3

Motion: Zeno's paradox, displacement, velocity, acceleration

Motion is a continuous process while our logic reasoning is discrete, *or*, step by step. How to use discrete steps of reasoning to precisely describe a continuous motion is a highly non-trivial problem. The ancient Greeks had already paid attention to this problem as represented by the Zeno paradox. In fact, in order for a deep understanding, an infinitesimal analysis is necessary, which are the watershed ridge between the advanced mathematics and elementary one.

3.1 Zeno's paradox

Zeno of Elea (490-430BC) is a Greek philosopher. He raised a paradox that Achilles, a hero of the Trojan War in Greek mythology, could not catch up with a tortoise. Later this paradox was recounted by Aristotle as "In a race, the

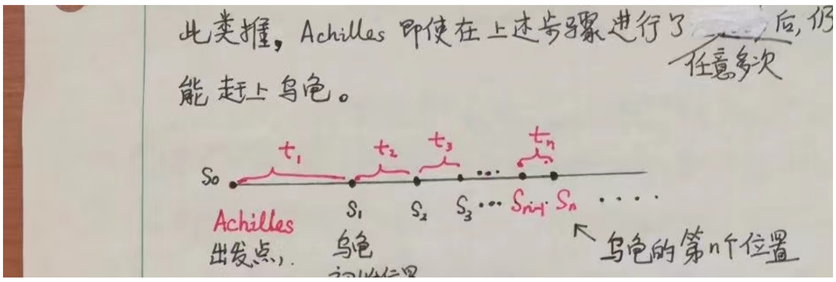

Figure 3.1 Zeno's paradox that Achilles cannot overtake a tortoise. Achilles' and the tortoise's initial positions are denoted x_0 and x_1, respectively. When Achilles arrives at x_1, the tortoise moves ahead to x_2. Then Achilles arrives at x_2, and the tortoise moves to x_3, and so on. This paradox shows the gap between our perception and the outside world in that our thinking is step by step while the motion is continuous.

quickest runner can never over-take the slowest, since the pursuer must first reach the point whence the pursued started, so that the slower must always hold a lead."

To be concrete, assume that Achilles' velocity is $v_a = 10$m/s and the tortoise's $v_t = 0.1$m/s. Initially, Achilles is located at $x_0 = 0m$ and the tortoise is ahead of Achilles at $x_1 = 99$m, and then the distance between them $s_1 = x_1 - x_0 = 99$m. This is a math problem that we learned how to solve in the elementary school. Assume that it takes Achilles the time T to overtake the tortoise, then it can easily derived that,

$$T = \frac{s_1}{v_a - v_t} = \frac{99}{9.9}s = 10s. \tag{3.1}$$

Nevertheless, Zeno provided a different perspective. He divided this chasing process into a series of steps: During step 1, Achilles reaches the initial position of the tortoise x_1. Meanwhile the tortoise moved ahead to x_2. During step 2, Achilles reaches Δt_2 and the tortoise moved to x_3, and so on. Since these steps can be kept on repeating forever, Zeno concluded that Achilles could never overtake the tortoise.

Certainly, this conclusion should not make sense. Where is the flaw in Zeno's reasoning? Let us denote the time interval spent during the n-the step as Δt_n, and then the total time spent should be,

$$T = \Delta t_1 + \Delta t_2 + \Delta t_3 + ..., \tag{3.2}$$

The question is that even though there is an infinite number of terms in this summation, does it really mean that the sum is infinite, or, could it still be finite?

To see what really happens, we need to analyze more carefully each step. During step one, the time spent is

$$\Delta t_1 = \frac{s_1}{v_a} = 9.9s. \tag{3.3}$$

Meanwhile the distance that the tortoise moved is

$$s_2 = x_2 - x_1 = v_t \Delta t_1 = s_1 \frac{v_t}{v_a} = 0.99\text{m}. \tag{3.4}$$

Then the time interval Δt_2 spent during step 2 is

$$\Delta t_2 = \frac{s_2}{v_a} = \Delta t_1 \frac{v_t}{v_a} = 0.099s, \tag{3.5}$$

and the distance during step two that the tortoise moved is

$$s_3 = x_3 - x_2 = v_t \Delta t_2 = 0.0099\text{m}. \tag{3.6}$$

By a similar reasoning, during the n-th step, Achilles takes the time Δt_n as

$$\Delta t_n = \Delta t_1 q^{n-1} \tag{3.7}$$

where $q = v_t/v_a$.

