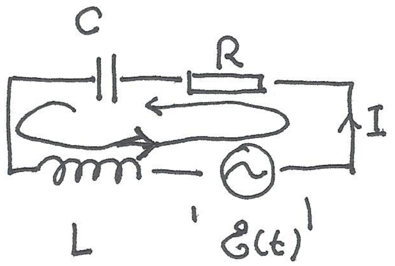
Lect 5 – continued Forced Oscillation, resonance

§ Driven damped Oscillations

Consider the LC circuit, in addition to the inductor, capacitor, and resistor, there's also a driving EMF.



$$IR + \frac{Q}{C} = -L\frac{dI}{dt} + \mathcal{E}(t)$$

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = \mathcal{E}(t)$$

$$\boxed{\frac{d^2Q}{dt^2} + \frac{R}{L}\frac{dQ}{dt} + \frac{1}{LC}Q = \frac{\mathcal{E}(t)}{L}.}$$

$$\frac{1}{\tau} = \frac{R}{L}$$

$$\omega_0^2 = \frac{1}{LC}$$

If for a mechanical oscillator,

$$\boxed{\ddot{x} + \frac{1}{\tau}\dot{x} + \omega_0^2 x = f(t). \quad \text{where } f(t) = F(t)/m.}$$

Essentially, these are the same equations: a constant coefficient, linear, inhomogeneous ODE. The solution can be obtained via the superposition principle.

$$x(t) = x_h(t) + x_p(t)$$

transient solution

$x_h(t)$ is the general solution to the homogeneous part of satisfying $\ddot{x}_h + \frac{1}{\tau}\dot{x}_h + \omega_0^2 x_h = 0.$

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$x_p(t)$ is a particular solution to the inhomogenous ODE, satisfying

$$\ddot{x}_p + \frac{1}{\tau}\dot{x}_p + \omega_0^2 x_p = f(t). \leftarrow \text{steady solution}$$

Since the general solution $x_h(t)$ decays with time, the long-time behavior is determined by x_p.

★ Consider a sinusoidal force driving : $f(t) = f_0 \cos\omega t$

A trick to solve this ODE is to promote it to the complex number field, we can also define another ODE

$$\ddot{y} + \frac{1}{\tau}\dot{y} + \omega_0^2 y = f_0 \sin\omega t, \quad \text{and define } z = x + iy$$

$$\Rightarrow \boxed{\ddot{z}_p + \frac{1}{\tau}\dot{z}_p + \omega_0^2 z_p = f_0 e^{i\omega t}}$$

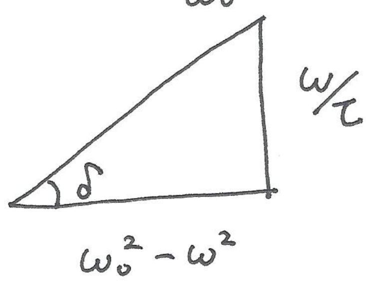
After solving z, then take its real part.

Try a particular solution : $z_p(t) = C e^{i\omega t}$

$$C\left[-\omega^2 + \frac{i\omega}{\tau} + \omega_0^2\right] = f_0$$

$$C = \frac{f_0}{\omega_0^2 - \omega^2 + i\omega/\tau}$$

$$\delta = \tan^{-1}\frac{\omega/\tau}{\omega_0^2 - \omega^2}$$



Let us express $C = A e^{-i\delta}$

$$A^2 = f_0^2 / \left[(\omega_0^2 - \omega^2)^2 + (\omega/\tau)^2\right].$$

hence $x_p(t) = \mathrm{Re}[A e^{i(\omega t-\delta)}] = A\cos(\omega t-\delta)$

$$x(t) = A\cos(\omega t-\delta) + \underbrace{C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}}_{\text{transient solution}} \to 0 \text{ as } t\to\infty.$$

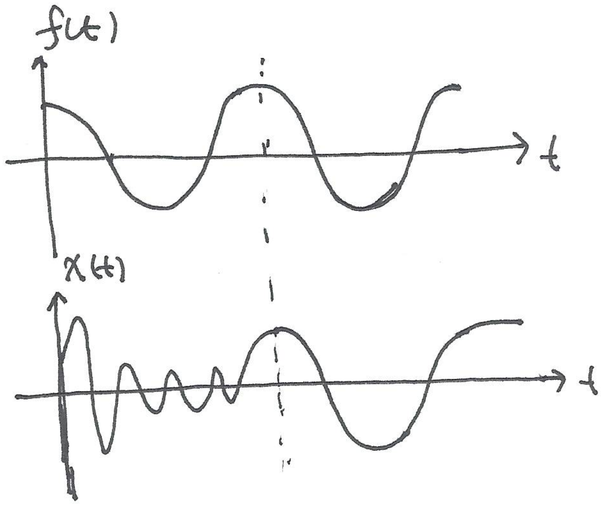
For a weakly damped system,

$$x(t) = A\cos(\omega t-\delta) + A_{tr}\, \underbrace{e^{-t/2\tau} \cos[\omega_1 t-\delta_{tr}]}_{\text{transient}}$$

where $\omega_1 = \sqrt{\omega^2 - \left(\frac{1}{2\tau}\right)^2}$.

