_s e^2}{mc}. \tag{9}$$

Hence, in the superconducting state, we have

$$\lim_{q\to 0}\Big(\chi^T_{JJ}(q,0) - \chi^L_{JJ}(q,0)\Big) \neq 0. \tag{10}$$

In other words, even in the long-wave length limit, the system can still distinguish the transverse and longitudinal $\mathbf{A}$ fields, which is a consequence of long-range phase coherence.

3.2 Penetration depth

Based on Eq. 3, we have

$$\nabla \times \nabla \times \mathbf{J_s} = -\frac{n_s e^2}{mc}\nabla \times \mathbf{B} = -\frac{4\pi n_s e^2}{mc^2}\mathbf{J_s} \Longrightarrow \nabla(\nabla \cdot \mathbf{J_s}) - \nabla^2 \mathbf{J_s} = -\lambda_L^{-2}\mathbf{J_s}, \tag{11}$$

where λ_L carrying the unit of length is called the penetration depth. Plugging the steady state condition $\nabla \cdot \mathbf{J_s} = 0$, we have

$$\nabla^2 \mathbf{J_s} = \frac{\mathbf{J_s}}{\lambda_L^2}, \quad \nabla^2 \mathbf{B} = \frac{\mathbf{B}}{\lambda_L^2}. \tag{12}$$

Consider a superconductor with a boundary lying in the zy-plane, it is easy to solve that $\mathbf{B}$ and $\mathbf{J_s}$ decay into the bulk exponentially as e^{-x/λ_L}. Hence, the measurement of λ_L is a way to determine the superfluid density n_s.

De Genne's derivation Write down the free energy of a superconductor as

$$\begin{aligned} f &= f_s + \frac{mn_s}{2} v_s(\mathbf{r})^2 + \frac{B^2(\mathbf{r})}{8\pi} = f_s + \frac{m}{2n_s} J_s^2(\mathbf{r}) + \frac{B^2(\mathbf{r})}{8\pi} \\ &= f_s + \frac{\lambda_L^2}{8\pi}(\nabla \times \mathbf{B})^2 + \frac{\mathbf{B}^2(\mathbf{r})}{8\pi}. \end{aligned} \tag{13}$$

Based on this, and perform the variational principle, we arrive at

$$\nabla \times \nabla \times \mathbf{B} + \frac{\mathbf{B}}{\lambda_L^2} = 0. \tag{14}$$

3.3 Pippard non-local form – coherence length ξ

The London equation assumes $\mathbf{J_s}(\mathbf{r})$ only depends on $\mathbf{A}(\mathbf{r})$ at the same spacial point. More generally, it should depend on $\mathbf{A}(\mathbf{r})$ in a vicinity of $\mathbf{r}$. In the Coulomb gauge, it is written as

$$\mathbf{J}(\mathbf{r}) = C \int d\mathbf{r}' \frac{(\mathbf{A}(\mathbf{r}') \cdot \mathbf{R})\mathbf{R}}{R^4} e^{-\frac{R}{\xi_0}}, \tag{15}$$

where $\mathbf{R} = \mathbf{r} - \mathbf{r}'$, and C is a constant. ξ_0 is a length scale, called Pippard's correlation length, or, coherence length. Later on, it can be microscopically show that

Figure 3: Pippard's non-local form for E&M response.

$$\xi_0 = \hbar v_f/(\pi\Delta), \tag{16}$$

where Δ is the zero temperature superconducting gap function. Later we will see that ξ_0 is roughly speaking the size of a Cooper pair.

When $\mathbf{A}$ is very slow varying, we need return back to the London equation. Set $\mathbf{A} \parallel \hat{z}$, we have

$$\begin{aligned} \mathbf{J}(\mathbf{r}) &= C\mathbf{A} \int \frac{\cos^2\theta}{R^2} e^{-R/\xi_0} R^2 dR \sin\theta d\theta d\phi = \frac{4\pi}{3} C\xi_0 \mathbf{A} = -\frac{n_s e^2}{mc}\mathbf{A}, \\ C &= -\frac{3}{4\pi}\frac{n_s e^2}{mc\xi_0}. \end{aligned} \tag{17}$$

Modification of λ in the dirty limit In the dirty superconductor, the mean-free path l needs to enter the Pippard formula. It was proposed as

$$\mathbf{J}(\mathbf{r}) = C \int d\mathbf{r}' \frac{(\mathbf{A}(\mathbf{r}') \cdot \mathbf{R})\mathbf{R}}{R^4} e^{-\frac{R}{\xi_0}} e^{-\frac{R}{l}}, \tag{18}$$

where C is still normalized as in Eq. 17. Then we have

$$\begin{aligned} \mathbf{J}(\mathbf{r}) &= \frac{1}{\lambda_L^2}\frac{l}{\xi_0 + l}\mathbf{A}(\mathbf{r}) \to \frac{l}{\xi_0\lambda_L^2}\mathbf{A}(\mathbf{r}) \\ &\Rightarrow \lambda = \lambda_L(\frac{\xi_0}{l})^{\frac{1}{2}} \quad (\lambda \gg l, \xi_0 \gg l). \end{aligned} \tag{19}$$

The physical meaning is that disorder suppresses the superfluid density, which increases the penetration depth.

Chambers formula Pippard proposed this non-local form is based on the Chambers formula for the normal state for the AC response,

$$\mathbf{J}(\mathbf{r},\omega) = \frac{e^2 v_f}{4\pi}\frac{\partial n}{\partial \epsilon}\int \mathbf{dr}' \frac{(\mathbf{E}(\mathbf{r}')\cdot\mathbf{R})\mathbf{R}}{R^4} e^{-i\frac{\omega R}{v_f}} e^{-\frac{R}{l}}. \tag{20}$$

We can see clearly the analogy.

Modification of penetration in the Pippard limit ($\xi_0 \gg \lambda_L$) The London equation only applies in the limit of $\lambda_L \gg \xi_0$. If in the opposite case of $\xi_0 \gg \lambda_L$, i.e., the Pippard limit, the actual penetration depth is modified, since $\mathbf{A}$ is not slow varying in the length scale of ξ_0.

