6
Kepler's Problem

Newton once said that he stood on the shoulders of giants. Although it was said that he was alluding Hook negatively, we could understand it in a positive way. Scientifically, the giants for Newton are Galileo and Kepler(1571-1630AD). Galieo's contribution was already presented in previous lectures, and now we proceed to study Kepler's problem. The solution to Kepler's problem was accomplished by Newton, based on which the concept of the universal gravity came into being. This is a most influential achievement of the human mind.

6.1 Kepler's story

Kepler summarized Tycho Brahe's observation data and proposed three laws of planet motions, based on which Newton identified the inverse-square law of gravity. This was actually a quite complicated and interesting story.

At Kepler's time, people only knew five planets except the earth: Mercury, Venus, Mars, Jupiter, and Saturn. Kepler was inspired by the fact of the existence of 5 convex polyhedral (Platonic solids): tetrahedron, cube, octahedron, dodecahedron, and icosahedron. He proposed that there would exist a one-to-one correspondence between the five planets to the five Platonic solids:

$$\begin{aligned} &\text{Mercury} \leftrightarrow \text{octahedron}, \text{Venus} \leftrightarrow \text{icosahedron}, \\ &\text{Mars} \leftrightarrow \text{dodecahedron}, \text{Jupiter} \leftrightarrow \text{tetrahedron}, \\ &\text{Saturn} \leftrightarrow \text{cube}. \end{aligned} \tag{6.1}$$

As shown in Fig. 6.1, the sphere of the earth orbit is set as a reference. The earth orbital sphere is circumscribed to a dodecahedron whose circumscribed sphere is the Mars orbit. The Mars orbital sphere is circumscribed to a tetrahedron whose circumscribed sphere is the Jupiter orbit. Furthermore, the Jupiter

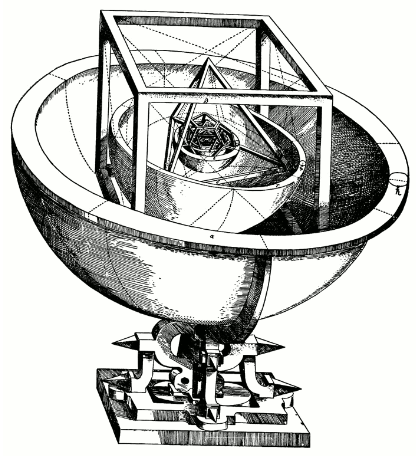


Figure 6.1 From Wikipediea. Kepler's model of solar system. He suggested the correspondences: Mercury ↔ octahedron, Venus ↔ icosahedron Mars ↔ dodecahedron, Jupiter ↔ tetrahedron, Saturn ↔ cube.

orbital sphere is circumscribed to a cube whose circumscribed sphere is the Saturn orbital sphere. On the other hand, the earth orbital sphere has an inscribed icosahedron whose inscribed sphere is the Venus orbital sphere. The Venus orbital sphere has an inscribed octahedron whose inscribed sphere is the Mercury orbital sphere.

Kepler wrote his theory in a book and sent it to Tycho Brahe, who spent his life on observing planet motion and accumulated enormous data. (He also sent it to Galileo but Galileo did not response.) Tycho Brahe welcomed and hired Kepler as his assistant. After Tycho Brahe's passing way, Kepler spent 20 years to analyze Tycho's data and hoped to verify his model. To his disappointment, Kepler failed to fit Mars' orbit by a circle. Finally, he reluctantly to recognize that planet orbits are ellipses in 1605, which is Kepler's first law. The largest eccentricity of planet orbits is that of Mercury, which is 0.2. Eccentricities for others are not large: $e_{mars} = 0.09$, $e_{jupiter} = 0.05$, $e_{saturn} = 0.05$, $e_{uranus} = 0.05$, $e_{earth} = 0.02$, $e_{neptune} = 0.008$, $e_{venus} = 0.007$, and $e_{moon} = 0.05$, which are in good approximations as circles. After further studies, he published Kepler's 2nd law, i.e., the area law, and the 3nd law which relates the radii and periods of different orbits. Nevertheless, Kepler did not feel pride for these discoveries since ellipse is not as perfect as circle.

