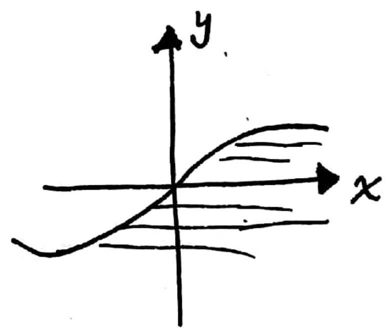
Water wave — tsunami & Storm waves

The water wave is actually a complicated problem. It's actually a wave in fluid but not due to compression. (Compression leads to sound wave whose sound velocity is much higher than water wave velocity).
The restoring forces are due to the gravity and the surface tension.
Even by the gravity itself, it can lead to wave wave. Actually, the water wave is mostly dominated by the fact that water is incompressible.
Water wave is a <u>surface wave</u>, which proporgates along the surface but decays exponetially as goes deeper.

For a large lake with depth of h, we set the coordinate frame.
For each water droplet whose equilibrium position located at (x, y), its displacement is denoted as $\vec{u}(x,y)$. We assume that waves vibrate in the xy-plane, and its proporgation along the x-axis. The water is uniform along the z-axis.



⊛ Properties of ideal incompressible liquid

For incompressible liquid, $\rho = $ const. The continuity equation

$$\nabla \cdot (\rho \vec{v}) + \frac{\partial \rho}{\partial t} = 0 \quad \text{simplifies to} \quad \nabla \cdot \vec{v} = 0.$$

$\vec{v} = \frac{d}{dt}\vec{u}(\vec{r},t) = \frac{\partial}{\partial t}\vec{u}(\vec{r},t)$. Since $\vec{r}$ is a label of water droplet, we do not take its time derivative.

Since $\nabla\cdot\vec{v} = 0 \Rightarrow \frac{\partial}{\partial t}(\nabla\cdot\vec{u}) = 0$, hence $\nabla\cdot\vec{u}$ = time-independent

if $\nabla\cdot\vec{u} \neq 0$, it means compression / expansion, hence,

$$\boxed{\nabla\cdot\vec{u} = 0 \qquad (*)}$$

We further assume that there are no vortices, i.e. $\nabla\times\vec{v} = 0$

$\rightarrow \frac{\partial}{\partial t}(\nabla\times\vec{u}) = 0$.

If $\nabla\times\vec{u}$ is non-zero and time-independent, we can absorb $\vec{u}$ in the definition of $\vec{r}$. Hence, we can choose

$$\boxed{\nabla\times\vec{u} = 0 \qquad (**)}$$

We can express $\vec{u} = \nabla\phi$, where ϕ is the potential function, It satisfies the Laplace equation.

$$\boxed{\nabla^2\phi = 0}$$

This isn't a dynamic equation. If without a surface, it will not generate wave propagation. — This is quite different from the usual elasticity wave equations, in which a bulk restoring force gives rise to Helmholtz equation $\left(\frac{\partial^2}{\partial t^2} - \nabla^2\right)\phi = 0$.

The general solution $\phi(x,y,t) = f(t)\, e^{ikx + k'y}$

plug in the Laplace equation $\Rightarrow k' = \pm k$, then

$$\boxed{\begin{aligned} u_x &= \nabla_x \phi = ik\, f(t)\, e^{ikx \pm ky} \\ u_y &= \nabla_y \phi = \pm k\, f(t)\, e^{ikx \pm ky} \end{aligned}}$$

$\leftarrow$ So far, no information comes from the restoring force.

Also notice "i"- factor, u_x and u_y have $90°$ phase difference.

⊛ Boundary condition

1. The bottom of the lake, the water cannot move vertically $u_y(-h) = 0$. This does not involve dynamics either.

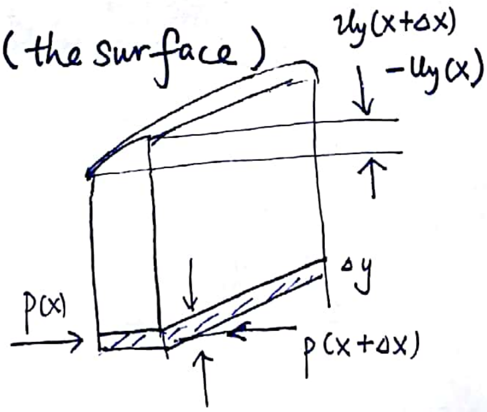
$$\phi = f(t)\left[A e^{ikx + ky} + B e^{ikx - ky}\right]$$

$$u_y = \nabla_y \phi = f(t)\, k \left[A e^{ky} - B e^{-ky}\right] e^{ikx}$$

$$u_y(-h) = 0 \Rightarrow B/A = e^{-2kh}$$

$$\boxed{\text{or } \phi(x,y,t) = f(t)\, e^{ikx} \cosh k(h+y)}$$

2. Dynamic boundary condition at $y=0$ (the surface)



ⓐ gravity force

$$F_x = -\left(p(x+\Delta x) - p(x)\right) \Delta y\, L_z$$
$$= -\rho g \left(u_y(x+\Delta x) - u_y(x)\right)\Big|_{y=0} \Delta y\, L_z$$