In elementary mathematics, we only learned how to sum finite terms. For example, we define

$$\begin{aligned} T_n &= \Delta t_1 \left(1 + q + q^2 + ... + q^n\right) \\ &= \Delta t_1 \frac{1 - q^n}{1 - q} \\ &= 10 \times (1 - (0.01)^n)\, s. \end{aligned} \tag{3.8}$$

Hence, we arrive at

$$\begin{aligned} T_1 &= \Delta t_1 = 9.9s \\ T_2 &= \Delta t_1 + \Delta t_2 = 9.999s \\ &... \\ T_n &= \Delta t_1 + \Delta t_2 + ... + \Delta t_n = 9.99...99s. \end{aligned} \tag{3.9}$$

So far everything is elementary mathematics.

The breakthrough actually arises when $n \to \infty$ is taken. In this case, literally we have

$$T = \Delta t_1 + \Delta t_2 + ..., \tag{3.10}$$

which gives rise to $T = 9.99.....s$. Since each term $\Delta t_n = \Delta t_1 q^{n-1}$, T is expressed order by order of q. Since $|q| < 1$, the higher order the term is, the smaller its contribution is. Hence, Eq. (3.10) is a perturbation theory. In contrast, Eq. (3.1) is a non-pertubative theory.

Compared to Eq. (3.1), the natural question is: Should 9.999.... be taken precisely as 10, or not? How to understand 9.99....? Let us check:

$$\begin{aligned} 10 - T_1 &= 10 - 9.9 = 0.1 \\ 10 - T_2 &= 10 - 9.9999 = 0.001 \\ 10 - T_3 &= 10 - 9.999999 = 0.00001 \\ &... \end{aligned} \tag{3.11}$$

Even exhausting our life, we could only perform the above process at finite steps. The great leap from elementary math to advanced one actually lies in that we are willing to accept the difference between 9.999..... and 10 is precisely 0, i.e, no approximation. This is because the difference between 10 and 9.99...

could be as small as you would like at any precision. Give me a precision ϵ, say, 10^{-2n}, we have $|10 - T_m| < \epsilon$ as long as $m > n$. Formally, it is denoted as

$$\lim_{n\to\infty} T_n = \lim_{n\to\infty} 9.99...9 = 10, \tag{3.12}$$

which should be viewed as a derivation, but rather as a definition.

Formally in the mathematics, there exists the following axiom: Any monotone bounded sequence $\{a_n, n = 1, 2, ...\}$ has a finite limit. Here we have

$$\begin{aligned} T &= \lim_{n\to\infty} T_n = \frac{\Delta t_1}{1-q} \lim_{n\to\infty}(1-q^n) = \frac{\Delta t_1}{1-q} \\ &= \frac{s_1}{v_a(1 - v_s/v_a)} = \frac{s_1}{v_a - v_s}. \end{aligned} \tag{3.13}$$

3.2 Analytic continuation – let divergent series make sense

The convergence of the geometric series relies on the common ratio $|q| < 1$. If $|q| > 1$, then the series diverges. Nevertheless, for physicists, a divergent series still makes sense in many situations. The key is the interpretation.

If we switch the positions of Achilles and the tortoise, then $q = v_a/v_s > 1$ and the geometric series of Eq. (3.10) diverges. Since the tortoise's speed is smaller than Achilles, there is no way for it to overtake Achilles. However, if we literarily take Eq. (3.13) to see what happens, it becomes

$$T = \frac{\Delta t_1}{1-q} = \frac{s_1}{v_s(1 - v_a/v_s)} = -\frac{s_1}{v_a - v_s}. \tag{3.14}$$

Although each term in the series is positive, we arrive at a negative T. It perfectly makes sense as long as we extrapolate the motions of Achilles and the tortoise from the past to the future, actually they meet before the time zero.

What is really happening here? Mathematically, this is called analytic continuation explained as follows. For simplicity, we define the dimensionless time $f(q) = T/\Delta t_1$, and assume that $f(q)$ should be analytic. Mathematically, there exist rigorous definitions of analytic functions. But for the moment, we do not need to be so rigorous. Roughly speaking, it just means that $f(q)$ can be expressed in terms of a regular form that we are used to.