Hence, Kepler discovered the laws of planet motions in a dramatic way. He

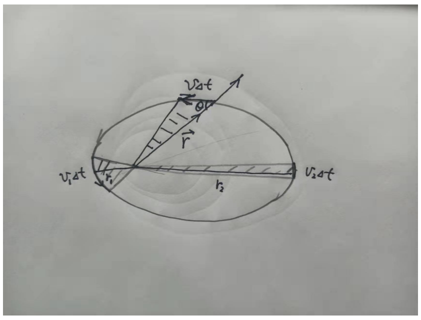


Figure 6.2 Kepler's 2nd law. The area that the earth-Sun line swept in a unit time Δt is proportional to the angular momentum of the earth.

was studying the wrong problem but arrived at the correct answer. Kepler's law served as the motivation and foundation for Newton's theory of gravity.

6.2 Kepler's three laws

6.2.1 The 1st law

The planet's orbit is a planar ellipse, and the Sun lies at one of the foci of the ellipse.

Kepler's first law is actually a very strong statement. Generally speaking, a planet moves in the dimensional (3D) space, but this law states that it is planar. A planar motion does not always form a closed orbit, but Kepler's first law assured that it is closed, and it is periodic. Copernicus thought that the planet orbit should be a circle as influenced by the aesthetic philosophy of Greeks. Nevertheless, Kepler figured out in general a planet orbit is an ellipse. A circular orbit is a special case in that the Sun lies in the center of the circle.

6.2.2 The 2nd law

Since the general orbital of a planet is elliptical rather than circular, the planet motion at each point in the orbit is different. To connect the planet motion along the orbit, that is what Kepler's 2nd law tells us.

The areas swept by the line connecting the Sun and a planet are equal in equal time intervals.

Assume that within a time interval Δt, the line running from the Sun to a planet sweep the area of ΔS. Kepler's 2nd law states that $\Delta S/\Delta T$ is a constant.

Consider two special positions in a planet orbit: the apogee (the farthest point from the Sun), and the perigee (the nearest point from the Sun). Denote the planet displacement vectors relative to the Sun at the apogee and perigee are $\mathbf{r_a}$ and $\mathbf{r_b}$, respectively. Correspondingly, the velocities are denoted as $\mathbf{v_a}$ and $\mathbf{v_b}$, then $\mathbf{v_a} \perp \mathbf{r_a}$, and $\mathbf{v_b} \perp \mathbf{r_b}$.

The arc length traveled around the apogee during time Δt is $\Delta s = v_1 \Delta t$, and the area swept at the apogee is $\Delta S = \frac{1}{2} r_1 \Delta s = \frac{1}{2} r_1 v_1 \Delta t$. Similarly, the same area should be swept during Δt around the perigee $\Delta S = \frac{1}{2} r_2 v_2 \Delta t$. Then

$$mr_1 v_1 = mr_2 v_2, \tag{6.2}$$

where the planet mass m is multiplied. The product of linear momentum and the displacement is actually the angular momentum, whose precise definition will be given later. It means that the angular momentum L_1 at the apogee equals that of L_2 at the perigee. Since $r_1 > r_2$, we have $v_1 < v_2$.

If the planet is at a general point in the orbit, then $\mathbf{v}$ and $\mathbf{r}$ are no longer perpendicular. Their relative angle is denoted by θ. Then the area swept during a small time-interval Δt is

$$\Delta S = \frac{1}{2} rv sin\theta. \tag{6.3}$$

If we add a direction to the area, it becomes $\mathbf{\Delta S} = \frac{1}{2} \mathbf{r} \times \mathbf{v}$. Hence it means the angular momentum conservation,

$$\mathbf{L} = m\mathbf{r} \times \mathbf{v}, \tag{6.4}$$

which does not change with time.

