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Lect 15 — Continuum Mechanics — Wave

- waves
- wave equation
- Right - left movers
- boundary condition
- normal modes

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Why study wave?

(x) point mass $\longrightarrow$ rigid body $\longrightarrow$ continuum mechanis

discrete number of coordinates

elesticity of solids
fluid mechanics

ordinary differential Eq $\longrightarrow$ partial different Eq

$$\ddot{x} = -\omega_0^2 x \qquad \frac{\partial^2}{c^2 \partial t^2} u(x,t) = \frac{\partial^2}{\partial x^2} u(x,t)$$

- Level of civilation — manipulation of waves

  ① Mechanical wave — sound wave, decays very qucikly
  easy to generate, easy to load information

  ② E & M wave — enough for global information communication
  but insufficient for interstellar communication

  ③ gravitation wave — next level of civilization

- Quantum mechanics — wave-particle duality
  particle is also wave, wave-mechanics, Schrödinger Eq.

- humaniety — music, harmonics, tones

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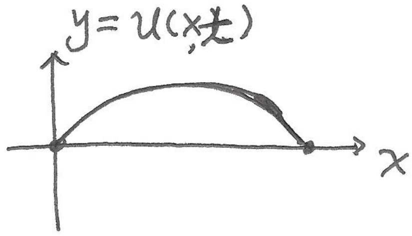
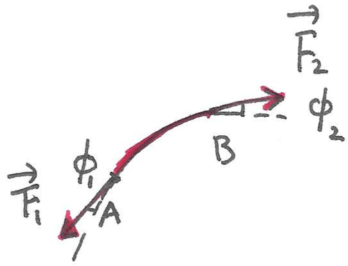
② Wave in 1 D — Gitar string



not just a mass point, but a continuum mass distribution. The displacement is a function of x and t.

$$y = y(x,t)$$

Pick up a linear segment AB



The tension T is roughly the same along the string. (we consider the vibration in the trasverse direction, not the longituidnal wave of compression.)

Along the x-direction: $F_x^{net} = T(\cos\phi_2 - \cos\phi_1) \approx \frac{T}{2}(\phi_1^2 - \phi_2^2) \simeq 0$

Along the y-direction: $F_y^{net} = T(\sin\phi_2 - \sin\phi_1) \approx T\, d\phi$

$$\phi \sim \frac{\partial u}{\partial x} \sim \tan\phi \quad (\text{slope})$$

$$\Rightarrow \quad F_y^{net} = T \frac{\partial \phi}{\partial x} dx = T \frac{\partial^2 u}{\partial x^2} dx$$

Newton's 2nd law: $\underbrace{\rho\, dx}_{\text{mass}} \underbrace{\frac{\partial^2 u}{\partial t^2}}_{\text{acceleration}} = F_y^{net} = T \frac{\partial^2 u}{\partial x^2} dx$

$$\Rightarrow \boxed{\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right) u(x,t) = 0, \text{ where } c = \sqrt{T/\rho}.}$$

ρ: linear density

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§ Right and left-movers

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) = \left(\frac{\partial}{\partial x} - \frac{1}{c}\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial x} + \frac{1}{c}\frac{\partial}{\partial t}\right)$$

define $\left.\begin{array}{l} \xi = x - ct \\ \eta = x + ct \end{array}\right\} \Rightarrow \begin{array}{l} x = \frac{\xi+\eta}{2} \\ ct = \frac{\xi-\eta}{2} \end{array}$

$$f(x, ct) = f(x(\xi,\eta), ct(\xi,\eta))$$

$$\left\{\begin{array}{l} \frac{\partial f}{\partial \xi} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial \xi} + \frac{\partial f}{\partial ct}\cdot\frac{\partial (ct)}{\partial \xi} \\ \frac{\partial f}{\partial \eta} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial \eta} + \frac{\partial f}{\partial (ct)}\cdot\frac{\partial (ct)}{\partial \eta} \end{array}\right. \Rightarrow \begin{array}{l} \frac{\partial}{\partial \xi} = \frac{1}{2}\left[\frac{\partial}{\partial x} - \frac{\partial}{\partial (ct)}\right] \\ \frac{\partial}{\partial \eta} = \frac{1}{2}\left[\frac{\partial}{\partial x} + \frac{\partial}{\partial (ct)}\right] \end{array}$$

$$\Rightarrow \frac{\partial}{\partial \xi}\frac{\partial}{\partial \eta} = \frac{1}{4}\left[\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]$$

