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Lect 13 Rigid body (I) – fixed axis rotation

§ Rigid body

§ Rotation around a fixed axis

§ Moment of inertia

§ equation of motion of fixed axis rotation

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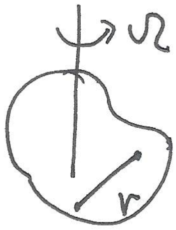
§ What's rigid body?

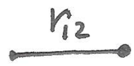
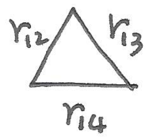
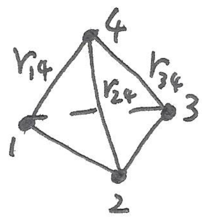


No elastisity!

The distance between any two points is fixed during the motion, i.e. no shape deformation.

how many degrees of freedom to describe a rigid body motion?

let's check

# of particles	# of degrees of freedom	if all distances fixed
1	3	
2	$2\times 3-1=5$	
3	$3\times 3-3=6$	
4	no extra (continuous) degrees of freedom (please prove!)	
5	6	
⋮	⋮	
N	6	

3 translational degrees of freedom

3 rotational degrees of freedom

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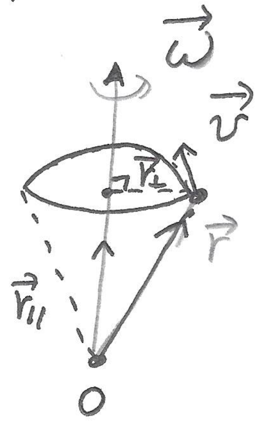
§ Laws of rotation around a fixed axis.

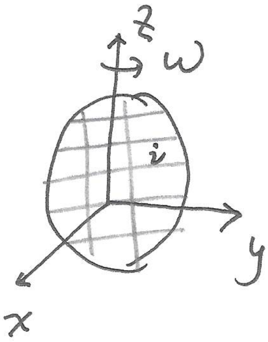


instantaneous velocity at a point $\vec{r} = \vec{r}_\parallel + \vec{r}_\perp$

$\vec{r}_\parallel$ is the component parallel to the rotation axis.

$\vec{r}_\perp$ is the component perpendicular to the axis

$$\boxed{\vec{v} = \vec{\omega} \times \vec{r}_\perp = \vec{\omega} \times (\vec{r}_\perp + \vec{r}_\parallel) = \vec{\omega} \times \vec{r}}$$



The angular momentum

$$\vec{L} = \sum_{i=1}^{N} \vec{l}_i = \sum_{i=1}^{N} m_i \vec{r}_i \times \vec{v}_i$$

$$\vec{v}_i = \vec{\omega} \times \vec{r}_i$$

$$\Rightarrow \vec{L} = \sum_{i=1}^{N} m_i \vec{r}_i \times (\vec{\omega} \times \vec{r}_i)$$

in our case $\vec{\omega} = \omega \hat{z}$, $\vec{r}_i = x_i \hat{x} + y_i \hat{y} + z_i \hat{z}$

$$\vec{r}_i \times (\vec{\omega} \times \vec{r}_i) = \vec{\omega} r_i^2 - \vec{r}_i (\vec{\omega} \cdot \vec{r}_i)$$

$$= (x_i^2 + y_i^2 + z_i^2) \omega \hat{z} - \omega (z_i x_i \hat{x} + z_i y_i \hat{y} + z_i^2 \hat{z})$$

$$= -\omega z_i x_i \hat{x} - \omega z_i y_i \hat{y} + \omega (x_i^2 + y_i^2) \hat{z}$$

$$L_z = \sum_{i=1}^{N} m_i \omega (x_i^2 + y_i^2) = I_z \omega, \text{ where } I_z = \sum_i m_i (x_i^2 + y_i^2)$$

$$\longrightarrow \int dx\, dy\, dz\, \rho (x^2 + y^2) = I_z \leftarrow \text{moment of inertia}$$

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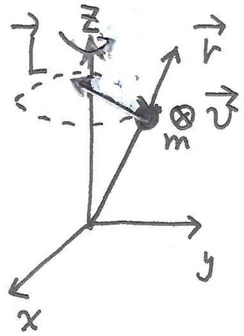
$$L_x = -\left(\sum_i m_i x_i z_i\right)\omega = I_{xz}\,\omega \neq 0$$

$$L_y = -\left(\sum_i m_i y_i z_i\right)\omega = I_{yz}\,\omega \neq 0$$

} in general.