In many complicated situations in the future we will face, in particular, in quantum field theory, we have little understanding in what happens at $q > 1$. But when $|q| < 1$, we can use the so called "perturbation theory" pretty much like Zeno's analysis. $f(q)$ is expanded order by order of q. Say, in Zeno's analysis, we arrive at,

$$f(q) = \sum_{n=1}^{\infty} q^n, \tag{3.15}$$

which converges at $|q| < 1$,

$$f(q) = \frac{1}{1-q}. \tag{3.16}$$

Actually, Eq. (3.16) has deeper meaning than the perturbation theory expression of Eq. (3.15). In many situations, the physical problem still has a solution at $|q| > 1$, the result is just non-perturbative. Assuming that the solution's dependence on q is analytical, we can use the perturbation theory to derive such an expression at $|q| < 1$, and it also works at $|q| > 1$.

This process is called analytic continuation, which is a remarkable method to explore the unknown from known. The validity can be justified when the uniqueness of analytic continuation can be proved. Indeed, this is the case under certain conditions in mathematics, and you will learn it in the class of "Mathematical Methods in Physics".

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§ Galileo's study on the uniform acceleration

Galileo set up a general methodlody for carrying on scientific researches, which release us from endless abstract speculation and debate. Galileo turned to focus on phenomenona and build up relations among phenomena. Then people can use mathematical modeling and reasoning to propose hypothesis. Based on hypothesis or theory, people can make prediction to be ~~tested~~ by desigining experiments. The success or failure of a theory is reflected by whether it can explain the pre-existing fact, and whether it has a predictive power.

The difficulties that Galileo faced were enormous:

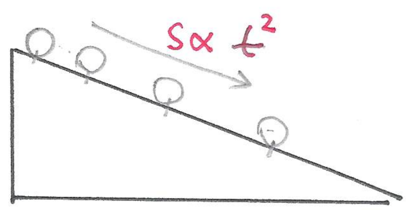
① experiment difficulty — lack of precision measurments of time

② conceptual level — instantanous velocity, acceleration, etc

③ math level — pre-calculus time.

Laws are hidden in phenomena!

Free fall motion is too fast to observe; hence, Galielo turned out to study motion on a slope to slow down the motion such that the observation is easier.



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$\boxed{\text{Galileo's data}}$

t	1	2	3	4	5	6	7	8	tick (time)
S	33	130	298	526	824	1192	1620	2104	punti
ΔS		97	168	228	298	368	428	484	
divided by 33	1	2.9	5.1	6.9	9.0	11.2	13	14.6	← nearly an arithematic progression

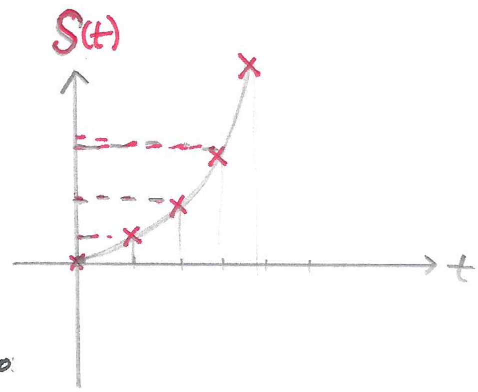
Considering possible experimental errors, Galileo observed the law that during equal small time interval, the distance traveled form an arithematic progression — idealization.

$$1 : 3 : 5 : 7 : 9 : 11 : 13 : 15$$

Then the distance traveled from the starting time is

$$1,\ 1+3=4,\ 1+3+5=9,\ 1+3+5+7=16,\ \cdots\cdots$$

$$\Rightarrow \quad \boxed{S(t) \propto t^2}$$



What's time? — If no one asks me, I know what it is. If I wish to explain to him who asks, I do not know — Saint Augustine.

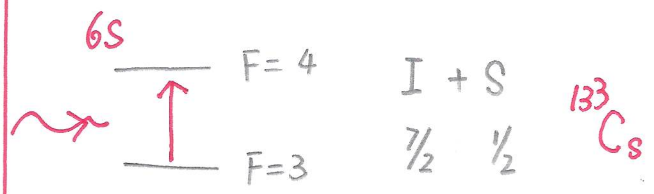
Then how to measure time? Use naturally existing peroids — year, month, day, etc. Or a better one easier to control — the hour glass. But they cannot provide the enough precision even for a free fall motion!

Galileo expressed that water clocks were used to measure time in his experiments. But it's hard to imagine that the necessary precision could be achieved without prior knowledge of the motion. Physics history experts suggested that he might clap to the beat to divide time into short equal intervals. He was a musician!