Consider a sample covers the half space of $z > 0$ with the boundary of the xy-plane. Assume that the actual penetration depth is λ, then $\mathbf{A}$ is only nonzero within a thickness of λ. This roughly speaking reduces the integral by a factor of λ/ξ_0, hence, $\mathbf{J} = -\frac{n_s e^2 \lambda}{mc\xi_0}\mathbf{A}$, and by self-consistency, we have

$$\lambda^{-2} = \frac{\lambda}{\xi_0}\lambda_L^{-2} \Rightarrow \lambda = (\xi_0 \lambda_L^2)^{1/3} \Rightarrow \frac{\lambda}{\lambda_L} = \left(\frac{\xi_0}{\lambda_L}\right)^{1/3}. \tag{21}$$

Here are some experimental data (see the Table. I)

	λ_L	ξ_0	λ_{th}	λ_{exp}
Al	157	16,000	530	490 $\sim$ 515
Sn	355	2300	560	510
Pb	370	830	480	390

Table 1: Penetration depth in the limit of $\xi_0 \gg \lambda$.

Temperature dependence Later on based on the microscopic theory, it can be derived that ξ_0 is nearly independent of temperature, C is temperature dependent and vanishes at T_c. Experimentally, an empirical law is

$$\frac{\lambda^2(T)}{\lambda^2(0)} = \frac{T_c^4}{T_c^4 - T^4} \sim \frac{1}{1 - T/T_c} \quad as \quad T \to T_c. \tag{22}$$

Hence C vanishes linearly as $T \to T_c$.

4 Thermodynamics of superconductor

4.1 Condensation energy

The superconducting state has free energy density f_s less than that of the normal state f_n. Now we relate their difference to the critical field H_c. Consider a superconducting cylinder with radius r_0 and length L, which is put inside a big solenoid with radius r, N turns of wire, and the current I. The magnetic field inside the solenoid can be calculated $H = \frac{4\pi NI}{cL}$. The total energy densities of the normal and superconducting states inside the region of the sample is

$$F_n = \pi r_0^2 L(f_n + \frac{H^2}{8\pi}), \quad F_s = \pi r_0^2 L f_s. \tag{23}$$

An additional work needs to be taken into account: When flux is repelled from the sample, there generated an emf. It does work to the current I, which could be used to lift some weight,

$$W = \int dt\epsilon(t)I = \int -N\frac{d\Phi}{dt}I = -N(\Phi_f - \Phi_i)I = NH\pi r_0^2 I = \pi r_0^2 L\frac{H^2}{4\pi}. \tag{24}$$

At the critical field H_c, we have $F_n = F_s + W$, i.e.,

$$f_n = f_s + \frac{H_c^2}{8\pi}, \tag{25}$$

hence $H_c^2/8\pi$ is the condensation energy.

We can also define the Gibbs function $G = F - HM$, and at H_c we have $G_n(H_c) = G_s(H_c)$. From thermodynamics, we have for the free energy and Gibb densities

$$df = HdM, \quad dg = -MdH, . \tag{26}$$

In the normal state, we have $M \approx 0$ and $B = H$, where in the superconducting state, since $B = H + 4\pi M = 0$, we have $M = -\frac{H}{4\pi}$.

Hence, as applying increasing H from 0 to H_c, we have

$$g_s(H_c) - g_s(0) = -\int_0^{H_c} MdH = -\frac{1}{4\pi}\int_0^{H_c} HdH = \frac{1}{8\pi}H_c^2 \tag{27}$$

where $g_s(H_c) = g_n(H_c) \approx g_n(0)$. Hence, $g_n(0) - g_s(0) = f_n - f_s = \frac{H_c^2}{8\pi}$.

4.2 Latent heat

Now we calculate the latent heat: Consider two close points 1 and 2 on the phase boundary $H_c(T)$. From 1 to 2, we have along both the normal and superconducting side

$$dG_s = -S_s dT - M_s dH, \quad dG_n = -S_n dT - M_n dH. \tag{28}$$

We arrive the

$$\frac{dH_c}{dT} = -\frac{S_n - S_s}{M_n - M_s} \approx -(S_n - S_s)\frac{4\pi}{H_c} \Rightarrow S_n - S_s = -\frac{H_c}{4\pi}\frac{dH_c}{dT}. \tag{29}$$

When the latent heat is zero, we have the 2nd order phase transition. This can occur either at $H_c = 0$ or at $\frac{dH_c}{dT} = 0$. The first case is at zero field where the superconducting transition is of the 2nd order. Since $H_c(T) = H_c(0)(1-(T/T_c)^2)$, the slope of H_c is zero at $T=0$, and the field-driven zero temperature transition is also of the 2nd order. Generally, at other places on $H_c(T)$, the transition is of the 1st order.

Let us calculate the specific heat jump at the zero field transition:

$$(C_n - C_s)|_{T_c} = T_c \frac{d(S_n - S_s)}{dT} = -\frac{T_c}{4\pi}\left(\frac{dH_c}{dT}\right)^2. \tag{30}$$

Phenomenology of Superconductivity (II) – Ginzburg-Landau formalism

Contents

In the general framework of Landau's 2nd order phase transition, an order parameter is used to describe an ordered state, and a free-energy is constructed based on symmetry. And the phase transition is determined by whether the order parameter reaches a non-zero expectation value. Many order parameters are easy to understand and are physically observables by themselves. For example, the order parameter for magnetism is typically the magnetic density, which can be a scale field for the Ising type magnetism, a 2-vector field (complex field) for the XY type magnetism, or, a 3-vector field for the isotropic case.

The nature of the superconducting order parameter However, the superconducting order parameter is non-intuitive, and it is difficult to have a physical picture. Indeed, even when it was constructed, people are not sure what it is since the microscopic theory was not available at that time. The situation was analogues to when Mendeleev discovered the periodical table without knowing quantum mechanics. Nevertheless, there are some guidance.