Example: If the mass distribution is unsymmetric with respect to the rotation axis, $\vec{L} \nparallel \vec{\omega}$.



§ Kinetic energy

$$E_K = \frac{1}{2}\sum_i m_i v_i^2 = \frac{1}{2}\sum_i m_i \omega^2 (x_i^2 + y_i^2) = \frac{I_z}{2}\omega^2$$

$$\vec{v} = -\omega y_i \hat{x} + \omega x_i \hat{y}$$

Parallel axis theorem:

If the rotation axis does not pass the center of mass, denote the distance between the mass center and the rotation axis as r_c, then $\boxed{I_z = I_{cz} + M r_c^2}$, where I_{cz} is the moment of inertia if the rotation axis passes the center of mass.

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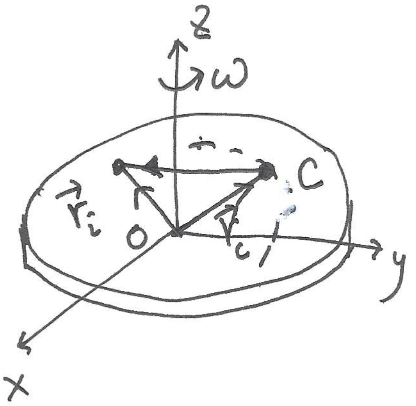


$$\vec{r}_i = \vec{r}_c + \vec{r}_i'$$

$\vec{r}_i'$ is the displacement of "ith point" relative to C

$$I_z = \sum m_i r_i^2$$

$$= \sum_i m_i |\vec{r}_c + \vec{r}_i'|^2 = \sum_i m_i (r_c^2 + r_i'^2 + 2\vec{r}_c \cdot \vec{r}_i')$$

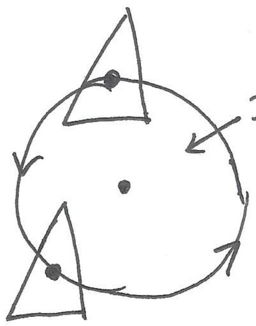
$$= M r_c^2 + I_{cz} + 2\vec{r}_c \cdot \sum_i m_i \vec{r}_i'$$

by definition $\sum_i m_i \vec{r}_i = (\sum_i m_i)\vec{r}_c \Rightarrow \sum_i m_i \vec{r}_i' = 0.$

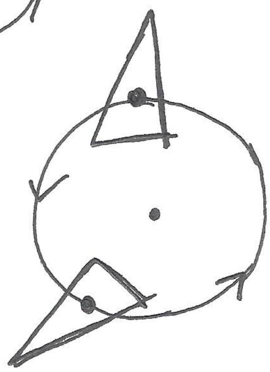
Hence $I_z = I_{cz} + M r_c^2$.

Then $\boxed{E_K = \frac{1}{2} I_z \omega^2 = \frac{1}{2} I_{cz}\omega^2 + \frac{1}{2} M (r_c \omega)^2.}$

Kinetic energy of a rotating slab = kinetic energy in the center of mass frame + the translation energy as a whole object



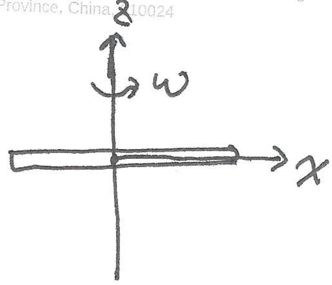
translation in a circular way



rotation around an axis away from the center of mass.