$\boxed{\text{time-unit}}$: Comité international des poids et mesures (CIPM)

2018: The second is defined by taking the fixed numerical value of the Cs frequency, $\Delta\nu_{Cs}$, the unperturbated ground ~~state~~ hyperfine transition frequency of the ^{133}Cs atom, to be 9,192,631,770 when expressed in the unit Hz, which is equal to s^{-1}.

(temperature $T = 0K$, average sea level, at rest, zero magnetic field).



$\lambda = 3.26\,cm$

$f = 9{,}192{,}631{,}770\ Hz$

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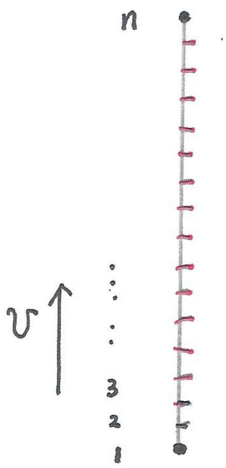
(x) Conceptual difficulty — instantaneous velocity

At Galileo's time, people had difficulty to define instantaneous velocity clearly. For example, consider the reverse motion of free fall, — the vertical projectile motion. We know that the motion is slowed down as the projectile reaches the peak.



Let us divide the motion into n parts equally, and the length of each step is h/n. As the projectile moves upward, finally $v \to 0$. It means it will take projectile more and more time to complete each step. Since we can divide the process into infinitly many steps, it's not clear whether the projectile would complete the motion in a finite period of time.

Or if we reverse the motion and consider the free fall, it means that if the falling object starts from the velocity zero, how could it start to fall at all!

Of course, we do know from our daily observation, both the projectile and the falling object will complete the motion. The ultimate solution is actually not simple, which rely on our concept of infinitismall quantities. — the sum of infinite number of time intervals can still converge to a finite amount of time period. — pretty much similiar to the Zeno's paradox.

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Galileo's response is that an object passes a point instantaneously. It occurs at a particular time "t", but it does not take an interval of time. Using the modern language, it means

$$\boxed{\begin{aligned} v(t) &= \lim_{\Delta t \to 0} \frac{S(t+\Delta t) - S(t)}{\Delta t} = \frac{dS(t)}{dt}, \\ \text{or} \quad \Delta t &= \frac{\Delta S}{v(t)} \to 0 \quad \text{as} \quad \Delta S \to 0. \end{aligned}}$$

⊛ Series expansion — frequently used formulae.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!} x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!} x^3 + \cdots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

Example of derivative:

$$f(x) = e^x \Rightarrow \frac{df}{dx} = \lim_{\Delta x \to 0} \frac{e^{x+\Delta x} - e^x}{\Delta x} = e^x \lim_{\Delta x \to 0} \frac{e^{\Delta x} - 1}{\Delta x} = e^x$$

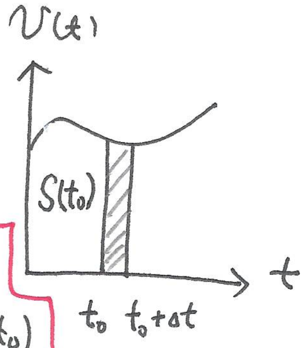
$$f(x) = a^x = e^{x \ln a} \Rightarrow \frac{df}{dx} = \ln a \, e^{x \ln a} = \ln a \cdot a^x$$

$$f(x) = x^n \Rightarrow \frac{df}{dx} = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^n - x^n}{\Delta x} = \lim_{\Delta x \to 0} \frac{x^n (1 + \frac{\Delta x}{x})^n - x^n}{\Delta x}$$

$$= \frac{x^n (1 + n \frac{\Delta x}{x}) - x^n}{\Delta x} = n x^{n-1}$$

$$f(x) = \ln x \quad \frac{df}{dx} = \lim_{\Delta x \to 0} \frac{\ln(x+\Delta x) - \ln x}{\Delta x} = \lim_{\Delta x \to 0} \frac{\ln x(1+\frac{\Delta x}{x}) - \ln x}{\Delta x} = \frac{1}{x}$$

✯ Velocity v.s. distance



distance (displacement)

$$S(t_0) = \sum_i v(t_i)\,\Delta t \xrightarrow{\Delta t\to 0} \boxed{\int_0^{t_0} dt\ v(t) = S(t_0)}$$