Angular momentum conservation is a fundamental law of nature as a consequence of spatial isotropy, which will be explained later. Simply put, it is because the gravity force passes the Sun center, therefore, it does not generate torque to change the angular momentum.

6.2.3 Kepler's third law

Different initial conditions can lead to different orbits. Kepler's 3rd law connects different orbits.
For different orbits, the ratio between the cube of half major axis and the period square is a constant.

For an elliptic orbit, see Fig. 6.3, the origin is set at the focus. The y-axis intersects the ellipse and cuts a cord (*latus rectum*), whose half length is denoted as h. The x-axis is the major axis intersecting the ellipse, whose half length is a.

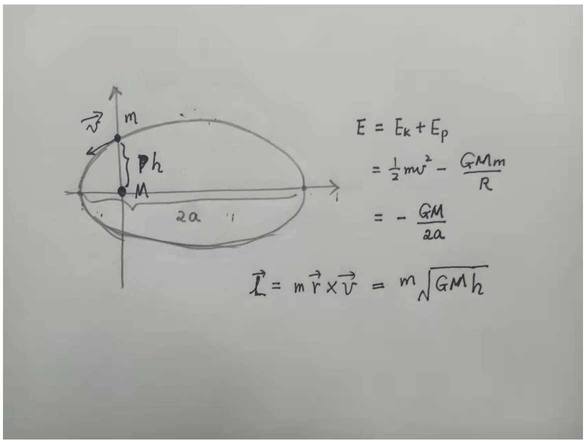


Figure 6.3 The total energy of an elliptic orbit is completely determined by the half major axis a as $E = -\frac{GMm}{2a}$. The angular momentum is completely determined by the half length of the cord passing the focus h as $l = m\sqrt{GMh} = mh\sqrt{GM/h}$.

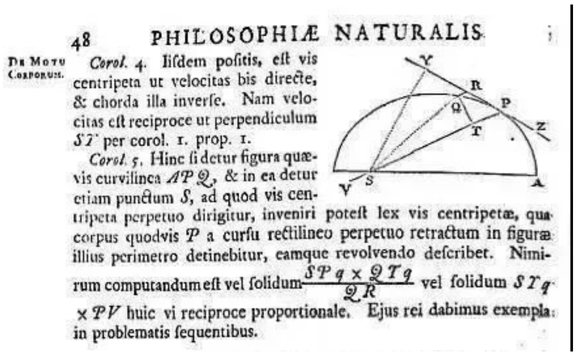
In fact, Kepler's 3rd law implies the inverse-square law: Consider the special case of a circular orbital, then $a = R$. A simple dimensional analysis, or, the scaling analysis, is given below. Due to the nature of the periodical motion, the acceleration, roughly speaking, scales as $F/m \sim v/T \sim R/T^2$. According to Kepler's 3rd law that $T^2 \sim R^3$, we arrive at

$$F \sim \frac{1}{R^2}. \tag{6.5}$$

Certainly, the above argument should not be viewed as a proof. Rather it should be viewed as a motivation for further exploration.

6.3 Solution to Kepler's problem by the geometric method

Kepler's laws are phenomenological laws based on astronomical observations, whose simplicity and beauty are already impressive. Such beautiful laws cannot just a coincidence, which stimulated physicists including Issac Newton to further explore the underlying law of gravity. By assuming the inverse-square law of gravity, Newton derived Kepler's three laws of planet motion. Although in the modern formalism, this can be done concisely via calculus, and indeed solving the motion under forces was the main motivation of Newton to invent his *fluxional* calculus.



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De Motu Corporum.

Corol. 4. Iiſdem poſitis, eſt vis centripeta ut velocitas bis directe, & chorda illa inverſe. Nam velocitas eſt reciproce ut perpendiculum *ST* per corol. 1. prop. 1.

