西湖大學
WESTLAKE UNIVERSITY

T: 0571-86886859
F: 0571-85271986
E: office@westlake.edu.cn

中国浙江省杭州市西湖区云栖小镇石龙山街18号，310024
18 Shilongshan Rd.,Cloud Town, Xihu District, Hangzhou,
Zhejiang Province, China, 310024

Lecture 11 Angular momentum conservation

Contents

① Symmetry and conservation

② Angular momentum — $L_{orb} + L_{spin}$

③ Examples : ① impact factor

② Centrifugal potential

西湖大學 WESTLAKE UNIVERSITY
T: 0571-86886859
F: 0571-85271986
E: office@westlake.edu.cn
中国浙江省杭州市西湖区云栖小镇石龙山街18号,310024
18 Shilongshan Rd.,Cloud Town, Xihu District, Hangzhou, Zhejiang Province, China, 310024

§ Conservation law due to symmetry — elementary

- energy conservation —— time translation symmetry
- momentum conservation — space translation symmetry

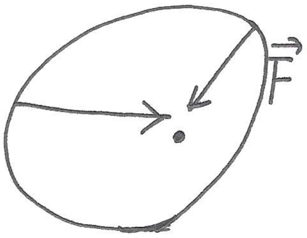
Every point in space is equivalent. An object has no preference on one place over the other. If its initial condition is at rest, it does not want to go other place. If it already moves, it'll keep its velocity. Let it be!

- angular momentum conservation — spacial isotropy
  (rotational symmetry)

西湖大學 WESTLAKE UNIVERSITY
T: 0571-86886859
F: 0571-85271986
E: office@westlake.edu.cn
中国浙江省杭州市西湖区云栖小镇石龙山街18号，310024
18 Shilongshan Rd.,Cloud Town, Xihu District, Hangzhou, Zhejiang Province, China, 310024

§ Central force field

$$\vec{F}(\vec{r}) = f(r)\,\hat{e}_r$$



Define $\vec{L} = \vec{r} \times \vec{p}$

$$\frac{d\vec{L}}{dt} = \dot{\vec{r}} \times \vec{p} + \vec{r} \times \dot{\vec{p}} = \vec{v} \times m\vec{v} + \vec{r} \times \vec{F} = \vec{r} \times \vec{F}$$

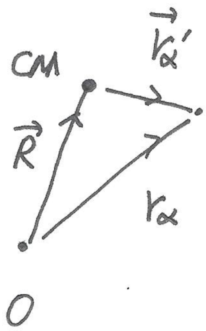
$\vec{r} \times \vec{F}$ is defined as torque, which vanishes for central force field.

we have known that Kepler's 2nd law is due to angular momentum conservation.

$$\boxed{\frac{d\vec{L}}{dt} = 0} \leftarrow \text{central force field.}$$

§ Angular momentum for serval particles

$\alpha = 1, 2, \cdots N$



$\vec{R}$: center of mass coordinates

$\vec{r}_\alpha'$: center of mass

vector in the frame of

$$\vec{l}_\alpha = \vec{r}_\alpha \times \vec{p}_\alpha \quad , \quad \vec{L}_{tot} = \sum_{\alpha=1}^{N} \vec{l}_\alpha = \sum_{\alpha=1}^{N} \vec{r}_\alpha \times \vec{p}_\alpha$$

$$\vec{L}_{tot} = \sum_{\alpha=1}^{N} (\vec{R} + \vec{r}_\alpha{}') \times \vec{P}_\alpha = \vec{R} \times \sum_{\alpha=1}^{N} \vec{P}_\alpha + \sum_{\alpha=1}^{N} \vec{r}_{\alpha'} \times m_\alpha \dot{\vec{r}}_\alpha$$

$$= \vec{R} \times \vec{P}_{tot} + \sum_{\alpha=1}^{N} \vec{r}_\alpha{}' \times m_\alpha (\dot{\vec{r}}_\alpha{}' + \dot{\vec{R}})$$

$$= \vec{L}_{orb} + \underbrace{\sum_{\alpha=1}^{N} \vec{r}_\alpha{}' \times m_\alpha \dot{\vec{r}}_\alpha{}'}_{\text{internal angular momentum}} + \underbrace{\left(\sum_{\alpha=1}^{N} m_\alpha \vec{r}_\alpha{}'\right)}_{0} \times \dot{\vec{R}}$$