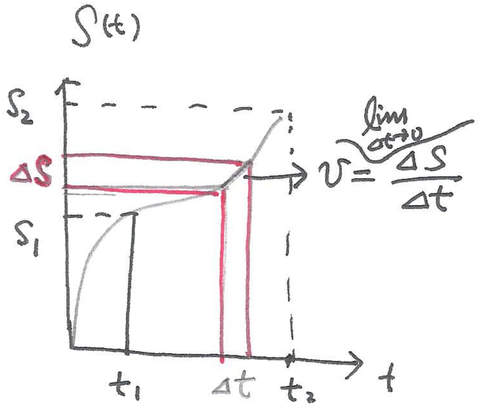
set $S(t=0)=0$.

$$S(t_0+\Delta t) - S(t_0) = \Delta t\, v(t_0) \Rightarrow \lim_{\Delta t} \frac{S(t_0+\Delta t)-S(t_0)}{\Delta t} = v(t_0)$$

$$\boxed{\frac{dS(t)}{dt} = v(t)}$$

Newton-Leibnitz formula

$$\boxed{\int_a^b f'(x)\,dx = f(b) - f(a)}$$



$$t_2 - t_1 = \sum_i \frac{\Delta S(t_i)}{v} = \int_{S_1}^{S_2} \frac{dS}{v}$$

✯ Acceleration

$$a(t) = \frac{dv(t)}{dt} = \lim_{\Delta t\to 0} \frac{v(t+\Delta t) - v(t)}{\Delta t} = \frac{d^2 S(t)}{dt^2}$$

$$\parallel$$

$$\lim_{\Delta t\to 0} \frac{S(t+\Delta t) - 2S(t) + S(t-\Delta t)}{(\Delta t)^2}$$

$$v(t_2) - v(t_1) = \int_{t_1}^{t_2} a(t)\,dt$$

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(*) motion with a constant acceleration

free fall : $S(t) = \frac{1}{2} g t^2$ with $\begin{cases} S(t=0) = 0 \\ v(t=0) = 0 \end{cases}$

$$\left.\begin{aligned} v(t) &= \frac{dS(t)}{dt} = gt \\ a &= \frac{dv(t)}{dt} = g \end{aligned}\right\}$$

or we consider a motion of ~~constant~~ acceleration

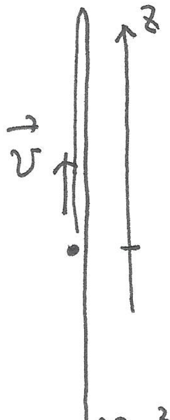
$$\frac{d^2 s}{dt^2} = g \longrightarrow \frac{dS(t)}{dt} = gt + C_1 \rightarrow S(t) = \frac{1}{2} g t^2 + C_1 t + C_2$$

C_1 and C_2 are constants determined by the intial conditions.

if $S(t=0) = 0$, $v(t=0) = 0 \Rightarrow C_1 = C_2 = 0.$

(*) displacement as a vector / velocity is also a vector

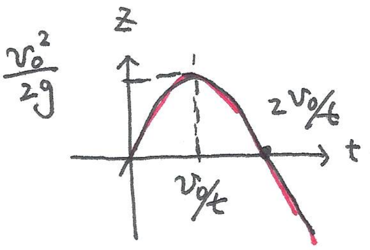
1d case: unification of the motion of the ascending and descending parts.



$$\begin{cases} \frac{d^2 z(t)}{dt^2} = -g \\ \left.\frac{dz}{dt}\right|_{t=0} = v_0 \\ z(t=0) = 0 \end{cases}$$

$$\Rightarrow z(t) = -\frac{1}{2} g t^2 + v_0 t = -\frac{g}{2}\left(t - \frac{v_0}{g}\right)^2 + \frac{v_0^2}{2g}$$

$$v(t) = v_0 - gt$$



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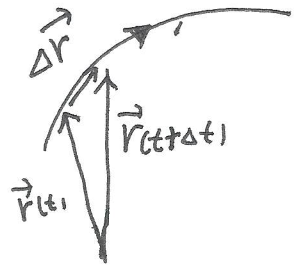
⊛ Motion in high-dimensions