☆ examples of moment of inertial

① uniform thin rod — rotation axis passes center of mass

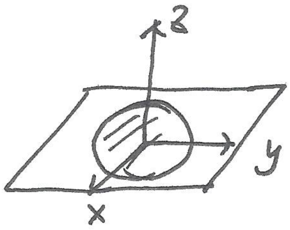


$$I_{cz} = \int_{-L/2}^{L/2} dx\, \rho x^2 = \frac{\rho}{3} x^3 \Big|_{-\frac{L}{2}}^{\frac{L}{2}} = \frac{\rho}{12} L^3 = \frac{(\rho L) L^2}{12} = \frac{M}{12} L^2$$

if the rotation axis is located at one end, then

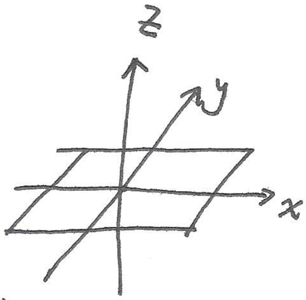
$$I_z = I_{cz} + M \cdot \left(\frac{L}{2}\right)^2 = \frac{M}{3} L^2$$

② circular disk — rotation axis passes its center



$$I_z = \int dx \int dy\, \rho (x^2 + y^2) = \rho \int_0^R r\,dr \int d\theta\, r^2 = 2\pi\rho \cdot \frac{R^2}{4}$$

$$= (\pi R^2 \rho) \frac{R^2}{2} = \frac{1}{2} M R^2$$



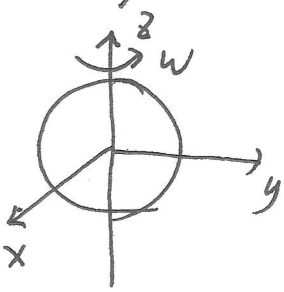
③ rectangular plate — axis passing the center

$$I_z = \int_{-a/2}^{a/2} dx \int_{-b/2}^{b/2} dy\, \rho (x^2 + y^2) = \rho \left(b \frac{x^3}{3} \Big|_{-\frac{a}{2}}^{\frac{a}{2}} + a \cdot \frac{y^3}{3} \Big|_{-\frac{b}{2}}^{\frac{b}{2}} \right)$$

$$= \rho \left(\frac{ba^3}{12} + \frac{ab^3}{12} \right) = \frac{\rho ab}{12} (a^2 + b^2) = \frac{M}{12} (a^2 + b^2)$$

④ uniform solid sphere

$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \end{cases}$$



$$I = \int r^2 dr \sin\theta\, d\theta\, d\varphi\, \rho (r^2 \sin^2\theta)$$

$$= \rho \int_0^R r^4 dr \int_0^\pi \sin^3\theta\, d\theta \int_0^{2\pi} d\varphi$$

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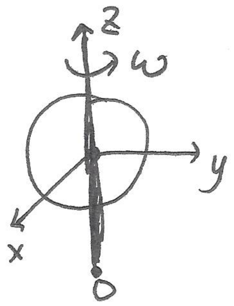
$$\int_0^{\pi} \sin^3\theta \, d\theta = -\int_0^{\pi} \sin^2\theta \, d\cos\theta = +\int_{-1}^{1} (1-x^2)\, dx = x - \frac{x^3}{3}\Big|_{-1}^{1}$$
$$= \frac{4}{3}$$

$$\Rightarrow I = \rho \cdot \frac{R^5}{5} \cdot \frac{4}{3} \cdot 2\pi = \left(\frac{4}{3}\pi R^3 \rho\right) \frac{2}{5} R^2 = \frac{2}{5} M R^2$$

§: Equation of motion for fixed axis rotation



Set $\vec{\omega}$ is along the $\hat{z}$-axis, for fixed axis rotation

$$L_z = I_z \omega,$$

Let's prove

$$dL_z/dt = \tau_z \leftarrow \text{torque}$$

Actually, we can pick up any point O at the rotation axis as the origin. We proved before

$$\frac{d}{dt}\vec{L} = \sum_{i=1}^{N} \vec{r}_i \times \vec{F}_i^{ex}$$

we take the z-component; ① $L_z = I_z \omega$ and $I_z = \int dx\,dy\,dz\, \rho(x^2+y^2)$

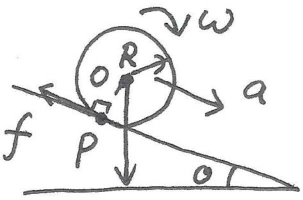
i.e. L_z does not depend on the location of "O" on the axis.