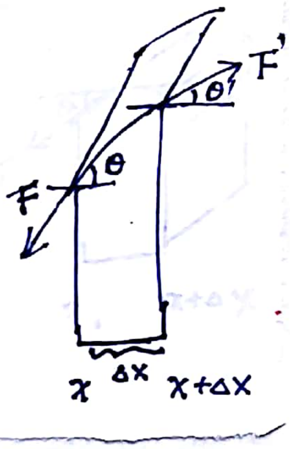
$$F_x = \Delta m \frac{\partial^2 u_x}{\partial t^2} \quad \text{and} \quad \Delta m = \rho g \Delta y L_z \Delta x$$

$$\Rightarrow \left.\frac{\partial^2 u_x}{\partial t^2}\right|_{y=0} = -g \left.\frac{\partial u_y}{\partial x}\right|_{y=0} \Rightarrow \left.\frac{\partial^2}{\partial t^2}\frac{\partial}{\partial x}\phi\right|_{y=0} = -g\left.\frac{\partial}{\partial x}\frac{\partial}{\partial y}\phi\right|_{y=0}$$

$$\text{or} \quad \left.\frac{\partial^2}{\partial t^2}\phi + g\frac{\partial}{\partial y}\phi\right|_{y=0} = S(t) \leftarrow$$

like a gauge not affecting u_x, u_y. Set it to zero.

(b) Surface tension



Consider the extra pressure due to surface tension.

$$F_y = TL(\sin\theta - \sin\theta')$$

$$P = \frac{TL}{L\cdot\Delta x}(-)\frac{\partial\theta}{\partial x}\cdot\Delta x = -T\frac{\partial\theta}{\partial x}$$

$$= -T\frac{\partial^2 u_y}{\partial x^2}$$

Add this to the gravity

$$F_x = -(p(x+\Delta x) - p(x))\Delta y L_z$$

$$= -\rho g \left.\frac{\partial u_y}{\partial x}\right|_{y=0}\Delta x \Delta y L_z + T\left.\frac{\partial^3 u_y}{\partial x^3}\right|_{y=0}\Delta x\Delta y L_z$$

$$= \Delta m \frac{\partial^2 u_x}{\partial t^2}$$

$$\Rightarrow \frac{\partial^2 u_x}{\partial t^2} = -g\frac{\partial u_y}{\partial x} + \left.\frac{T}{\rho}\frac{\partial^3 u_y}{\partial x^3}\right|_{y=0}$$

$$\Rightarrow \frac{\partial}{\partial x}\frac{\partial^2}{\partial t^2}\phi = -g\frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}\phi\right] + \frac{T}{\rho}\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x^2}\frac{\partial}{\partial y}\phi\right]$$

$$\Rightarrow \frac{\partial^2}{\partial t^2}\phi + g\frac{\partial}{\partial y}\phi - \frac{T}{\rho}\frac{\partial}{\partial x^2}\frac{\partial}{\partial y}\phi = 0$$

$$\frac{\partial^2}{\partial t^2}\phi + \frac{\partial}{\partial y}\left[g\phi - \frac{T}{\rho}\frac{\partial}{\partial x^2}\phi\right] = 0$$

plug in $\phi = e^{-i\omega t} e^{ikx} \cosh(h+y)k$

$$-\omega^2\phi + \frac{\partial}{\partial y}\left[g\phi + k^2\frac{T}{\rho}\phi\right] = 0$$

$$-\omega^2\phi + \left(g + k^2\frac{T}{\rho}\right)\phi \cdot \tanh(h+y)\cdot k = 0 \Big|_{y=0}$$

$$\Rightarrow \omega^2 = \left(g + k^2\frac{T}{\rho}\right)\cdot k \cdot \tanh hk$$

$$\boxed{\omega^2 = (gk + T/\rho\, k^3)\cdot\tanh(kh)}$$

hence, at long wave length, the gravity dominates, while the surface tension is more important at short-wave length

check $g \sim T/\rho\, k^2 \Rightarrow k^2 \sim \rho g/T = \frac{1\times10^3\cdot10}{7.2\times10^{-2}} = 1.4\times10^4\,m^{-2}$

$$T = 72\,dyn/cm = \frac{72\times10^{-5}}{10^{-2}}\,N/m = 7.2\times10^{-2}\,N/m$$

$$\Rightarrow k \sim 1.2\times10^2\,m^{-1} \Rightarrow \lambda = \frac{2\pi}{k} \simeq \frac{6.28}{120} \simeq 5\,cm.$$

Hence, for the wave water by a stone, its wave length at "cm"

$\Rightarrow$ Hence, the surface tension is important!