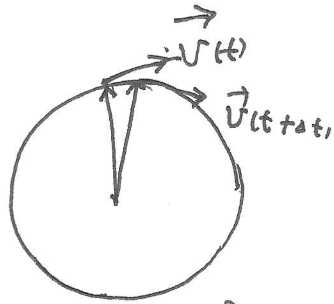
We use the vector notation to represent displacement, $\vec{r}$, velocity $\vec{v}$, and acceleration $\vec{a}$. The corresponding concepts can be generalized from the straightline motions

$$\vec{v} = \frac{d\vec{r}}{dt} = \lim_{\Delta t \to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t}$$



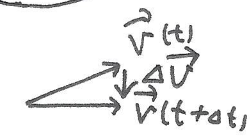
$$\vec{a} = \frac{d\vec{v}}{dt} = \lim_{\Delta t \to 0} \frac{\vec{v}(t+\Delta t) - \vec{v}(t)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{\vec{r}(t+\Delta t) + \vec{r}(t-\Delta t) - 2\vec{r}(t)}{(\Delta t)^2}$$



★ circular motion with uniform speed.

but it has the centripedal acceleration!



★ we can write down the definition interms of components

$$v_x = \frac{dx}{dt} \qquad a_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2}$$

$$v_y = \frac{dy}{dt} \qquad a_y = \frac{dv_y}{dt} = \frac{d^2y}{dt^2}$$

$$v_z = \frac{dz}{dt} \qquad a_z = \frac{dv_z}{dt} = \frac{d^2z}{dt^2}$$

$$\Rightarrow v = \sqrt{v_x^2 + v_y^2 + v_z^2}$$

$$a = \sqrt{a_x^2 + a_y^2 + a_z^2}$$

★ Examples

① uniform speed circular motion

$$\begin{cases} x = r\cos\omega t \\ y = r\sin\omega t \end{cases} \Rightarrow \begin{cases} v_x = -\omega r\sin\omega t \\ v_y = \omega r\cos\omega t \end{cases} \Rightarrow \begin{cases} a_x = -\omega^2 r\cos\omega t \\ a_y = -\omega^2 r\sin\omega t \end{cases}$$

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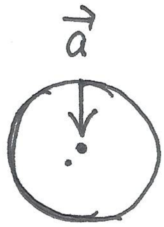
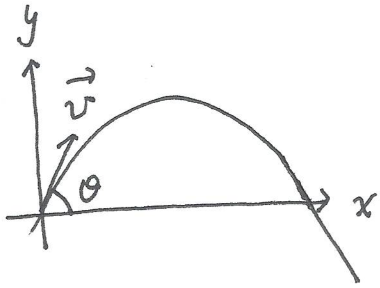


$$\Rightarrow \quad \vec{a} = -\omega^2 \vec{r}$$



② Projectile motion

$v_x = v\cos\theta \qquad v_y = v\sin\theta$

$$\frac{dx}{dt} = v_x \qquad \Rightarrow \begin{cases} x = v_x t + C_1 \\ y = v_y t - \frac{1}{2} g t^2 + C_2 \end{cases}$$

$$\frac{dy}{dt} = v_y - gt$$

plug in $x(0) = 0$, $y(0) = 0$

$$\Rightarrow \begin{cases} x = v\cos\theta\, t \\ y = v\sin\theta\, t - \frac{1}{2} g t^2 \end{cases}$$

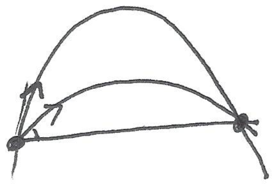
$\Rightarrow$ equation of trajactory

$$y = x\tan\theta - \frac{g}{2v^2\cos^2\theta} x^2$$

$$= \frac{-g}{2v^2\cos^2\theta}\left[x - \frac{\sin 2\theta}{2g} v^2\right]^2 + \frac{(v\sin\theta)^2}{2g}$$

Range of shooting : $\frac{\sin 2\theta}{g} v^2$

shooting altitude : $\frac{(v\sin\theta)^2}{2g}$



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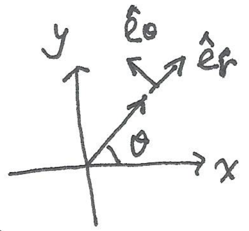
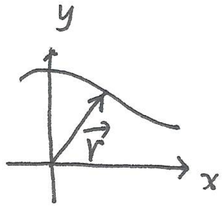
⊛ Motion in polar coordinates



$$\begin{cases} \hat{e}_r = \hat{x}\cos\theta + \hat{y}\sin\theta \\ \hat{e}_\theta = -\hat{x}\sin\theta + \hat{y}\cos\theta \end{cases} \Rightarrow \frac{d\hat{e}_r}{d\theta} = \hat{e}_\theta,\quad \frac{d\hat{e}_\theta}{d\theta} = -\hat{e}_r$$