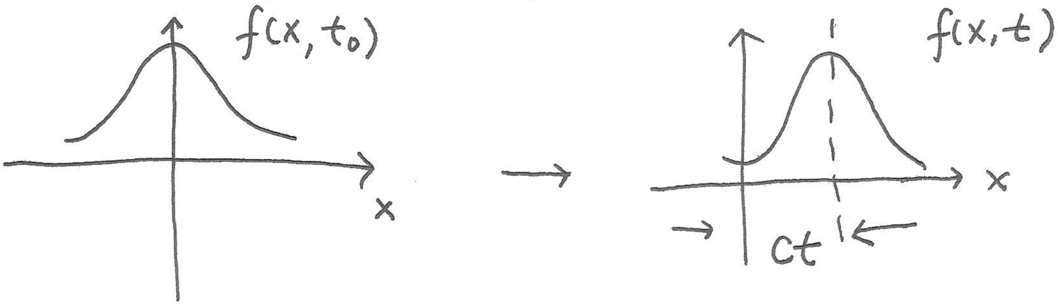
wave Eq $\Rightarrow$ $\frac{\partial}{\partial \xi}\frac{\partial}{\partial \eta} u = 0$

$$\frac{\partial}{\partial \eta} u(\xi,\eta) = \ell(\eta)$$

$$u(\xi,\eta) = f(\xi) + g(\eta), \quad \text{where } g'(\eta) = \ell(\eta).$$

$$\Rightarrow u(x,t) = \underbrace{f(x-ct)}_{\text{right mover}} + \underbrace{g(x+ct)}_{\text{left mover}}$$

where f and g are arbitany smooth functions



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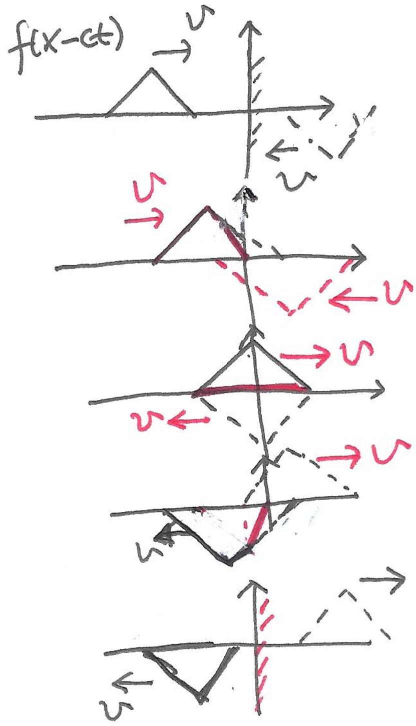


$-f(-x-ct)$

⊛ boundary condition — reflection

For a single mass point problem, we consider the initial conditions, i.e. $x(t=0)$, and $\dot{x}|_{t=0}$.

For wave problem, in addition to the initial conditions, we have also the boundary conditions.

Consider the general solution

$$u(x,t) = f(x-ct) + g(x+ct)$$

under $u(x,t)\big|_{x=0} = 0 \quad \Rightarrow -f(-ct) = g(ct)$.

define ~~$x+ct = y$~~ $\Rightarrow$ ~~$g(y) = -f(-y)$~~ $= -$~~$f(x$~~

In other words, for any value of y, we have $g(y) = -f(-y)$.

plug in $y = x+ct \quad \Rightarrow g(x+ct) = -f(-x-ct)$.

$$\Rightarrow \boxed{u(x,t) = f(x-ct) - f(-x-ct)}$$

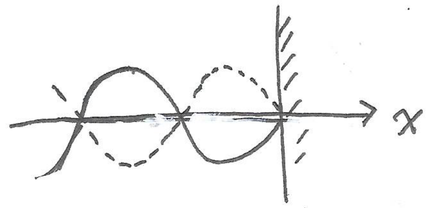
incident wave — reflected wave

if $f(x-ct) = \cos(k(x-ct)) = \cos(kx-\omega t)$

$-f(-x-ct) = -\cos(kx+\omega t) \quad \leftarrow k = \frac{2\pi}{\lambda}$ wave number

$$\Rightarrow u(x,t) = \cos(kx-\omega t) - \cos(kx+\omega t) = -2\sin kx \sin \omega t$$

Standing wave: At each point x



$u(x,t) = A\cos\omega t \leftarrow$ harmonic oscilation

and the amplitude $A = 2\sin kx$, which is spatially dependent.

