INSTITUTE FOR THEORETICAL SCIENCES
理论科学研究院

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Lecture 10 Conservation laws — Momentum conservation

- Newton's 3rd law

- symmetry aspect of momentum conservation

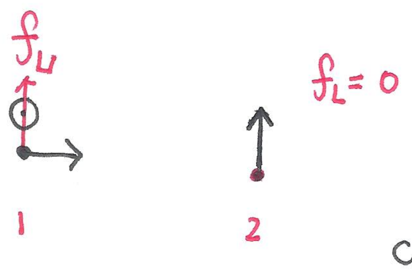
- Collisions

- Tsiolkovsky equation

§ Newton's 3rd law and its limitations

- Action equals reaction. 1 • 2 • $\vec{F}_{12} = -\vec{F}_{21}$

$\boxed{\text{problem:}}$ Newton's 3rd law assumes instantaneous interactions, but this is inconsistent with relativity. In E&M, the forces between two moving charges, even the force directions are not parallel to each other.



The key of these limitations are solved by the modern perspective: fields.

charge or mass generates fields. Fields propagate at the light velocity, and fields act on charge, or, mass. Nevertheless, we still assume that within the Newtonian mechanics, at speed much less than the light velocity, the Newton's 3rd law is valid.

- Conside two objects 1 and 2, $\vec{F}_{12} = -\vec{F}_{21}$.

$$\begin{cases} d\vec{P}_1/dt = \vec{F}_{12} \\ d\vec{P}_2/dt = \vec{F}_{21} \end{cases} \Rightarrow \frac{d(\vec{P}_1 + \vec{P}_2)}{dt} = 0$$

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§ Symmetry and Galilean transformation (perspective)

- Consider a mass of 2m. Imagine there's an explosion that it spilts two parts of equal mass. The system has the reflection symmetry, hence, the velocities are equal in magnitude and opposite in direction. Hence, the momentum before and after the explosion is zero.

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Consider two equal masses with velocities v_1 and v_2. They collide and stick togethe, what's their final velocity?

We switch to a reference frame F', which has the velocity $v' = \frac{v_1+v_2}{2}$. In this reference, two masses have the opposite velocites $\pm \frac{v_1-v_2}{2}$, hence. they exhibit the reflection symmetry. Then if they stick together, the velocity has to be zero. Then transform back to the lab frame $v = \frac{v_1+v_2}{2}$.

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Consider a mass of 3m. Then it splits into two non-equal masses with m and 2m, then what's the relation between v and v'?

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we view $\boxed{3m} = \boxed{2m}\,\boxed{m}$. Them $\boxed{2m}$ splits into two equal masses,

then $\boxed{2m}\,\boxed{m} \to \overset{v}{\leftarrow}\boxed{m}\ \ \boxed{m}\overset{v}{\rightarrow}\boxed{m} \to \overset{v}{\leftarrow}\boxed{m}\ \ \ \boxed{2m}\rightarrow v' = \frac{v}{2}$.

Then based on symmetry considerations, we cunculde that if the mass ratio of spliting $m_1 : m_2 = 1 : 2$, then $v_1 : v_2 = 2 : 1$ with opposite directions. Hence, the totle momentum remains zero.

If we reverse the process, and perform Galilean transformation, we can arrive that if the mass ration is $1:2$, then after collision, if then stick ~~together~~ we have $3mv = mv_1 + 2mv_2$, i.e. $v = \frac{v_1 + 2v_2}{3}$.

- Using the similar method, we can prove that if a mass split into two parts with the ratio $m_1 : m_2 = p : q$ (p and q integers), then their velocites are opposite in direction $\boxed{v_1 : v_2 = q : p.}$

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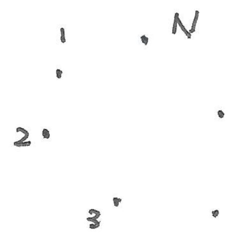
§: N-interacting point particle systems



$$\begin{cases} m_1 \ddot{\vec{r}}_1 = \vec{F}_1^{ex} + \vec{F}_{12} + \vec{F}_{13} + \cdots \vec{F}_{1n} \\ m_2 \ddot{\vec{r}}_2 = \vec{F}_2^{ex} + \vec{F}_{21} + \vec{F}_{23} + \cdots \vec{F}_{2n} \\ \vdots \\ m_n \ddot{\vec{r}}_N = \vec{F}_N^{ex} + \vec{F}_{N1} + \cdots + \vec{F}_{N,N-1} \end{cases}$$