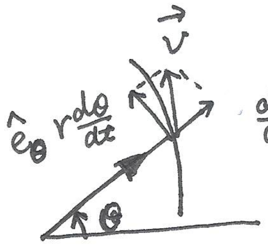
$$\Rightarrow \frac{d\hat{e}_r}{dt} = \hat{e}_\theta \frac{d\theta}{dt},\quad \frac{d\hat{e}_\theta}{dt} = -\hat{e}_r \frac{d\theta}{dt}$$



$$\vec{r}(t) = r(t)\,\hat{e}_r(t)$$

$$\vec{v}(r) = \frac{d\vec{r}}{dt} = \frac{dr}{dt}\hat{e}_r + r\frac{d\hat{e}_r}{dt}$$

$$= \hat{e}_r\, dr/dt + \hat{e}_\theta\, r\frac{d\theta}{dt}$$



$$\vec{a}(r) = \frac{d\vec{v}}{dt} = \frac{dr}{dt}\frac{d\hat{e}_r}{dt} + \hat{e}_r\frac{d^2r}{dt^2} + \frac{d\hat{e}_\theta}{dt} r\frac{d\theta}{dt} + \hat{e}_\theta\frac{dr}{dt}\frac{d\theta}{dt} + \hat{e}_\theta r\frac{d^2\theta}{dt^2}$$

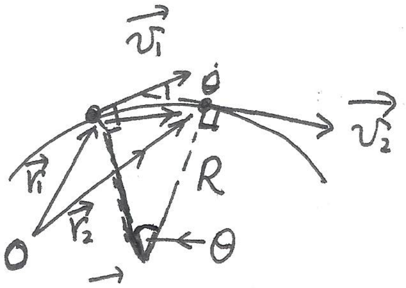
$$= \left(2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\right)\hat{e}_\theta + \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\hat{e}_r$$

$$= \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\hat{e}_r + \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)\hat{e}_\theta$$

For circular motion with fixed r, $\Rightarrow$

$$\vec{a} = \underbrace{-r\left(\frac{d\theta}{dt}\right)^2\hat{e}_r}_{\text{centripendal acceleration}} + \underbrace{r\frac{d^2\theta}{dt^2}\hat{e}_\theta}_{\text{acceleration along the tangential direction}}$$

※ Acceleration in the tangent direction and the normal direction

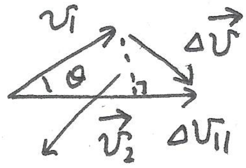


Consider a motion along a curve.

At $\vec{r}_1$, its velocity is $\vec{v}_1$,

At $\vec{r}_2$, its velocity is $\vec{v}_2$.

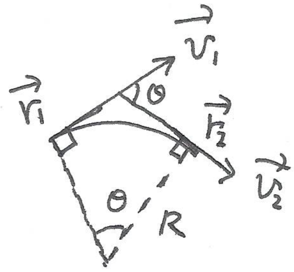
The angle difference between $\vec{v}_1$ and $\vec{v}_2$ is θ.



$\Delta v_{\perp} = v\theta$

$$\Rightarrow \Delta\vec{v} = \vec{v}_2 - \vec{v}_1 = \Delta v_{\parallel}\,\hat{e}_t + \Delta v_{\perp}\,\hat{e}_n .$$

$$\Rightarrow \vec{a} = \lim_{\Delta t \to 0} \frac{\Delta\vec{v}}{\Delta t} = \frac{dv}{dt}\,\hat{e}_t - v\frac{d\theta}{dt}\,\hat{e}_n$$



We can appoximate the trajectory from $\vec{r}_1$ to $\vec{r}_2$ as a circle with a radius of R. 1/R is the curvature

Then $\dfrac{d\theta}{dt} = \dfrac{v}{R}$

$$\Rightarrow \vec{a} = \underbrace{\frac{dv}{dt}\,\hat{e}_t}_{\text{tangential acceleration}} - \underbrace{\frac{v^2}{R}\,\hat{e}_n}_{\text{normal acceleration}}$$