At $kx = -n\pi$, here, $x = -n\pi/k$, $A(x) = 0$.
$(n = 1, 2, \cdots)$

These positions are static, without vibaration — nodes!

Between two nodes, all the points move up and down in the same phase. And this pattern is fixed in space.

Mode: if any point moves sinusoidally at the same frequency, then this pattern is called a mode.

(*) Complex number representation

$$f(x-ct) = e^{i(kx-\omega t)}$$

$$u(x,t) = e^{i(kx-\omega t)} - e^{-i(kx+\omega t)} = \underline{2i\sin kx\, e^{-i\omega t}}$$

Standing wave.

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⊛ Waves on a guitar string

If both ends are fixed, $u(x,t)\Big|_{\substack{x=0 \\ \text{and } x=L}} = 0.$

then let's try a mode solution

$$u(x,t) = X(x)\cos(\omega t - \delta)$$

(every where moves sinusoidally with the same ω)

plug in $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$

$$\Rightarrow -\omega^2 X(x)\cos(\omega t - \delta) = c^2 \frac{d^2 X}{dx^2}\cos(\omega t - \delta)$$

$$\frac{d^2 X(x)}{dx^2} = -k^2 X(x), \text{ where } k = \omega / c$$

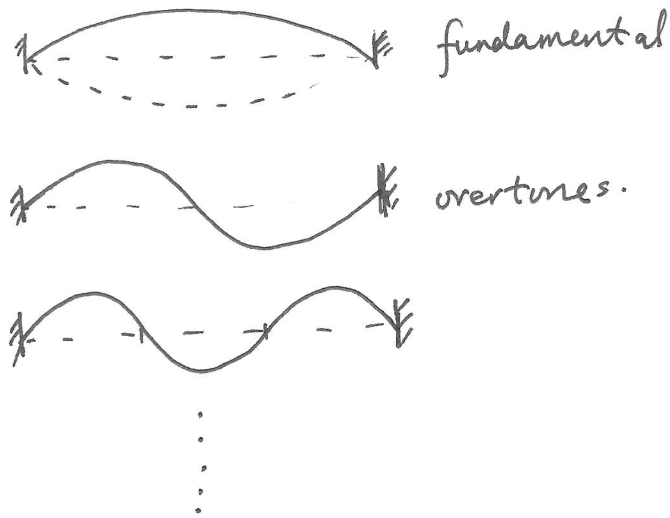
$$\Rightarrow X(x) = A\sin kx + B\cos kx$$

plug in boundary condition $X(0) = 0 \Rightarrow B = 0$

$X(L) = 0 \Rightarrow k_n L = n\pi, \quad n = 1, 2, \cdots$

$$\omega_n = \frac{n\pi}{L} c$$

$$\Rightarrow \boxed{u_n(x,t) = A_n \sin k_n x \cos(\omega_n t - \delta_n)}$$



Quantization

Similar to quantum mechanics.

waves for matter particles,

say, electrons.

All the mode solutions form a complete set, such at any waves in this system can be expanded as linear superpositions of modes, i.e.

$$\boxed{u(x,t) = \sum_n A_n \sin k_n x \cos(\omega_n t - \delta_n)}$$

We may view $u_n = \sin k_n x \cos \omega_n t$ and $\sin k_n x \sin \omega_n t$

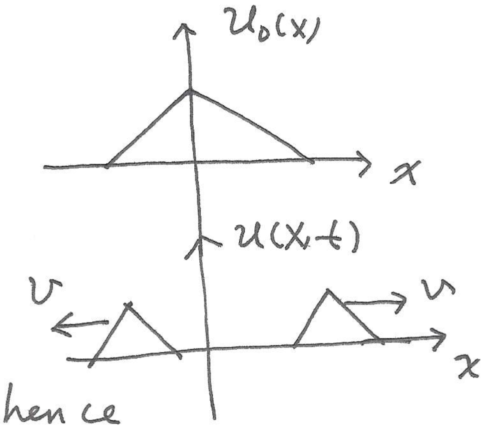
as bases of an abstranct space.