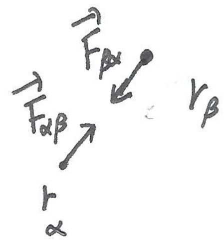
$$\vec{L}_{orb} = \vec{R} \times \vec{P}_{tot}, \qquad \vec{L}_{spin} = \sum_{\alpha=1}^{N} \vec{r}_\alpha{}' \times m_\alpha \dot{\vec{r}}_\alpha{}'$$

$$\frac{d}{dt} \vec{L}_{orb} = \underbrace{\dot{\vec{R}} \times \vec{P}_{tot}}_{= 0} + \vec{R} \times \dot{\vec{P}}_{tot} = \vec{R} \times \vec{F}^{ext}$$

$$\frac{d}{dt} \vec{L}_{tot} = \sum_{\alpha=1}^{N} \vec{r}_\alpha \times m \dot{\vec{r}}_\alpha = \sum_{\alpha=1}^{N} \vec{r}_\alpha \times \left(\vec{F}_\alpha^{ex} + \sum_{\beta \neq \alpha} \vec{F}_{\alpha\beta} \right)$$

$$= \sum_{\alpha=1}^{N} \vec{r}_\alpha \times \vec{F}_\alpha^{ex} + \sum_{\alpha \neq \beta} \vec{r}_\alpha \times \vec{F}_{\alpha\beta}$$

Due to Newton's 3rd law. $\vec{F}_{\alpha\beta} = - \vec{F}_{\beta\alpha}$



$$\frac{1}{2} \left(\sum_{\alpha \neq \beta} \vec{r}_\alpha \times \vec{F}_{\alpha\beta} + \vec{r}_\beta \times \vec{F}_{\beta\alpha} \right) = \frac{1}{2} \sum_{\alpha \neq \beta} (\vec{r}_\alpha - \vec{r}_\beta) \times \vec{F}_{\alpha\beta}$$

$$\sum_{\alpha \neq \beta} \vec{r}_\alpha \times \vec{F}_{\alpha\beta} \quad //$$

For central force field

$$\vec{r}_\alpha - \vec{r}_\beta \parallel \vec{F}_{\alpha\beta} \Rightarrow \sum_{\alpha \neq \beta} \vec{r}_\alpha \times \vec{F}_{\alpha\beta} = 0$$

西湖大學
WESTLAKE UNIVERSITY
T: 0571-86886859
F: 0571-85271986
E: office@westlake.edu.cn
中国浙江省杭州市西湖区云栖小镇石龙山街18号, 310024
18 Shilongshan Rd.,Cloud Town, Xihu District, Hangzhou, Zhejiang Province, China, 310024

Hence $\frac{d}{dt}(\vec{L}_{tot}) = \sum_{\alpha=1}^{N} \vec{r}_\alpha \times \vec{F}_\alpha^{ex} = \sum_{\alpha=1}^{N} \vec{r}_\alpha{}' \times \vec{F}_\alpha^{ex} + \vec{R} \times \vec{F}_{tot}$

$$\Rightarrow \frac{d}{dt}(\vec{L}_{tot} - \vec{L}_{orb}) = \sum_{\alpha=1}^{N} \vec{r}_\alpha{}' \times \vec{F}_\alpha^{ex}$$

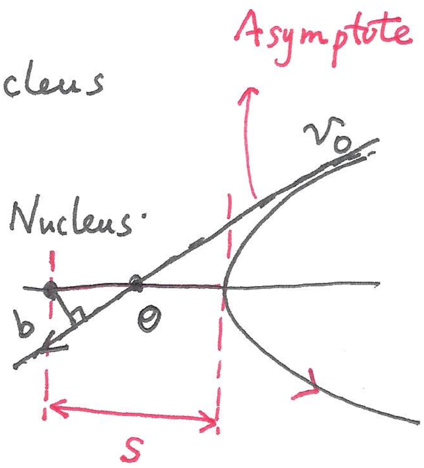
$$\boxed{\dot{L}_{spin} = \sum_{\alpha=1}^{N} \vec{r}_\alpha{}' \times \vec{F}_\alpha^{ex}}$$

torque respect to Center of mass.

In quantum mechanics, angular momentum includes orbital one and spin. Spin is quantized at $\hbar/2$.

orbital angular momentum is quantized at $\hbar$.

• example : proton scattering by a heavy nucleus



A proton's trajectory in the repulsive electric force field by a nucleus is a hyperbola. It's asymptote passes the ~~origin~~ center of the hyperbola, but not the Nucleus.