1. It should be a complex field $\Psi(\mathbf{r})$ rather than a real field, since superconductivity

strongly couples to E&M fields. Hence, it has a phase degree of freedom. Then its coupling to the E&M field can be added by the minimal substitution.

2. Then naturally it can be viewed as the macroscopic wavefunction for the superconducting component of electrons, such that $|\Psi(r)|^2 \propto n_s(r)$. Please note that actually this is not very precise. The superfluid density is a description of phase stiffness of superconductivity, and $|\Psi(r)|^2$ is directly related to the gap function. For a fixed $|\Psi(r)|^2$, the superfluid density $n_s(r)$ can be suppressed by other factors, such as disorder, etc.

Ginzburg-Landau formalism at the beginning was not received much attention, but later Gor'kov found that it can be derived from the microscopic BCS Hamiltonian, in which the GL wavefunction is the anomalous Green's function. The order parameter can be understood as the macroscopic wavefunction of the center of mass motion of Cooper pairs.

1 Ginzburg-Landau free energy

1.1 The GL equation

Assume that the superconducting electrons are described by a complex order parameter $\Psi(r)$ with an effective free energy density

$$f[\Psi, \mathbf{A}] = \frac{1}{2m^*}|\Big(-i\hbar\nabla - \frac{e^*}{c}\mathbf{A}\Big)\Psi|^2 + \alpha|\Psi|^2 + \frac{\beta}{2}|\Psi|^4 + \frac{|\nabla\times\mathbf{A}|^2}{8\pi}. \tag{1}$$

where α is temperature dependent, and β is temperature independent. Here $e^* = 2e, m^* = 2m$ due to the pairing nature of the order parameter. Actually this was not known when the theory was constructed.

In the vicinity of T_c, we expand

$$\alpha = \alpha_0\Big(\frac{T}{T_c} - 1\Big). \tag{2}$$

In a fixed external magnetic field $\mathbf{H}$, the correct thermodynamic potential to consider is the Gibbs function

$$G = \int dv\Big(f[\Psi, \mathbf{A}] - \frac{1}{4\pi}\mathbf{B}\cdot\mathbf{H}\Big) = \int dv\Big(f[\Psi, \mathbf{A}] - \frac{1}{4\pi}(\nabla\times\mathbf{A})\cdot\mathbf{H}\Big). \tag{3}$$

Performing the variation with respect to Ψ^*, we arrive at the

$$-\frac{1}{2m^*}(-i\hbar\nabla - \frac{e^*}{c}\mathbf{A})^2\Psi + \alpha\Psi + \beta|\Psi|^2\Psi = 0, \tag{4}$$

which is called the GL equation (I). The corresponding boundary condition is that

$$\hat{\mathbf{n}}\cdot(-i\hbar\nabla - \frac{e^*}{c}\mathbf{A})\Psi = 0, \tag{5}$$

which was used by Landau and Ginzburg. Later on, based on microscopic theory calculation, the boundary condition is

$$\hat{n} \cdot (\nabla - \frac{ie^*}{\hbar c}\mathbf{A})\Psi = \frac{\Psi}{b}, \tag{6}$$

where b is a constant depending on material properties. For the superconductor-insulator interface, $b \approx \xi_0^2/a$ with a the lattice constant, and thus it is enormously large even compared to $\xi(T)$. But for the superconductor-metal interface, b can be short compare to $\xi(T)$, hence the boundary effect is very important. By multiplying Ψ^* to the left hand side of the equation, and take its imaginary part, it gives rise to that the supercurrent density can only flow along the surface, i.e.,

$$\hat{\mathbf{n}} \cdot \mathbf{J_s} = 0. \tag{7}$$

As for performing the variation with respect to $\mathbf{A}$, we need to use the following identity

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B}), \tag{8}$$

we have

$$\begin{aligned} &(\nabla \times \mathbf{A}) \cdot (\nabla \times \delta\mathbf{A}) = \nabla \cdot (\delta\mathbf{A} \times (\nabla \times \mathbf{A})) + \delta\mathbf{A} \cdot \nabla \times (\nabla \times \mathbf{A}), \\ &-\mathbf{H} \cdot \nabla \times \delta\mathbf{A} = -\nabla \cdot (\delta\mathbf{A} \times \mathbf{H}) + \delta\mathbf{A} \cdot (\nabla \times \mathbf{H}) \end{aligned} \tag{9}$$

Then we have

$$\begin{aligned} &\delta\mathbf{A} \cdot \Big(\frac{1}{4\pi}\nabla \times \nabla \times \mathbf{A} - [\frac{-i\hbar}{2m^*}\frac{e}{c}(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*) - \frac{e^2}{m^*c^2}|\Psi|^2\mathbf{A}]\Big) = 0 \\ &\mathbf{J_s} = \frac{c}{4\pi}\nabla \times \nabla \times \mathbf{A} = \frac{-ie^*\hbar}{2m^*}(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*) - \frac{e^{*2}}{m^*c}\mathbf{A}|\Psi|^2, \end{aligned} \tag{10}$$

where we use the condition that $\nabla \times \mathbf{H} = 0$ in the interior of a superconductor.