$\vec{F}^{ex}_{tot}$

$$\Rightarrow \quad \sum m_i \ddot{\vec{r}}_i = \frac{d}{dt}\left(\sum_i m_i \dot{\vec{r}}_i\right) = \sum_{i=1}^{N} \vec{F}_i^{ex} + \sum_{i\neq j} \vec{F}_{ij}$$

Since $\vec{F}_{ij} + \vec{F}_{ji} = 0$, $\boxed{\Rightarrow \frac{d}{dt}\left(\sum_{i=1}^{N} m_i \dot{\vec{r}}_i\right) = \vec{F}^{ex}_{tot}}$

$$\vec{P}_{tot} = \sum_{i=1}^{N} m_i \dot{\vec{r}}_i = M\frac{d}{dt}\sum_{i=1}^{N}\left(\frac{m_i}{M} r_i\right) \leftarrow M = \sum_{i=1}^{N} m_i$$

$$\boxed{\vec{P}_{tot} = M\dot{\vec{R}}} \leftarrow \vec{R} = \sum_{i=1}^{N} \frac{m_i}{M} r_i \leftarrow \text{center of mass position.}$$

This is the origin of the validity of mass point although when applying for an extended object. For translation, an extended object can be viewed all the mass is concentrated at its center of mass.

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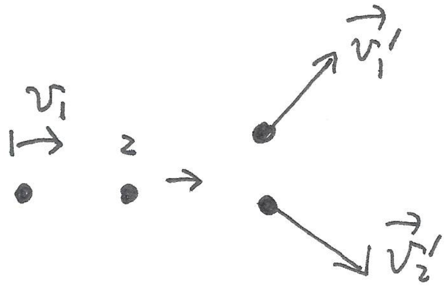
- Equal mass, elastic collision



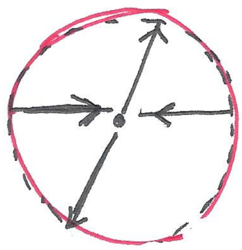
energy conservation

① $\frac{1}{2}mv_1^2 = \frac{1}{2}mv_1'^2 + \frac{1}{2}mv_2'^2$

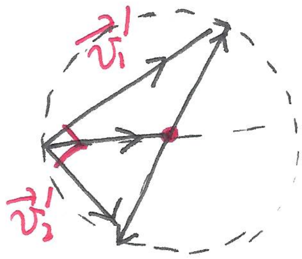
Momentum conservation ② $m\vec{v}_1 = m\vec{v}_1' + m\vec{v}_2'$

$$\left.\begin{array}{l} ② \Rightarrow v_1^2 = v_1'^2 + v_2'^2 + 2\vec{v}_1'\cdot\vec{v}_2' \\ ① \Rightarrow v_1^2 = v_1'^2 + v_2'^2 \end{array}\right\} \Rightarrow \boxed{\vec{v}_1' \perp \vec{v}_2'}$$

or v_1' or $v_2' = 0$.



Collision in the center of mass frame.



$\vec{v}_1'$, $\vec{v}_2'$ the velocity in the lab frame!

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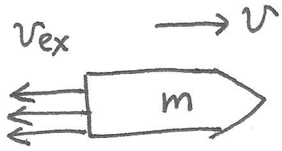
· Tsiolkovsky (Rocket equation) 1903 (mass-change)



v_{ex}: effective exhaust velocity

the velocity of exhaust gas relative to the rocket.

time t: $\quad p(t) = mv$

time $t+\Delta t$: $\quad$ rocket mass $\quad m + dm \quad (dm < 0)$

fuel burnt : $-dm$

$$p(t+\Delta t) = (m+dm)(v+dv) + (-dm)(v - v_{ex})$$

$$\approx mv + m dv + v_{ex} dm$$

$p(t+\Delta t) = p(t) \quad \Rightarrow \quad m dv = -v_{ex} dm$

$$dv = -v_{ex}\frac{dm}{m} \Rightarrow \int_{v_0}^{v} dv = -v_{ex}\int_{m_0}^{m}\frac{dm}{m}$$

$$v - v_0 = v_{ex}\ln\frac{m_0}{m}$$

if 90% of a rocket's mass is fuel, $\Rightarrow m_0/m = 10 \Rightarrow$ $\boxed{\begin{aligned} v - v_0 &= v_{ex}\ln 10 \\ &= 2.3\, v_{ex}\end{aligned}}$