② $(\vec{r}_i \times \vec{F}_i^{ex})_z = x_i F_i^y - y_i F_i^x$, which does not depend on the z-location of "O" either. Hence, we have

$$dL_z/dt = \tau_z$$

← define for the rotation axis, independent of the location of "O" along the axis.

Example: rolling without slipping



① Constraint: $\vec{v}_p = \vec{v}_o + \vec{\omega} \times \vec{r}_{op} = 0$

$$\Rightarrow \quad v_o = \omega R \quad \Rightarrow \quad a = R\frac{d\omega}{dt}$$

with respect to center of Mass.

$$\begin{cases} f \cdot R = I\frac{d\omega}{dt} \\ Mg\sin\theta - f = Ma \end{cases} \Rightarrow \quad Mg\sin\theta - I\frac{d\omega}{dt}\cdot\frac{1}{R} = Ma$$

$$Mg\sin\theta = \left(\frac{I}{R^2} + M\right) a$$

$$\Rightarrow \boxed{a = \frac{1}{1 + I/MR^2}\, g\sin\theta}$$

② energy conservation

$$Mgh = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}Mv^2 + \frac{1}{2}I\left(\frac{v}{R}\right)^2$$

$$\Rightarrow \quad g\frac{dh}{dt} = v\frac{dv}{dt}\left(1 + I/MR^2\right)$$

$$dh/dt = v\sin\theta, \quad a = \frac{dv}{dt} \quad \Rightarrow \quad a = \frac{g\sin\theta}{1 + I/MR^2}$$

Comment: ① $f < Mg\cos\theta \cdot \mu$, hence, μ cannot to be too small to achieve roll(ing) without slipping.

② the largest angle for rolling without slipping

$$\boxed{\tan\theta \leq \mu\left(\frac{MR^2}{I} + 1\right)}$$

$$g\sin\theta\left(1 - \frac{1}{1 + I/MR^2}\right) < g\cos\theta \cdot \mu \Rightarrow \tan\theta < \frac{\mu\left(1 + I/MR^2\right)}{I/MR^2}$$

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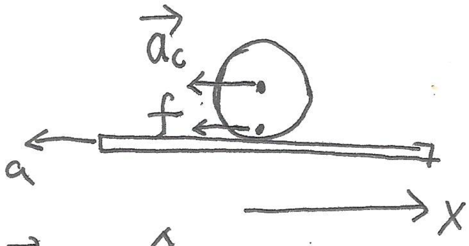
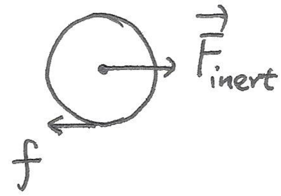
Example: A cylinder is put on a rug. Pull the rug with an acceleration $\vec{a}$, and the cylinder does not slip. Then describe the motion of the cylinder.



$\vec{a} = -a\hat{x}$

\# We use the non-inertial frame of the rug. Then in this frame, there's the inertial force



$$\vec{F}_{inert} = Ma\hat{x}$$

The torque: $\frac{d\omega}{dt} = \frac{a'}{R} = \frac{f \cdot R}{I}$ — a' is the acceleration of the center of mass of the cylinder in the rug frame.

and $\vec{F} + \vec{f} = M\vec{a}'$

$$\Rightarrow Ma' = Ma - \frac{Ia'}{R^2}$$

$$\Rightarrow a' = \frac{M}{M + I/R^2} a = \frac{1}{1 + \frac{I}{MR^2}} a = \frac{2}{3} a \quad \text{if } I = \frac{1}{2}MR^2$$

The rug acceleration $\vec{a} = -a\hat{x} \Rightarrow$ the cylinder

$$\vec{a}_c = \vec{a} + \vec{a}' = -\frac{I/MR^2}{1 + I/MR^2} a\hat{x} = -\frac{I}{MR^2 + I} a\hat{x}$$

$\Rightarrow I = \frac{1}{2}MR^2 \Rightarrow$ $\boxed{\vec{a}_c = -\frac{a}{3}\hat{x}, \text{ and } f = -\frac{M}{3}a\hat{x}}$