The corresponding boundary condition is

$$\begin{aligned} &\hat{n} \cdot (\delta\mathbf{A} \times (\nabla \times \mathbf{A}) - \delta\mathbf{A} \times \mathbf{H}) = 0 \Rightarrow \delta\mathbf{A} \cdot \Big((\mathbf{B} - \mathbf{H}) \times \hat{\mathbf{n}}\Big) = 0 \\ &\hat{\mathbf{n}} \times (\mathbf{B} - \mathbf{H}) = 0 \end{aligned} \tag{11}$$

1.2 Flux quantization

Consider a multiple-connected geometry of a ring, and the thickness of the ring is much larger than λ_L. Consider a loop inside the bulk of a ring, such that the supercurrent along the loop vanishes. The flux trapped inside such a loop is calculated as

$$\mathbf{J_s}(\mathbf{r}) = 0 \Rightarrow \oint d\mathbf{r}\nabla\varphi(r) = \frac{e^*}{\hbar c}\oint d\mathbf{r}\mathbf{A} \Rightarrow \Phi = \frac{2n\pi}{e^*/\hbar c} = \frac{hc}{2e} = 2 \times 10^{-7} G.cm^2 \tag{12}$$

2 Characteristic length scales

2.1 Healing length $\xi(T)$ and penetration depth $\lambda(T)$

Plugging in $\Psi(\mathbf{r}) = \sqrt{\rho_s(\mathbf{r})}e^{i\varphi(r)}$, we have

$$\mathbf{J_s} = \frac{e^*\hbar}{m^*}\rho_s(r)\Big(\nabla\phi(r) - \frac{e^*}{\hbar c}\mathbf{A}(\mathbf{r})\Big). \tag{13}$$

If we set $e^* = 2e, m^* = 2m$, and $\rho_s(r) = n_s/2$, the London equation is recovered. Here, ρ_s is the density of superfluid Cooper pairs, hence it is half of the number of electrons.

From $\frac{c}{4\pi\lambda_L^2} = \frac{e^{*,2}}{m^* c}\rho_s$, we have

$$\lambda_L^2(T) = \frac{m^* c^2}{4\pi e^{*2}}\frac{\beta}{|\alpha|} \propto |T_c - T|, \tag{14}$$

where

$$\rho_s(T) = |\Psi_0|^2 = \frac{|\alpha|}{\beta} \propto |T_c - T|, \quad \text{at} \quad T \to T_c. \tag{15}$$

We can also define a length scale, called the healing length

$$\xi^2(T) = \frac{\hbar^2}{2m^*|\alpha|} \Rightarrow \xi(T) \propto |T - T_c|^{-\frac{1}{2}}. \tag{16}$$

Hence, we have the mean-field values of two critical exponents

$$\beta = \frac{1}{2}, \quad \nu = \frac{1}{2}. \tag{17}$$

(Here β is a standard notation of a critical exponent for the sharp increase of order parameter just below T_c. Please do not confuse it with the coefficient of the GL equation.)

2.2 Thermodynamic critical field H_c

We define the thermodynamic critical field H_c as related to the condensation energy,

$$\frac{H_c^2}{8\pi} = \frac{\alpha^2}{2|\beta|} \Rightarrow \frac{H_c^2}{4\pi} = \frac{\alpha^2}{|\beta|}. \tag{18}$$

Then

$$\begin{aligned} \frac{1}{\lambda_L(T)\xi(T)} &= \frac{e^*}{\hbar c}\Big(\frac{8\pi|\alpha|^2}{\beta}\Big)^{\frac{1}{2}} = \frac{\sqrt{2}e^*}{\hbar c}H_c \\ H_c(T) &= \frac{\Phi_0}{2\pi\sqrt{2}\xi(T)\lambda(T)} \end{aligned} \tag{19}$$

2.3 Upper critical field H_{c2}

We consider a critical field at which superconductivity is completely suppressed. Such a field is called H_{c2}. Later on, we will see that it actually happens when the vortex cores touch each other. In this case, H_{c2} is at the order of Φ_0/ξ^2. Now we start with GL equation. Since at H_{c2}, the order parameter nearly is nearly zero, we can neglect the $|\Psi|^4$ term, and arrive at

$$f_s = -\frac{\hbar^2}{2m^*}\Psi^*\Big(-i\hbar\nabla - \frac{e^*}{c}\mathbf{A}(\mathbf{r})\Big)^2\Psi + \alpha\Psi^*\Psi. \tag{20}$$

The cyclotron frequency $\omega_c = \frac{eB}{m*c}$, which set up the Landau level gap $\hbar\omega$. The zero point motion energy is $\frac{1}{2}\hbar\omega_c$. Hence, when

$$\begin{aligned} \frac{1}{2}\hbar\omega_c &= |\alpha|, \quad \Rightarrow \frac{\hbar}{2}\frac{e^*B}{m^*c} = |\alpha|, \quad \Rightarrow \frac{\hbar^2}{2m^*|\alpha|} = \frac{\hbar c}{e^*B} \\ B &\approx H_{c2} = \frac{hc/e*}{2\pi\xi^2} = \frac{\Phi_0}{2\pi\xi^2}. \end{aligned} \tag{21}$$

2.4 The GL parameter κ: type I and II

Now we can define a dimensionless quantity,

$$\begin{aligned} \kappa &= \frac{\lambda_L(T)}{\xi(T)} = \Big(\frac{\beta}{2\pi}\Big)^{\frac{1}{2}}\frac{m^*c}{\hbar e^*} = \frac{\lambda_L^2(T)}{\lambda_L(T)\xi(T)} \\ \kappa &= \sqrt{2}H_c\lambda_L^2(T)/(\frac{\hbar c}{e^*}) = 2\sqrt{2}\pi\frac{H_c\lambda_L^2(T)}{\Phi_0}, \end{aligned} \tag{22}$$

which is called the Ginzburg-Landau parameter. κ plays an important role in determining type I and II superconductors.

It is easy to check that

$$\frac{H_{c2}}{H_c} = \sqrt{2}\kappa. \tag{23}$$

We will have the following two situations, denoted as the type (II) and (I) superconductors, respectively.

1. Type II: If $\kappa > \frac{1}{\sqrt{2}}$, then $H_{c2} \geq H_c$. When $H_{c2} > H > H_c$, we still have superconducting condensation, since H is not enough to kill Ψ completely. Nevertheless, it cannot completely expel the magnetic flux either. Otherwise, the Meissener phase will not be energetically favorable at $H > H_c$. Hence, magnetic flux will partly enter the bulk, and it is called the mixed state. Later on, we will that it corresponds to form vortices.