This distance from the nucleus to the asymptote "b" is called the impact parameter. ~~The~~ If we know the velocity v_0 at long distance, i.e. $r \to \infty$, and the impact parameter b, then what's

西湖大學 WESTLAKE UNIVERSITY

T: 0571-86886859
F: 0571-85271986
E: office@westlake.edu.cn

中国浙江省杭州市西湖区云栖小镇石龙山街18号，310024
18 Shilongshan Rd.,Cloud Town, Xihu District, Hangzhou, Zhejiang Province, China, 310024

the closet distance that the proton can approach the nucleus?

① Angular momentum conservation

$$m v_0 b = m v' \cdot s \Rightarrow v' = v_0 b/s$$

② energy conservation $\frac{1}{2} m v_0^2 = \frac{1}{2} m v'^2 + \frac{Qe}{s}$

$$\Rightarrow \frac{1}{2} m v_0^2 = \frac{1}{2} m v_0^2 (b/s)^2 + \frac{Qe}{s} \quad \text{or} \quad \boxed{\frac{Qe}{s} = \frac{1}{2} m v_0^2 \left[1 - \left(\frac{b}{s}\right)^2\right]}$$

• Example: shape of galaxy.

Consider a large cluster of dust with spherical shape with an initial angular momentum $\vec{L}$. Then under gravity, the dust cluster, or, the galaxy begins to collapse. Then the collapses along and perpendicular to the direction of $\vec{L}$ are different.

For a simplified model, consider a mass point m, moving around the center with mass M. The angular momentum L is fixed. Then we can write down an effective potential:

In the polar coordinate:

$$E_k = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2), \qquad \begin{array}{ll} x = r\cos\theta & \Rightarrow \dot{x} = \dot{r}\cos\theta - r\sin\theta\,\dot{\theta} \\ y = r\sin\theta & \quad\ \dot{y} = \dot{r}\sin\theta + r\cos\theta\,\dot{\theta} \end{array}$$

$$= \frac{1}{2} m (\dot{r}^2 + r^2\dot{\theta}^2)$$

西湖大學
WESTLAKE UNIVERSITY
T: 0571-86886859
F: 0571-85271986
E: office@westlake.edu.cn
中国浙江省杭州市西湖区云栖小镇石龙山街18号，310024
18 Shilongshan Rd.,Cloud Town, Xihu District, Hangzhou,
Zhejiang Province, China, 310024

$$E = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) - \frac{GMm}{r}$$

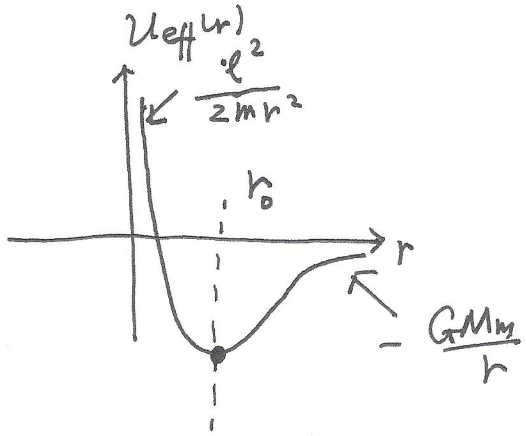
· Angular momentum $l = L_z = m(x\dot{y} - y\dot{x}) = m\left(r\cos\theta(\dot{r}\sin\theta + r\cos\theta\,\dot{\theta}) - r\sin\theta(\dot{r}\cos\theta - r\sin\theta\,\dot{\theta})\right)$

$$= m r^2 \dot{\theta}$$

$\Rightarrow \dot{\theta} = l/mr^2$. plug in the expression of E

$$E = \frac{1}{2} m \dot{r}^2 + \underbrace{\frac{1}{2}\frac{l^2}{mr^2} - \frac{GMm}{r}}_{U_{eff}}$$

Centrifugal potential

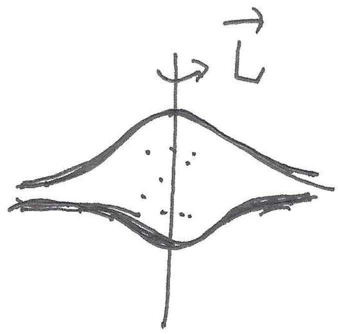


to minimize $U_{eff}(r)$

$$\frac{dU_{eff}(r)}{dr} = -\frac{l^2}{mr^3} + \frac{GMm}{r^2}\Bigg|_{r=r_0} = 0$$

$$\Rightarrow \boxed{r_0 = \frac{l^2}{GMm^2}}$$

$\rightarrow$



Hence, the collapse in the perpedicular direction is limited to r_0,

but the collapse along the direction of $\vec{L}$ has no such a constraint.