Consider the purely gravity wave case, and perform dimensional analysis. $\omega^2 = gk\, f(kh)$, where kh is dimensionless.

Hence, "h" provids a length scale, through which we can obtain anomalous dimension. If $f(x) \to \begin{cases} \text{const} & \text{as } x \to \infty \\ x^\eta & x \to 0 \end{cases}$, we arrive at

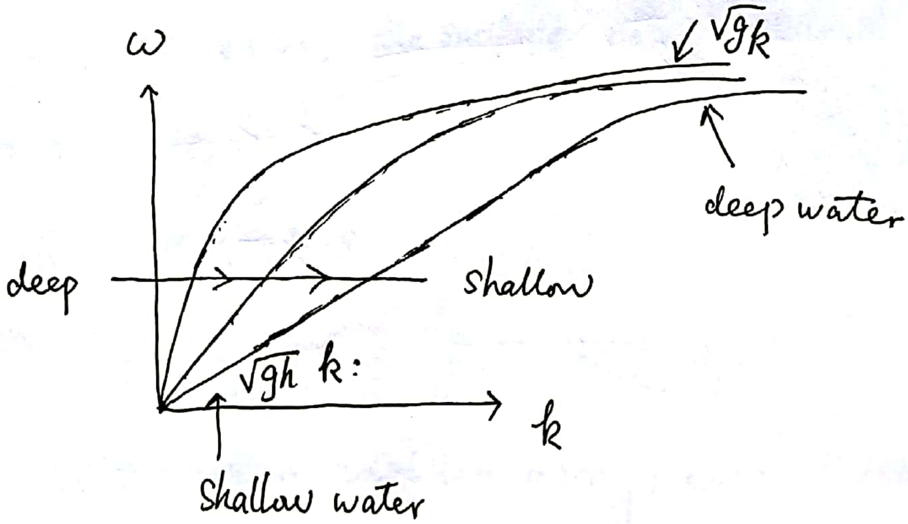
$$\omega^2 \propto \begin{cases} k & \text{as } k \to \infty \\ k^{1+\eta} & \text{as } k \to 0 \end{cases}.$$

Hence, the dispersion power ~~els~~ evolves between the long and short-wave region. $\Rightarrow$

$$\omega^2 \propto \begin{cases} k & \text{— short wave} \quad \lambda \ll h \\ k^2 & \text{— long wave} \quad \lambda \gg h \end{cases}$$

---

dispersion - relation



(x) Analysis to wave configurations

$$\phi(x,y,t) = A\, e^{i(kx-\omega t)} \cosh k(y+h)$$

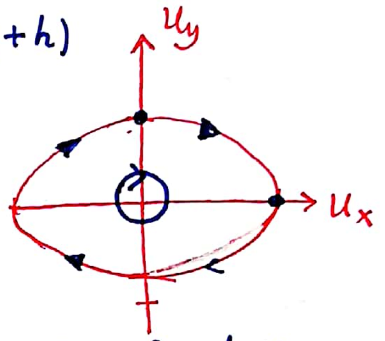
$$u_x = \nabla_x \phi = i B\, e^{i(kx-\omega t)} \cosh k(y+h)$$

$$u_y = \nabla_y \phi = B\, e^{i(kx-\omega t)} \sinh k(y+h)$$

$\xrightarrow{\text{take real part}}$

$$\begin{cases} u_x = B \sin(\omega t - kx) \cosh k(y+h) \\ u_y = B \cos(\omega t - kx) \sinh k(y+h) \end{cases}$$

$$\left(\frac{u_x}{\cosh k(y+h)}\right)^2 + \left(\frac{u_y}{\sinh k(y+h)}\right)^2 = B^2$$

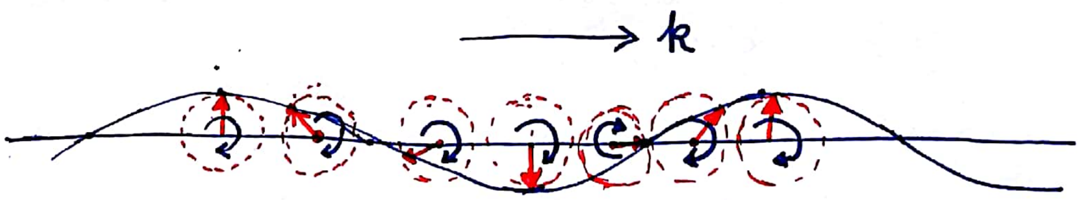


At a fixed x, location of water-droplet v.s. time.