2. Type I: If $\kappa < \frac{1}{\sqrt{2}}$, then $H_c > H_{c2}$. If we lower down the magnetic field, it will first reach H_c, then it becomes the complete Meissner phase, and it will not change as further lowering the field.

3 Interface

In this part, I will talk about preliminary applications of the GL formalism to spatially inhomogeneous systems.

3.1 Interface solution

Let us solve the first GL equation for a semi-infinite sample in the absence of the magnetic field, which cover the space of $x > 0$ and uniform along y and z-directions. We renormalize $f = \Psi/\Psi_0$, then the GL equation

$$\xi^2(T)\nabla^2 f = -f(1-f^2) \tag{24}$$

with the boundary condition

$$\begin{aligned} \text{at } x \to 0, &\qquad f \to 0 \\ \text{at } x \to \infty, &\qquad f \to 1, \ \frac{df}{dx} \to 0, . \end{aligned} \tag{25}$$

Here we assume in the limit $b/\xi(T) \to 0$, such that the $f \to 0$ at $x \to 0$ is compatible to the boundary condition Eq. 6.

We have

$$\begin{aligned} \frac{df}{dx}\frac{d^2 f}{dx^2} &= \xi^{-2} f \frac{df}{dx}(1-f^2) \\ \frac{d}{dx}(\frac{df}{dx})^2 &= -\frac{\xi^{-2}}{2}\frac{d}{dx}(1-f^2)^2, \\ \frac{df}{dx} &= \frac{(1-f^2)}{\sqrt{2}\xi(T)} \Rightarrow \frac{df}{1-f^2} = \frac{dx}{\sqrt{2}\xi(T)}, \\ f(x) &= \tanh\frac{x}{\sqrt{2}\xi(T)}. \end{aligned} \tag{26}$$

Healing length v.s. Pippard's correlation length The physical meaning of the healing length $\xi(T)$ is different from the Pippard correlation lenght ξ_0. Healing length is the length scale over which the superconducting order parameter varies. ξ_0 in the clean system describes the size of the Cooper pair, and roughly

$$\xi(T) = \xi_0(1 - T/T_c)^{-\frac{1}{2}}. \tag{27}$$

Since $\Psi(r)$ describes the center of mass motion of a Cooper pair, ξ_0 defines the short range length cut off of the GL theory. It makes no sense to apply the GL theory for length scale smaller that ξ_0. Roughly speaking, ξ_0 can be viewed as the healing length at $T = 0$. The GL theory only works in the vicinity of T_c, and it does not apply to the zero temperature.

3.2 Positive and negative surface energy

The boundary suppresses the superconducting order parameter, hence, this effect costs the energy. If the system were uniform, the energy from the superconducting order parameter is

$$\frac{H_c^2}{8\pi} = \frac{\alpha^2}{2\beta}. \tag{28}$$

We assume the limit that $H_c \to 0$ at $T \to T_c$, such that the condensation energy is simply

$$E/A = \int_0^\infty dx\Big(\frac{\hbar^2}{2m^*}|\frac{d\Psi}{dx}|^2 + \alpha|\Psi|^2 + \frac{\beta}{2}|\Psi|^4\Big) = \frac{\alpha^2}{2\beta} \times 2\int_0^\infty dx\Big(\xi^2(\frac{df}{dx})^2 - f^2 + \frac{1}{2}f^4\Big)$$

Compare to the condensation energy in the uniform case, we have

$$\frac{\Delta E_c}{A} = \frac{E}{A} - \int dx\Big(-\frac{H_c^2}{8\pi}\Big) = \frac{H_c^2}{8\pi}\delta, \tag{29}$$

where δ carry the length unit defined as

$$\begin{aligned}\delta &= 2\int_0^\infty dx\Big(\xi^2(\frac{df}{dx})^2 - f^2 + \frac{1}{2}f^4 + \frac{1}{2}\Big) = 2\int_0^\infty dx(1-f^2)^2 \\ &= 2\int_0^\infty df\frac{(1-f^2)^2}{\frac{df}{dx}} = 2\sqrt{2}\xi(T)\int_0^1 df(1-f^2) = \frac{4\sqrt{2}}{3}\xi(T).\end{aligned} \tag{30}$$

Now we consider in the case with the magnetic field $H < H_c$. Assume that $H_c \to 0$, such that we can use Eq. 30 for the condensation energy, which neglects the vector field dependence. On the other hand, the magnetic field contribution can penetrate within a region at the order of $\lambda(T)$. The magnetic field contribution to the Gibbs function is

$$\frac{E_H}{A} = \int dx\frac{B^2}{8\pi} - \frac{BH}{4\pi} = -\frac{H^2}{8\pi}\int dx\Big(1 - \Big(1 - \frac{B}{H}\Big)^2\Big) \sim -\frac{H^2}{8\pi}c_1\lambda_L(T). \tag{31}$$

where c_1 is a numeric coefficient at the order of 1, and its actually value is not important. Hence, the interface energy is

$$\Delta E/A = \frac{H_c^2}{8\pi}\Big(\frac{4\sqrt{2}}{3}\xi(T) - c_1\frac{H^2}{H_c^2}\lambda_L(T)\Big) \tag{32}$$

Set $H = H_c$, then in the limit of $\kappa \gg 1$, the surface energy is negative. This means that when magnetic field enters the sample, it will fragment into many vortices. Each vortex carries the minimum number of flux allowed by the flux quantization, i.e., $\Phi_0 = hc/(2e)$. On the other hand, if $\kappa \ll 1$, the surface energy is positive. It does not favor to form vortices. The magnetic field goes into the bulk by suppressing superconductivity. This divides the system into two different region called type (II) and type (I) respectively. A more careful study of the surface energy shows that the precise boundary is that $\kappa < \frac{1}{\sqrt{2}}$ (type (I)), and $\kappa > \frac{1}{\sqrt{2}}$ (type (II)).