Corol. 5. Hinc ſi detur figura quævis curvilinea *APQ*, & in ea detur etiam punctum *S*, ad quod vis centripeta perpetuo dirigitur, inveniri poteſt lex vis centripetæ, qua corpus quodvis *P* a curſu rectilineo perpetuo retractum in figuræ illius perimetro detinebitur, eamque revolvendo deſcribet. Nimirum computandum eſt vel ſolidum $\frac{SPq \times QTq}{QR}$ vel ſolidum $STq \times PV$ huic vi reciproce proportionale. Ejus rei dabimus exempla in problematis ſequentibus.

Figure 6.4 A page of Newton’s principia.

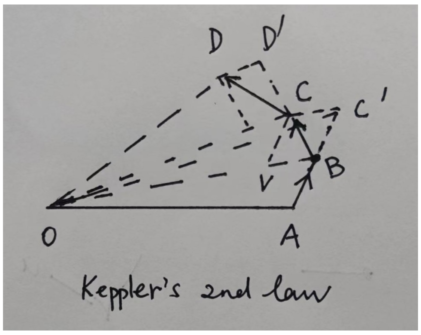

Figure 6.5 The geometric proof of Kepler’s 2nd law.

Nevertheless, interestingly, in Newton’s *Principia*, he did not use calculus. Rather, he adopted the style of Eculid’s *Elements* by using the geometric method. At that time, the mathematical foundation of calculus was not rigorously established. In fact, its establishment was deferred to the 19th century. Newton wanted to avoid the criticism of the infinitesimal analysis to hind the recognition of his theory of gravity.

6.3.1 Proof to Kepler’s 2nd law

Assume that the gravity is a central force field, which is sufficient for the proof to Kepler’s 2nd law. This was a very important problem historically – what drives the planets to move around? There were people at that time believed that planets are propelled by invisible angles’ beating wings. Hence the driving force should be along the tangential direction. However, we will show that actually the force is centripetal rather than tangential.

The geometric picture to prove Kepler's 2nd law is presented in Fig. (6.5). Suppose that the Sun is located at O and the planet starts from A. Within a small time interval Δt which is a first order infinitesimal, it moves to B. We use a short line segment AB to approximate its trajectory, and the error is at the 2nd order infinitesimal. The velocity is $\mathbf{v_{AB}} = \frac{\mathbf{AB}}{\Delta t}$. If there was no gravity, in the next time interval Δt, the planet would continue its straight-line motion toward C', such that $\mathbf{AB} = \mathbf{BC'}$. It is obvious that

$$S_{\triangle OAB} = S_{\triangle OBC'}. \tag{6.6}$$

However, the gravity pulls the planet back. The attraction is along the direction of $\mathbf{OB}$, such that the planet is pulled from C' to C, hence, $\mathbf{C'C} \parallel OB$, and $\mathbf{v_{BC}} = \frac{\mathbf{BC}}{\Delta t}$. It is easy to show that

$$S_{\triangle OBC} = S_{\triangle OBC'}, \tag{6.7}$$

since they share the same base and the same height. Hence within the same length of time interval Δt, the areas swept by the Sun-planet line are equal,

$$S_{\triangle OAB} = S_{\triangle OBC}. \tag{6.8}$$

This process can be further repeated: after consecutive intervals of Δt, the planet arrives at D, E, ..., then

$$S_{\triangle OAB} = S_{\triangle OBC} = S_{\triangle OCD} = S_{\triangle ODE} = ... \tag{6.9}$$

Moreover, the above process can be easily convinced that the trajectory $ABCD$... is a planar curve. This completes the proof of the Kepler's 2nd law.

$\Delta S/\Delta t$ can be figured out as $\frac{\Delta S}{\Delta t} = \frac{1}{2m}\frac{mr^2\Delta\theta}{\Delta t} = \frac{L}{2m}$, where L is the magnitude of the orbital angular momentum. If adding the direction to the area, we arrive at

$$2m\frac{\Delta \mathbf{S}}{\Delta t} = \frac{1}{2m}\frac{mr^2\Delta\theta}{\Delta t} = \mathbf{L}. \tag{6.10}$$

6.3.2 Proof to Kepler's first law

Next we prove Kepler's first law – the trajectory is generally speaking an ellipse following the method presented in "*Feynman's lost lecture: the motion of planet around the sun*".

