西湖大學
WESTLAKE UNIVERSITY

Lect 7. Galilean transformation — inertial, non-inertial frames

Outline

1. Inertial frames, Galilean's relativity

All inertial frames are equal. One cannot distinguish one inertial frame from another.

2. non-inertial frame — modification of Newton's law

- tidal effect
- 12 hour period of tide — tide as a gudrapolar respose
- Solar tide and lunar tide

constructively superposition — spring tide

destructively — neap tide

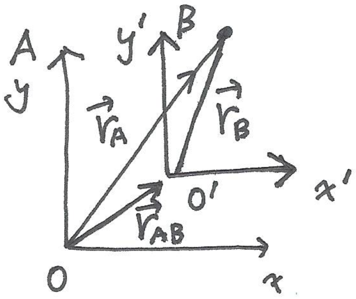
§ Inertial frame.

A frame in which Newton's 1st and 2nd laws are valid is called the inertial frame. Newton's 1st law essentially states the existance of the inertial frame. Actually an inertial frame is also an idealization, since we do not know which concrete frame is exactly inertial. The earth is a pretty good inertial frame approximatly, but it's spinning and rotated around the sun. The sun is also move around in the milky way.

① Suppose the reference frame A is an inertial frame. Another reference frame B is static with respect to A, but are related by a spacial shift. The transformation between the displacement vector in two frames are

$$\begin{cases} \vec{r}_A(t) = \vec{r}_{AB} + \vec{r}_B(t') \\ t = t' \end{cases}$$

t, and t' are time in reference frames A and B, respectively.



In this case,

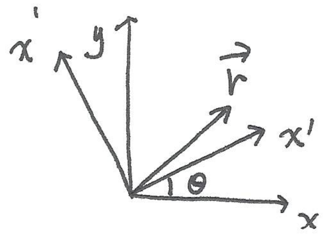
$$\begin{cases} \frac{d}{dt}\vec{r}_A(t) = \frac{d}{dt'}\vec{r}_B(t') \\ \frac{d^2}{dt^2}\vec{r}_A(t) = \frac{d^2}{dt'^2}\vec{r}_B(t') \end{cases} \Rightarrow \begin{aligned} &\text{In A-frame: } \vec{F}_A = m\vec{a}_A \\ &\text{In B-frame } \vec{F}_B = m\vec{a}_B \end{aligned}$$

$$\vec{F}_A = \vec{F}_B$$

Hence. Newton's 2nd law is also valid in the frame B,

Hence, the frame B is also inertial.

② suppose that the reference frame B is rotated at an angle with respect to frame A.



The coordinate transframatin is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} \quad \text{and} \quad t = t'$$

hence $$\begin{pmatrix} v_x \\ v_y \end{pmatrix} = \frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}}_{R_z(\theta)} \underset{\frac{d}{dt}}{\begin{pmatrix} x' \\ y' \end{pmatrix}} = R_z(\theta) \begin{pmatrix} v_{x'} \\ v_{y'} \end{pmatrix}$$

$$\begin{pmatrix} a_x \\ a_y \end{pmatrix} = R_z(\theta) \begin{pmatrix} a_{x'} \\ a_{y'} \end{pmatrix}.$$

On the other hand, force is also a vector. It's component is the two frames transform similarly as

$$\begin{pmatrix} F_x \\ F_y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} F_{x'} \\ F_{y'} \end{pmatrix}$$

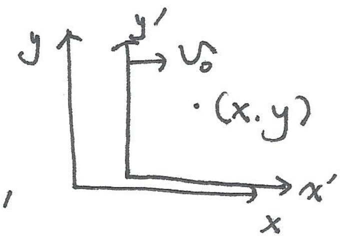
Then if in frame A, Newtons 2nd law is valid, i.e.

$$\begin{pmatrix} F_x \\ F_y \end{pmatrix} = m \begin{pmatrix} a_x \\ a_y \end{pmatrix}, \quad \text{then} \quad R_z(\theta) \begin{pmatrix} F_{x'} \\ F_{y'} \end{pmatrix} = m \, R_z(\theta) \begin{pmatrix} a_{x'} \\ a_{y'} \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} F_{x'} \\ F_{y'} \end{pmatrix} = m \begin{pmatrix} a_{x'} \\ a_{y'} \end{pmatrix},$$ i.e. it's also valid in frame B.

i.e. B is also an inertial frame.

③ If the frame B is moving with respect to A, they satisfy the Galileo transformation ← Galilean boost



$$\begin{cases} x = x' + v_0 t \\ y = y' \\ t = t' \end{cases} \Rightarrow \begin{cases} v_x = v_x' + v_0 \\ v_y = v_y' \end{cases} \Rightarrow \begin{cases} a_x = a_x' \\ a_y = a_y' \end{cases}$$

$\Rightarrow$ If $\vec{F} = m\vec{a}$ is valid in the frame A, it remains valid in the frame B.