3.3 Condition for zero interface energy

Consider the GL equation,

$$\frac{1}{2m^*}\Big(\Pi_x^2 + \Pi_y^2\Big)\Psi + \alpha\Psi + \beta|\Psi|^2\Psi = 0, \tag{33}$$

where

$$\Pi_x = -i\hbar\nabla_x - \frac{e^*}{c}A_x, \quad \Pi_y = -i\hbar\nabla_y - \frac{e^*}{c}A_y \tag{34}$$

Define $\Pi^\pm = \Pi_x \pm \Pi_y$, then we have

$$[\Pi^+, \Pi^-] = i\hbar\frac{e^*}{c}B_z. \tag{35}$$

Then the GL equation becomes

$$\frac{1}{2m^*}\Pi^-\Pi^+\Psi(\mathbf{r}) + \Big(\frac{\hbar e B_z(\mathbf{r})}{2m^*c} + \alpha + \beta|\Psi|^2(\mathbf{r})\Big)\Psi(\mathbf{r}) = 0. \tag{36}$$

We seek the solution that

$$\begin{aligned} \Pi^+\Psi(\mathbf{r}) = \Big(-i\hbar(\nabla_x + i\nabla_y) - \frac{e^*}{c}(A_x + iA_y)\Big)\Psi(r) &= 0, \\ \nabla_y\Psi - i\nabla_x\Psi = \frac{e^*}{\hbar c}(A_x + iA_y)\Psi, \end{aligned} \tag{37}$$

and this solution is called Sarma's solution. Then

$$\begin{aligned} \nabla\times\mathbf{B} &= \frac{\partial B_z}{\partial y}\hat{x} - \frac{\partial B_z}{\partial x}\hat{y} = \frac{4\pi}{c}\mathbf{J} = \frac{-2i\pi e^*}{m^*c}(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*) - \frac{4\pi e^{*2}}{m^*c^2}\mathbf{A}|\Psi|^2 \\ \frac{\partial B_z}{\partial y} &= \frac{2\pi e^*}{m^*c}(\Psi^*\Pi_x\Psi + \Psi\Pi_x^*\Psi^*) \\ \frac{\partial B_z}{\partial x} &= -\frac{2\pi e^*}{m^*c}(\Psi^*\Pi_y\Psi + \Psi\Pi_y^*\Psi^*) \\ \frac{\partial B_z}{\partial y} - i\frac{\partial B_z}{\partial x} &= \frac{2\pi e^*}{m^*c}\Big(\Psi^*\Pi^+\Psi + \Psi(\Pi^-\Psi)^*\Big) = \frac{2\pi e^*}{m^*c}\Psi(\Pi^-\Psi)^*. \end{aligned} \tag{38}$$

On the other hand,

$$\begin{aligned} \frac{e\hbar}{2m^*c}\mathbf{B} &= -(\alpha + \beta\Psi^*\Psi)\hat{z} \\ \frac{e\hbar}{2m^*c}\Big(\frac{\partial B_z}{\partial y} - i\frac{\partial B_z}{\partial x}\Big) &+ \beta\Psi^*(\frac{\partial\Psi}{\partial y} - i\frac{\partial\Psi}{\partial x}) + \beta\Psi(\frac{\partial\Psi^*}{\partial y} - i\frac{\partial\Psi^*}{\partial x}) = 0 \\ \frac{\partial B_z}{\partial y} - i\frac{\partial B_z}{\partial x} &= \frac{2m^*c\beta}{e\hbar^2}(\Psi(\Pi^-)^*\Psi^*) \end{aligned} \tag{39}$$

In order for Eq. 38 and Eq. 39 to be consistent with each other, we need to have

$$\frac{2\pi e^*}{m^*c} = \frac{2m^*c\beta}{e^*\hbar^2} \Rightarrow 2\kappa^2 = \Big(\frac{m^*c}{e^*\hbar}\Big)^2\frac{\beta}{\pi} = 1 \Rightarrow \kappa = \frac{1}{\sqrt{2}}. \tag{40}$$

Now consider a wall in the yz-plane. Consider the boundary condition that

$$x \to -\infty, \quad \Psi \to 0, \quad \mathbf{B} = H_c \hat{z}$$
$$x \to +\infty, \quad \Psi \to \frac{|\alpha|}{\beta}, \quad \mathbf{B} = 0. \tag{41}$$

Sarma's solution does apply to these boundary conditions. Hence

$$\begin{aligned} G &= \int dx \Big(\alpha |\Psi|^2 + \frac{\beta}{2} |\Psi|^4 + \frac{1}{2m^*} \Big(|\Pi_x \Psi|^2 + |\Pi_y \Psi|^2 \Big) + \frac{B^2}{8\pi} - \frac{BH_c}{4\pi} \\ &= \int dx \Psi^* (-\frac{1}{2m^*} (\Pi_x^2 + \Pi_y^2) \Psi) + \alpha |\Psi|^2 + \frac{\beta}{2} |\Psi|^4 + \frac{B^2}{8\pi} - \frac{BH_c}{4\pi} \\ &= \int dx \Big(-\frac{\beta}{2} |\Psi|^4 + \frac{B^2 - 2H_c B}{8\pi} \Big) \end{aligned} \tag{42}$$

Now subtract the Gibbs function of the uniform case at H_c, which is $-\frac{H_c^2}{8\pi}$. Then the wall energy is

$$\Delta G = \int dx \Big(-\frac{\beta}{2} |\Psi|^4 + \frac{(B - H_c)^2}{8\pi} \Big). \tag{43}$$

Since

$$B = \frac{2m^* c}{e\hbar} (|\alpha| - \beta |\Psi|^2) = H_c - \frac{2\beta m^* c}{e\hbar} |\Psi|^2 = H_c - \sqrt{8\pi\beta} \kappa |\Psi|^2, \tag{44}$$

where we have used $\frac{2m^* c}{e\hbar} |\alpha| = H_c$ at $\kappa = \frac{1}{\sqrt{2}}$. Then

$$\Delta G = \int dx (-\frac{\beta}{2} + \frac{\beta}{2}) |\Psi|^4 = 0. \tag{45}$$

This means the vanishing of surface energy at $\kappa = 1/\sqrt{2}$, and $H = H_c$.