Actually, Kepler's 2nd law does not ensure a closed orbit. For simplicity, we

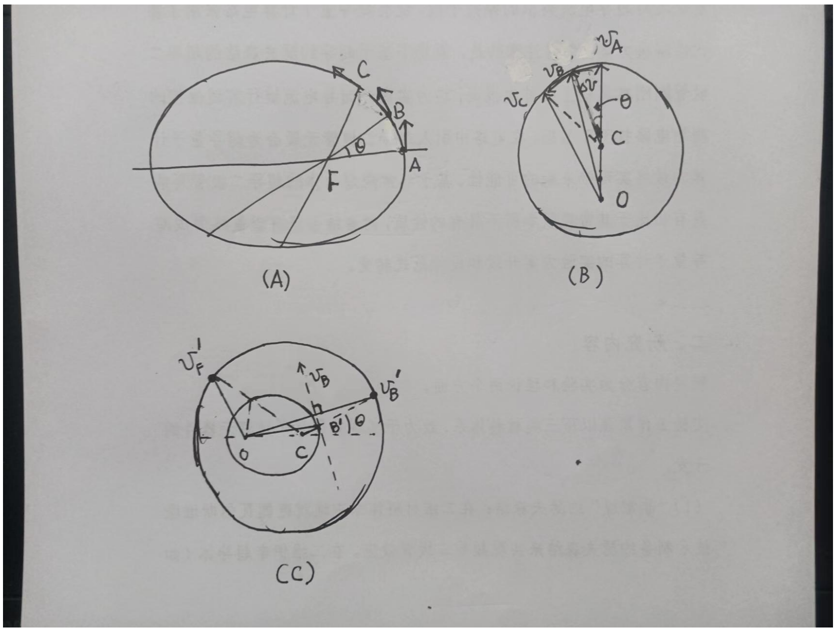

Figure 6.6 The geometric proof of the elliptical orbit of planet motion. A) F is the location of the Sun and A is the perigee. The angle between two neighboring radii is fixed at $\Delta\theta$. B) The velocity vectors $\mathbf{v}_A$, $\mathbf{v}_B$, $\mathbf{v}_C$ ... are plotted from the origin. Their ending points span a circle with the radius $v = \frac{GMm}{L}$, whose center is denoted as C. C) Rotate the velocity circle by 90°. Use the locations of O and C as the two foci, and v as the major axis length to construct an ellipse. Such an ellipse is similar to the planet orbit.

assume it is for the moment and will justify it later. Assume that the gravity is a central force field of the inverse-square law central,

$$\mathbf{F} = -\frac{GMm}{r^2}\mathbf{e_r}, \tag{6.11}$$

where M is the Sun mass, and m is the planet mass.

Newton realized the inverse-square law by comparing with the satellite motion – Moon's orbit to the free fall motion the ground. This is probably the origin of the legend of Newton's apple. Ancient people already measured the earth-moon distance compared to the earth radius by the method of parallax. The results of d_{em}/r_e measured by Ptolemy, Huygens, and Tcyho Brahe are very close to each other as 59, 60, 60.5, respectively. We know that the falling distance in one second on the ground of the earth is about 5m. How much is the "falling distance" of the Moon in one second? If without the earth gravity, the

Moon would fly along the tangent direction with a distance $\Delta x = v\Delta t$. To pull it back to the circular orbit, the Moon needs to fall at a distance s satisfying

$$\frac{v\Delta t}{s} = \frac{2d_{em}}{v\Delta t}$$
$$s = \frac{1}{2}\frac{v^2}{d_{em}}t^2 = \frac{1}{2}\omega^2 d_{em}t^2 = \frac{2\pi^2}{T^2}d_{em}t^2. \tag{6.12}$$

Plugging in the period $T = 27.3days$, and $d_{em} = 60 \times r_e$, we arrive at $s = 1.36mm$, which is about $1/3676 \approx 1/60^2$ of the falling distance on the earth. This crude estimation yields quite accurately the inverse-square relation, based on which Newton built up his confidence. He realized that the satellite motion and the planet motion satisfies the same law in both cases the gravity exhibits the inverse-square relation. Then the concept of "universal gravity" came to birth.