Transient motion depends on the initial conditions, but it decays.

Different initial conditions lead to the same steady motion — attractor.



§ Resonance

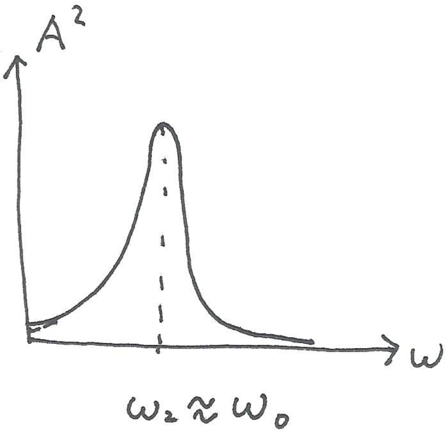
under the driving force $f = f_0 e^{i\omega t}$, we have $z_p(t) = \dfrac{f_0 e^{i\omega t}}{\omega_0^2-\omega^2+i\omega/\tau}$.

① If ω_0 and ω are very different, we have $\omega \to 0$, $\quad z_p(t) \to f_0/\omega_0^2 \, e^{i\omega t}$

$\omega \to \infty \quad z_p(t) \to - f_0/\omega^2 \, e^{i\omega t}$ } the amplitudes are small.

② If $\omega_0 \to \omega$, the Amplitude $\quad A = \dfrac{f_0}{\omega_0^2 - \omega^2 + i\omega/\tau}$

$$|A^2| = \frac{f_0^2}{(\omega_0^2-\omega^2)^2 + \omega^2/\tau^2}$$



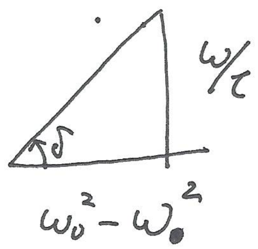
Fix ω_0, the maximum A occurs at slightly smaller frequency

$$\frac{\partial}{\partial(\omega^2)} \left[(\omega_0^2-\omega^2)^2 + \omega^2/\tau^2 \right] \Big|_{\omega=\omega_2} = 0$$

$$-2(\omega_0^2 - \omega_2^2) + \frac{1}{\tau^2} = 0 \quad \Rightarrow \quad \omega_2 = \left[\omega_0^2 - \frac{1}{2\tau^2} \right]^{1/2}.$$

$$= \omega_0 \left[1 - \frac{1}{2\omega_0^2\tau^2} \right]^{1/2} \approx \omega_0 \left[1 - \frac{1}{4\omega_0^2\tau^2} \right]$$

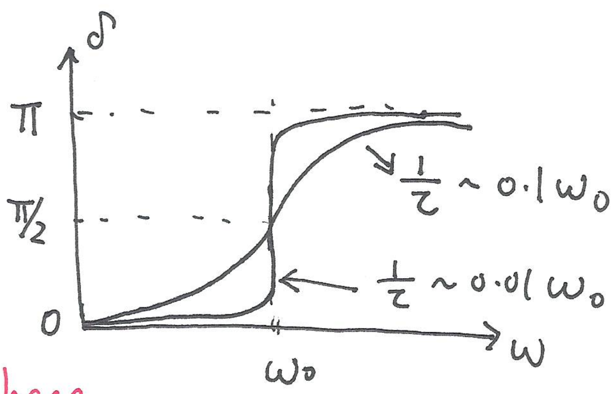
③ phase difference $\quad \delta = \tan^{-1} \dfrac{\omega/\tau}{\omega_0^2 - \omega^2}$



- at $\omega \ll \omega_0$, $\delta \to 0$
- at $\omega \gg \omega_0$, $\delta \to \pi$

} far from reasonance, A is real.

please note the "π" phase difference at $\omega \gg \omega_0$.



dissipation causes the delay of phase.

- Width of the resonance

At $\frac{1}{\tau} \ll \omega_0$, $|A|^2$ reaches half maximum at $(\omega_0^2 - \omega^2)^2 \simeq \omega_0^2/\tau^2$.

$$\Rightarrow \omega_0^2 - \omega^2 = \pm \frac{\omega_0}{\tau}$$

$$\omega^2 = \omega_0^2 \pm \frac{\omega_0}{\tau} \approx \left(\omega_0 \pm \frac{1}{2\tau}\right)^2$$

$$\Rightarrow \omega \approx \omega_0 \pm \frac{1}{2\tau}$$