4 Vortex solution

4.1 A single vortex line

We consider in the extreme type II limit, i.e, $\lambda \gg \xi$, such that the vortex core radius ξ is very small. We can neglect the cost of energy due to the suppression of superconducting order parameter at the core. Then the free energy becomes

$$\begin{aligned} F &= \int_{r>\xi} dv \; \frac{B^2}{8\pi} + \frac{\hbar^2 \rho_s}{2m^*} (\nabla \phi - \frac{e^*}{c} \mathbf{A})^2 = \int_{r>\xi} dv \; \frac{B^2}{8\pi} + \frac{m^*}{2\rho_s e^{*2}} J_s^2 \\ &= \int_{r>\xi} dv \; \frac{B^2}{8\pi} + \frac{m^* c^2}{32\pi^2 \rho_s e^{*2}} (\nabla \times B)^2 = \int_{r>\xi} dv \; \frac{1}{8\pi} \Big(B^2 + \lambda_L^2 (\nabla \times B)^2 \Big), \end{aligned} \tag{46}$$

with $\lambda_L^{-2} = \frac{4\pi\rho_s e^*}{m^* c^2}$. After performing the variational principle, we arrive at

$$\mathbf{B} + \lambda_L^2 \nabla \times \nabla \times \mathbf{B} = \mathbf{B} - \lambda_L^2 \nabla^2 \mathbf{B} = \Phi_0 \hat{z} \delta^2(\mathbf{r}), \tag{47}$$

where the δ-function on the right hand represents the vortex core at the size of ξ. Take a loop around the core, we have

$$\int d\sigma \cdot \mathbf{B} + \lambda_L^2 \oint \nabla \times \mathbf{B} \cdot d\mathbf{l} = \Phi_0. \tag{48}$$

At $r \gg \lambda_L$, $\nabla \times \mathbf{B} = 4\pi/c\mathbf{J_s}(\mathbf{r})$ decays exponentially, then the loop integral is negligible, and, hence Φ_0 is the fundamental flux quanta. If $\xi < r \ll \lambda_L$, the contribution from the first time scales as ξ^2/λ_L^2, then we have

$$\nabla \times (B(r)\hat{z}) = -\frac{dB}{dr}\hat{e}_\theta = \frac{\Phi_0}{2\pi\lambda_L^2}\frac{1}{r}\hat{e}_\theta \Rightarrow dB = \frac{\Phi}{2\pi\lambda_L^2} d\ln r^{-1} \Rightarrow B = \frac{\Phi}{2\pi\lambda_L^2}\ln(\frac{\lambda}{r}) + const$$

Set $\mathbf{B} = B\hat{z}$, we use the cylindrical coordinate,

$$\frac{\partial^2 B}{\partial (r/\lambda)^2} + \frac{1}{r/\lambda}\frac{\partial B}{\partial (r/\lambda)} - B = 0, \tag{49}$$

which is the zeroth order imaginary argument Bessel function. At $r \to \infty$, $B \to 0$, hence, we use the solution that

$$B(r) = \frac{\Phi_0}{2\pi\lambda_L^2} K_0(\frac{r}{\lambda_L}) \tag{50}$$

where $K_0(x) \to \ln\frac{2}{x}$ at $x \to 0$. Then we also have the long distance behavior

$$B(r) = \frac{\Phi_0}{2\pi\lambda_L^2}\sqrt{\frac{\pi\lambda_L}{2r}} e^{-\frac{r}{\lambda_L}} \tag{51}$$

at $r \gg \lambda_L$.

Figure 1: Structure of a single vortex line. The magnetic filed and screening current distribute within a distance at the order of λ from the core, and the order parameter is suppressed within a distance within ξ.

Now we calculate the free energy of a single vortex line. By using the classic equation Eq. 47, we have

$$\begin{aligned} F &= \frac{\lambda_L^2}{8\pi}\int_{r\geq\xi} -\mathbf{B}\cdot\nabla\times\nabla\times\mathbf{B} + (\nabla\times\mathbf{B})^2 \\ &= \frac{\lambda_L^2}{8\pi}\int_{r\geq\xi} \nabla\cdot(\mathbf{B}\times\nabla\times\mathbf{B}) = \frac{\lambda_L^2}{8\pi}\int_{r\geq\xi} d\mathbf{s}\cdot(\mathbf{B}\times\nabla\times\mathbf{B}) \\ &= \frac{\lambda_L^2}{8\pi} 2\pi\xi B(\xi)|\nabla\times B(\xi)|L, \end{aligned} \tag{52}$$

where L is the length of the vortex line, and the integral surface is the side of a cylinder with a small radius ξ. Plugging $B(\xi) = \frac{\Phi_0}{2\pi\lambda^2}\ln(\lambda/\xi)$, $|\nabla \times B(\xi)| = \Phi/(2\pi\xi\lambda_L^2)$, we have

$$F/(L\lambda_L^2) = \Big(\frac{\Phi_0}{4\pi\lambda_L^2}\Big)^2 \ln\frac{\lambda_L}{\xi}. \Rightarrow F/L = \Big(\frac{\Phi_0}{4\pi\lambda_L}\Big)^2 \ln\frac{\lambda_L}{\xi} \tag{53}$$

4.2 The lower critical field H_{c1}

For the type (II) case, there exist a lower and an upper critical fields, H_{c1} and H_{c2}, respectively. Above H_{c1}, vortices start to form. Consider the dilute limit, or, so that we can neglect the interaction among vortex lines. Consider the Gibbs function per unit length of the vortex line,

$$G/L = NF/L - A\frac{\bar{B}H}{4\pi} \tag{54}$$

where N is the number of vortices, and A is the area, $\bar{B}$ is the average value over the unit cell of vortex. The advantage to introduce $\bar{B}$ is that we do not need to examine the detailed spatial distribution of B-field.