Now let us prove the elliptic orbit. As shown in Fig. (8.1) A, the Sun is located at F. Start from the perigee A on the orbit, and assign points B, C, and so on, such that $\angle BFA = \angle CFB = ... = \Delta\theta$, and $\Delta\theta$ is small. The radii of these points are denoted as r_A, r_B, r_C, ..., respectively, and the corresponding velocity vectors are denoted $\mathbf{v}_A$, $\mathbf{v}_B$, $\mathbf{v}_C$, etc, respectively.

Now let us compare $\Delta\mathbf{v}_{BA} = \mathbf{v}_B - \mathbf{v}_A$, and $\Delta\mathbf{v}_{CB} = \mathbf{v}_C - \mathbf{v}_B$. The time interval spent in sweeping ΔBOA is

$$\Delta t = \frac{S_{\Delta BOA}}{\Delta S/\Delta t} = \frac{1}{2}r_A^2\Delta\theta/(L/2m) = \frac{mr_A^2\Delta\theta}{L}. \tag{6.13}$$

Hence, $\Delta\mathbf{v}_{BA}$ can be calculated via Newton's 2nd law as

$$\Delta\mathbf{v}_{BA} = \frac{\mathbf{F}\Delta t}{m} = -\frac{GMm\Delta\theta}{L}\mathbf{e_A}, \tag{6.14}$$

which is along the opposite of the radial direction. Nicely, the dependence on the radius cancels. Since $\Delta\theta$ is fixed for each small triangle, $\Delta\mathbf{v}$ has a fixed amplitude, and its direction changes $\Delta\theta$ in each step. Hence, the ending points of the velocity vectors lie on a circle in the velocity space. According to Eq. 6.14, the tangential direction of the velocity circle actually reflects the direction of the displacement vector in real space. This means that the motion in the velocity space is actually the dual of that in the real space.

The circle center generally speaking is not located at the origin O, and is denoted as C. According to the planar geometric knowledge, the angle of $\Delta\mathbf{v}$ with respect to C actually equals to the angle change of $\Delta\mathbf{v}$ during each step. (Please note that here I mean the direction change of $\Delta\mathbf{v}$, not the direction change of **v** itself.) The radius of this circle is

$$v_r = \frac{\Delta v}{\Delta \theta} = \frac{GMm}{L}. \tag{6.15}$$

Now it is evident that the time evolution of the velocity vector **v** is periodic. Does it mean that the motion in the real space is also periodic? Generally it is not true. Nevertheless, based on the geometric correspondence between the real space and the velocity space motions, we conclude the answer is yes. Look at the velocity space: As the velocity vector completes a circle and moves back to $\mathbf{v}_A$ and its tangent direction also comes back, it means that in real space the planet is in the direction of FA. However, we need to determine the planet's location along the direction of FA. If it had not come back to the same point of A but it still possesses the same $\mathbf{v}_A$, it would mean that it angular momentum does conserve.

However, how to reconstruct the real space trajectory starting from the velocity space circle remains non-trivial. Here is a trick introduced by Feynamn: Rotate Fig. 6.6 B by 90° as presented in Fig. 6.6 C, and reconstruct the real-space trajectory in the same figure. In the rotated figure, $\mathbf{v}_{B'}$ is perpendicular to $\mathbf{v}_B$, which is along the tangent direction of the real space trajectory in Fig. 6.6 A. But the difficulty is how to locate the tangent point.