$$u(x,t) = \sum_n B_n \underline{\sin k_n x \cos \omega_n t} + C_n \underline{\sin k_n x \sin \omega_n t}$$

and B_n and C_n are coordinates. c.f. $\vec{r} = x\,\underline{\hat{x}} + y\,\underline{\hat{y}} + z\,\underline{\hat{z}}$

⊛ Example a triangular wave in a finite string

before solving this problem, let's consider the case without a boundary.



Initial condition $\begin{cases} a)\ u(x,t=0) = u_0(x), \\ b)\ \dfrac{\partial u(x,t=0)}{\partial t} = 0. \end{cases}$

Set $u(x,t) = f(x-ct) + g(x+ct)$, hence

$$f(x) + g(x) = u_0(x)$$

$$\frac{\partial}{\partial t} u(x,t) = \frac{\partial}{\partial t} f(x-ct) + \frac{\partial}{\partial t} g(x+ct) = c\left[\frac{\partial f(x-ct)}{\partial ct} + \frac{\partial g(x+ct)}{\partial ct}\right]$$

$$= c\left[-\frac{\partial f}{\partial x} + \frac{\partial g}{\partial x}\right] \Rightarrow \frac{\partial}{\partial t} u\Big|_{t=0} = c\left[-\frac{\partial f(x,t)}{\partial x} + \frac{\partial g(x,t)}{\partial x}\right]\Big|_{t=0} = 0$$

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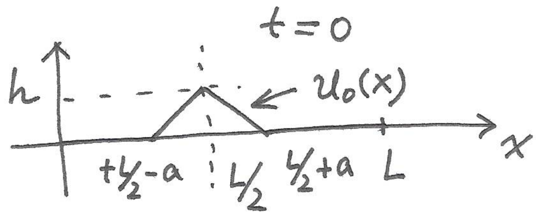
$$\Rightarrow \quad f(x) - g(x) = \text{Const} = C$$

$$\Rightarrow \quad f(x) = \frac{u_0(x) + C}{2} \qquad \Rightarrow \quad u(x,t) = f(x-ct) + g(x+ct)$$

$$g(x) = \frac{u_0(x) - C}{2} \qquad = \frac{1}{2}\left\{u_0(x-ct) + u_0(x+ct)\right\}$$

(*) Now consider to impose a boundary $x \in [0, L]$



we take the expansion

$$u(x,t) = \sum_n \sin k_n x \,(B_n \cos \omega_n t + C_n \sin \omega_n t)$$

$$\frac{\partial}{\partial t} u(x,t) = \sum_n \sin k_n x \;\omega_n \,[-B_n \sin \omega_n t + C_n \cos \omega_n t)$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = 0 \quad \Rightarrow \quad \sum_n \omega_n \sin k_n x \cdot C_n = 0 \quad \text{for all } x$$

hence $C_n = 0$.

$$\Rightarrow \quad u(x,t) = \sum_n B_n \sin k_n x \cos \omega_n t$$

At $t=0$, $\boxed{u(x,0) = \sum_n B_n \sin k_n x = u_0(x)}$ $\quad k_n = \frac{n\pi}{L} x$

$$\boxed{\begin{aligned} \frac{2}{L}\int_0^L \sin k_n x \sin k_m x \, dx &= \frac{1}{L}\int_0^L dx \left[\cos(k_n - k_m)x - \cos(k_n + k_m)x\right] \\ &= \delta_{n,m} \end{aligned}}$$

$$B_n = \frac{2}{L}\int_0^L u_0(x)\sin\frac{\pi n x}{L}\,dx = \frac{2}{L}\int_{-L/2}^{\frac{L}{2}} u_0(x')\sin\left(\frac{n\pi}{L}x' - \frac{n\pi}{2}\right)dx'$$

$$x' = x - \frac{L}{2}$$

$u_0(x')$ is even

$$\Rightarrow \quad B_{2m} = 0$$

$$B_{2m+1} = \frac{4}{L}\int_0^a u_0(x')\,(-)^{m+1}\cos\left(\frac{(2m+1)\pi}{L}x'\right)dx'$$