Since each vortex carry the fixed flux Φ_0, $\bar{B}A/\Phi_0 = N$. We have

$$g = G/(LA) = \bar{B}\Big(\frac{F/L}{\Phi_0} - \frac{H}{4\pi}\Big) = \frac{\bar{B}}{4\pi}\Big(\frac{\Phi_0}{4\pi\lambda_L^2}\ln\frac{\lambda_L}{\xi} - H\Big), \tag{55}$$

where n is the vortex density.

Hence, we have the following two cases

1. The Meissner state: $H < H_{c1} = \frac{\Phi_0}{4\pi\lambda_L^2}\ln\frac{\lambda_L}{\xi}$. g is an increasing function of B, hence, the minimum g state is with $B = 0$.

2. The mixed state: $H > H_{c1}$. g decreases as B increases, and this marks the onset of the mixed state, the state with vortices.

4.3 Interaction between vortex lines

Consider two vortex lines along the z-axis located at $\mathbf{r_1}$ and $\mathbf{r_2}$. Then the magnetic field distribution is determined by

$$\begin{aligned} \mathbf{B} + \lambda_L^2\nabla\times\nabla\mathbf{B} &= \Phi_0(\delta(\mathbf{r}-\mathbf{r_1}) \\ &+ \delta(\mathbf{r}-\mathbf{r_2})) \end{aligned} \tag{56}$$

hence the solution of $B(r)$ is a superposition

$$\begin{aligned} \mathbf{B}(\mathbf{r}) &= \mathbf{B_1}(\mathbf{r}) + \mathbf{B_2}(\mathbf{r}), \\ \mathbf{B}_i(\mathbf{r}) &= \frac{\Phi_0}{2\pi\lambda^2}K_0\Big(\frac{|\mathbf{r}-\mathbf{r_i}|}{\lambda_L}\Big)\hat{z}. \end{aligned} \tag{57}$$

Similarly, the energy of the system is

$$F = \frac{\lambda_L^2}{8\pi}\int(\mathbf{B}\times\nabla\times\mathbf{B})\cdot d\mathbf{S} = \frac{\lambda_L^2}{8\pi}\int(d\mathbf{S_1}+d\mathbf{S_2})\cdot(\mathbf{B_1}+\mathbf{B_2})\times(\nabla\times\mathbf{B}_1+\nabla\times\mathbf{B}_2), \tag{58}$$

where $d\mathbf{S_{1,2}}$ are cylinder with very small radius ξ around $\mathbf{r}_{1,2}$, respectively. Among the 8 terms, a few terms are non-vanishing as $\xi \to 0$, since B_i's are regular, $\mathbf{J_i} = \frac{c}{4\pi}\nabla\times\mathbf{B}$ diverges, we should combine $\mathbf{dS_i}$ and $\nabla\times\mathbf{B_i}$ together. There will be 4 terms, and two of them are the self-energies of each vortex which are The following two terms describe the interaction between vortices 1 and 2,

$$
\begin{aligned}
U_{12}/L &= \frac{\lambda_L^2}{8\pi}\Big(\int d\mathbf{S_2}\cdot(\mathbf{h_1}\times\nabla\times\mathbf{h_2}) + \int d\mathbf{S_1}\cdot(\mathbf{h_2}\times\nabla\times\mathbf{h_1})\Big) \\
&= \frac{\lambda_L^2}{8\pi}\Big(h_1(r_2)\hat{\mathbf{z}}\cdot\int(\nabla\times\mathbf{h_2})\times d\mathbf{S_2} + (1\leftrightarrow 2)\Big) \\
&= \frac{\lambda_L^2}{8\pi}h_1(r_2)\frac{\Phi}{2\pi\xi\lambda_L^2}2\pi\xi\hat{\mathbf{z}}\cdot(\hat{\mathbf{e}}_\theta\times-\hat{\mathbf{e}}_\mathbf{r})) + (1\leftrightarrow 2) \\
&= \frac{\Phi}{8\pi}(h_1(r_2)+h_2(r_1)) = \frac{\Phi_0}{4\pi}h_{12} = \frac{\Phi_0^2}{8\pi\lambda_L^2}K_0(\frac{r_{12}}{\lambda_L})
\end{aligned}
\tag{59}
$$

Hence, the interaction is repulsive. It diverges as $\ln(|\lambda_L/r_{12}|)$ at short distance, while at large distance, it decays $1/\sqrt{r_{12}}e^{-r_{12}/\lambda}$ due to screening currents.

When $H = H_{c1} + 0^+$, the vortex lines are dilute. We only need to include the nearest neighbour vortex lines for the Gibbs function

$$
\begin{aligned}
G/L &= NF/L - A\frac{\bar{B}H}{4\pi} + \frac{zN}{2}\frac{\Phi_0}{2\pi\lambda_L^2}K_0\Big(\frac{d}{\lambda_L}\Big) \\
g &= \frac{\bar{B}}{4\pi}\Big(H_{c1} - H + \frac{z}{2}\frac{\Phi_0}{2\pi\lambda_L^2}K_0\Big(\frac{d}{\lambda_L}\Big)\Big) \\
&= \frac{\bar{B}}{4\pi}\Big(H_{c1} - H + \frac{z}{2}\frac{\Phi_0}{2\pi\lambda_L^2}K_0\Big(\sqrt{\frac{\Phi_0}{\bar{B}\lambda^2}}\Big)\Big)
\end{aligned}
\tag{60}
$$

where the last term is the H-field produced from another vortices, and the distance $d \sim \sqrt{\frac{\Phi_0}{B}}$.

The effect of the vortex interaction is exponentially small in the dilute limit. But it becomes significant when the inter-vortex distance becomes comparable with the penetration depth. The sketch of the the Gibbs function v.s. $\bar{B}$ is plotted in Fig. 2. There is a minimum of position of $\bar{B}$, which determines the length of the vortex lattice.

Figure 2: The Gibbs function of the vortex lattice.