The trajectory is an ellipse.

The amplitude of u_x is larger than that of u_y. As y goes deeper, i.e. more negative, the aspect ratio becomes more and more narrow.

(a) In the case of deep water, i.e. $kh \gg 1$, the surface $\cosh kh \sim \sin kh$ the ellipse becomes a circle



As y goes deeper, the vibration amplitud decays exponentionally.

And the decay length is $\sim \lambda$. Below λ, the ~~wave~~ water becomes negeligible.

Scanned by CamScanner

For a given droplet, it does circularly motion in the xy-plane. The droplet moves forward along $\vec{k}$ at the water crest, and it moves backward at the trough.

(b) Shallow-water wave, i.e. $kh \ll 1$

$$\begin{cases} u_x \sim \sin(\omega t - kx) & \text{— const independent of } y \\ u_y \sim \cos(\omega t - kx)\, k(y+h) & \text{— decays linearly} \end{cases}$$

$$\left|\frac{u_y}{u_x}\right| \sim kh = 2\pi h/\lambda \ll 1$$

In the case of friction, the forward motion of water at the crest is easier. The backward motion of water at the trough is affected by frictions, hence, there's a net motion forward.

Hence, near the shore, we do see the water wave coming to the shore.

Application

⊛ Tsunami wave (津波) — gravity shallow water wave.

When earthquake occurs, its low frequency part is effective to excite tsunami wave. The period is about 1000s, i.e. $f \sim 10^{-3}$ Hz.

Due to the extreme low frequency, even the deep ocean $h = 5$ km is shallow water.

$$\omega = k\sqrt{gh} \Rightarrow k = \frac{\omega}{\sqrt{gh}} \quad \text{i.e. } \lambda = \frac{\sqrt{gh}}{f}$$

plug in $g = 10\, m/s^2$, $h = 5 \times 10^3\, m$, $f = 10^{-3}$ Hz $\Rightarrow$ $\boxed{\lambda \simeq \frac{2\times 10^2}{10^{-3}}\, m \simeq 200\, km \gg h.}$

Tsunami wavelength in the deep ocean is at the order of 10^3 km !!

The wave velocity $\boxed{v = \frac{\omega}{k} = \sqrt{gh} \simeq 200 m/s,}$

Of course, this is still much smaller than the sound velocity in water, but is much more than the storm wave.

Nearly no decay: The amplitude decays very slow with depth. The water droplet almost has no vertical motion, and the motion is nearly horizontal. The wavelength is extremely long. Hence, the entire ocean, from the surface to the bottom, over the length scale of 100 km, is doing an overall motion. This kind of collective motion carries a large amount of energy. Since it's nealy a global motion, the internal friction is small.

As the tsunami waves go ashore, its amplitude goes large.

The energy associated within one wave length

$$E \propto \omega^2 A^2 \lambda \cdot h \Rightarrow A \sim \lambda^{-1/2} h^{-1/2} \Rightarrow A \propto h^{-3/4}$$

$$k = \frac{\omega}{v} = \frac{\omega}{\sqrt{gh}} \Rightarrow \lambda^{-1} \sim h^{-1/2}$$

Hence, from deep ocean $h = 5\,km$. to the coast $h = 50m$

then A is amplified $(100)^{3/4} \simeq 30$.

Hence, a half meter high wave in the ocean, becomes a 15 meters high tsunami!

② Storm wave — deep water wave

The period of storm wave $\sim 6 \sim 10s$. i.e. the frequency $f \sim 0.1\,Hz$.

$$\omega^2 = kg \Rightarrow k = \frac{\omega^2}{g} \Rightarrow \lambda = \frac{g}{2\pi f^2}$$

if $f = 0.1\,Hz$. $\lambda = \frac{10}{6 \times 10^{-2}} \simeq 100m$

if in deep ocean. $h \simeq 5 \times 10^3\,m$, then $\lambda \ll h \Rightarrow$ deep water wave.

Then storm wave decays quickly with depth. Below 100 meters. the ocean is not affected! Its velocity

$$v = \frac{d\omega}{dk} = \frac{1}{2}\frac{\omega}{k} = \frac{1}{2}\frac{g}{\omega} = \frac{g}{4\pi f}$$

plug in $f = 0.1\,Hz$

$$v \simeq \frac{10}{12 \times 0.1} \simeq 10\,m/s$$

which is much slower than tsunami wave!