The fundemental frequency $\omega_1 = \frac{\pi}{L}c$

periodicity $\quad \tau = \frac{2\pi}{\omega_1}$.

time-evolution: $t=0$

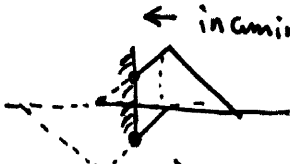
static

$t = \tau/8$

$t = \tau/4$

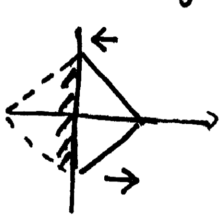
when reach boundary

← incoming



→ reflected

interference

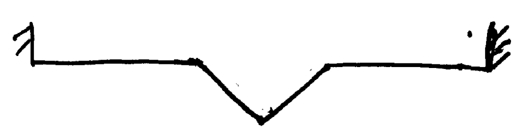


add to zero

$t = \tau/4$

$t = \frac{3\tau}{8}$

$t = \tau/2$



§ 3D wave equation

1D: $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ $\longrightarrow$ 3D $\frac{\partial^2 p}{\partial t} = c^2 \left(\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} \right)$

p- pressure, sound wave

$$C = \sqrt{\frac{BM}{\rho_0}}$$

BM ← bulk modulus

ρ_0 ← density

plane wave solution: (free-space)

① Centainly $p = f(x-ct) + g(x+ct)$ remains a possible solution, which means its propagatin along $\pm \hat{x}$ direction.

Generally speaking, we can choose an arbitary propagation direction $\hat{n}$.

$$p = f(\hat{n} \cdot \vec{r} - ct)$$

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \nabla^2 = \vec{\nabla} \cdot \vec{\nabla}$$

$$\nabla \cdot p = \frac{df}{d(\hat{n} \cdot \vec{r})} \hat{n} = -\frac{1}{c} \frac{\partial f}{\partial t} \hat{n}, \Rightarrow \nabla(\nabla \cdot p) = -\frac{\hat{n}}{c} \nabla\left(\frac{\partial f}{\partial t}\right)$$

$$\nabla^2 p = -\frac{\hat{n}}{c} \cdot \left(-\frac{\hat{n}}{c}\right) \frac{\partial}{\partial t} \frac{\partial f}{\partial t} = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} f$$

In particular, the plane-wave $p = \cos[k(\hat{n} \cdot \vec{r} - \omega t)]$

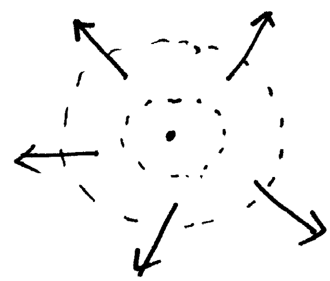
Spherical coordinate

$$\nabla^2 f = \frac{1}{r}\frac{\partial^2}{\partial r^2}(rf) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}$$

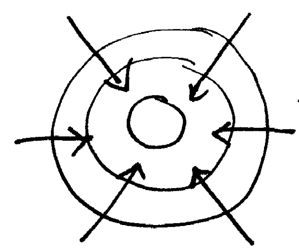
we can choose

$$r\,p(r,t) = f(r-ct) + g(r+ct)$$

$$\Rightarrow \quad p(r,t) = \underbrace{\frac{1}{r} f(r-ct)}_{\text{out-going spherical wave}} + \underbrace{\frac{1}{r} g(r+ct)}_{\text{in-coming spherical wave}}$$



$\frac{f(r-ct)}{r}$



$\frac{g(r+ct)}{r}$

power $\sim \int r^2 dr \left(\frac{1}{r}\right)^2$

no-power divergence.

quantum mechanical

Schrödinger Eq

$$\frac{\partial}{\partial t}\psi(\vec{r},t) = \underbrace{-\frac{\hbar^2}{2m}\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right]}_{\text{Kinetics energy}} \psi(\vec{r},t)$$

$$+ \underbrace{V(\vec{r},t)}_{\text{potential}}\ \psi(\vec{r},t)$$

+ Boundary ~~eq~~ condition:

Free space $V=0 \Rightarrow$ free-wave equation

$\boxed{\text{Wave mechanics}}$