Let us try to locate the tangent point by construct the bisector of Ov'_B, which is parallel to $\mathbf{v}_B$. As $\mathbf{v_B}'$ runs over around the circle, the envelop of the sweeping bisector actually forms an ellipse. This ellipse can be stated even more explicitly, whose foci are located at O and C, and the major axis length is just the radius of the velocity circle.

How to see this? We relate it to a celebrated geometric property of ellipse: A light ray emitted from one focus is reflected to the other. Actually we will prove it by a geometric construction.

Let us connect Cv'_B. The bisector line of $\mathbf{v}_B$ intersect Cv'_B at point B'. Connect OB' and CB'. Since O and v'_B are reflectionally symmetric with respect to $\mathbf{v}_B$, the light ray OB' is reflected back to C, and $B'v'_B$ is the image of OB'. Hence

$$|CB'| + |OB'| = |Cv'_B| = \frac{GMm}{L}, \tag{6.16}$$

hence, B' is located on the ellipse defined above.

Actually the bisector line of $\mathbf{v}_B$ is a tangent line to the ellipse: For any other point P on this bisector line, the sum of the distance

$$|CP| + |OP| = |CP| + |v'_B P| > |Cv'_B|. \tag{6.17}$$

Now we can build up a point-to-point mapping between the constructed ellipse in Fig. 8.1 and the planet trajectory by setting up the polar coordinate. For the real space trajectory, the origin is set at F. For the reconstructed ellipse, we set the origin at the focus C. Then at each polar angle θ, the tangent lines on these two curves are parallel to each other. In the polar coordinate, the tangential vector of a curve represented by $\mathbf{r}(\theta) = r\mathbf{e_r}$

$$\frac{d\mathbf{r}}{dt} = \frac{dr}{d\theta}\mathbf{e_r} - r\mathbf{e}_\theta. \tag{6.18}$$

For the above two curves, we assume that their equations are

$$\mathbf{r_1} = r_1(\theta)\mathbf{e}_r, \quad \mathbf{r_2} = r_2(\theta)\mathbf{e}_r. \tag{6.19}$$

Since at the same polar angle, the tangent lines on the two curves are parallel to each other, i.e.,

$$\frac{1}{r_1}\frac{dr_1}{d\theta} = \frac{1}{r_2}\frac{dr_2}{d\theta}, \tag{6.20}$$

then we arrive at

$$\mathbf{r_1}(\theta) = C\mathbf{r_2}(\theta), \tag{6.21}$$

where C is a constant. This means that these two curves are similar to each other. In other words, the trajectory in real space is also an ellipse.

6.4 Comment on Kepler's 3rd law

The beauty of the geometric proof is impressive, and it also reveals the nature of the planet motion in a profound way. Next we will use a more advanced method of calculus for further explorations.

Kepler's 3rd law can be shown by a scaling method. Suppose $\vec{r}(t)$ is a solution to

$$\frac{d^2\mathbf{r}(t)}{dt^2} = -\frac{GM}{r^2}\mathbf{e_r}. \tag{6.22}$$

Perform a scaling transformation that

$$\mathbf{r}^s(t) = \lambda_1\mathbf{r}(\lambda_2 t). \tag{6.23}$$

It is easy to show that

$$\frac{d^2\mathbf{r^s(t)}}{dt^2} + \frac{GM}{r^{s,2}}\mathbf{e_r} = \lambda_1\lambda_2^2\frac{d^2\mathbf{r}(t)}{dt^2} + \lambda_1^{-2}\frac{GM}{r^{s,2}}\mathbf{e_r} = 0, \tag{6.24}$$

on condition that

$$\frac{\lambda_2^2}{\lambda_1^3} = 1. \tag{6.25}$$

This means that the spacial size of the orbit and the period of the orbit exhibit

$$L^2/T^3 = \text{const.} \tag{6.26}$$

Actually, Kepler's 3rd law is stronger than the above result. The length scale is only related to the major axis but independent on the minor axis.