> So far, we have proved that under translation, rotation, and Galilean boost transformations Newton's 2nd law is maintained!

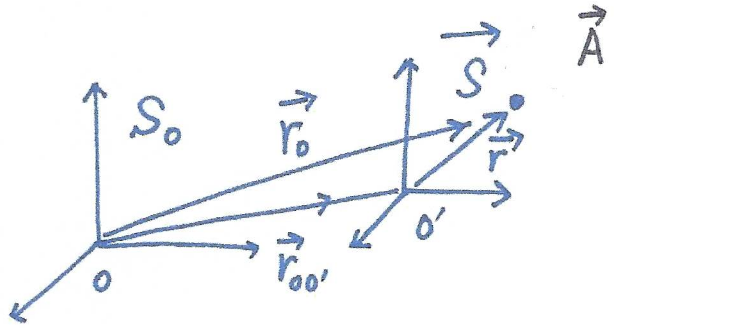
The collection of these operations form the symmetry group — Galilean group!

Galilean Relativity: Mechanical laws are the same in all inertial reference frames. A reference frame is not special than any other.

§ Non-inertial frames — no rotation

If a reference frame is rotating or acceleration with respect to an inertial frame, then Newton's laws of motion is no longer valid. It's then called the non-inertial frame. An accelerating car, train, elevator, are in non-inertial frames. And our earth rigorously speaking is also non-inertial due to its spin and orbital motion. It's convenient to use non-inertial frames since measurements are performed with respect to them. Then we have to modify Newton's 2nd law by adding fictitious forces, which are called inertial forces!

2. acceleration without rotation



S_0: inertial frame

S: non-inertial frame

$\vec{r}_0$: coordinate in S_0

$\vec{r}$: coordinate in S

$\vec{r}_{OO'}$: relative vecter between S_0.

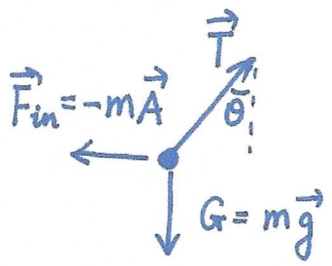
newton's law in S_0: $m\ddot{\vec{r}}_0 = F$

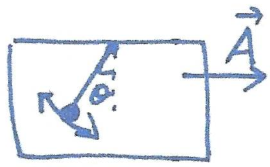
relation between S & S_0: $\vec{r}_0 = \vec{r}_{OO'} + \vec{r} \Rightarrow \ddot{\vec{r}} = \ddot{\vec{r}}_{OO'} + \ddot{\vec{r}}$
$= \vec{A} + \ddot{\vec{r}}$

$\Rightarrow$ law in frame S: $m\ddot{\vec{r}} = \vec{F} - m\vec{A} = \vec{F} + F_{inertial}$.

where $\boxed{\vec{F}_{inertial} = -m\vec{A}}$

Simple example: pendulum in an accelerating car



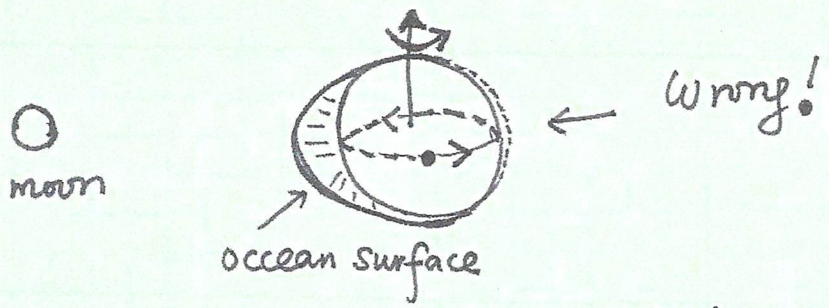


$\Rightarrow \quad m\ddot{\vec{r}} = \vec{T} + \vec{G} + \vec{F}_{in} = \vec{T} + m(\vec{g} - \vec{A})$

$\vec{g}_{eff} = \vec{g} - \vec{A} \Rightarrow |g_{eff}| = \sqrt{g^2 + A^2}, \quad tg\,\theta = \frac{A}{g}$

frequency $\omega = \sqrt{\frac{g_{eff}}{l}} = \sqrt{\frac{g}{l}}\sqrt{1+\left(\frac{A}{g}\right)^2} = \omega_0\sqrt{1+\left(\frac{A}{g}\right)^2}$

Application: tides — gravity of moon



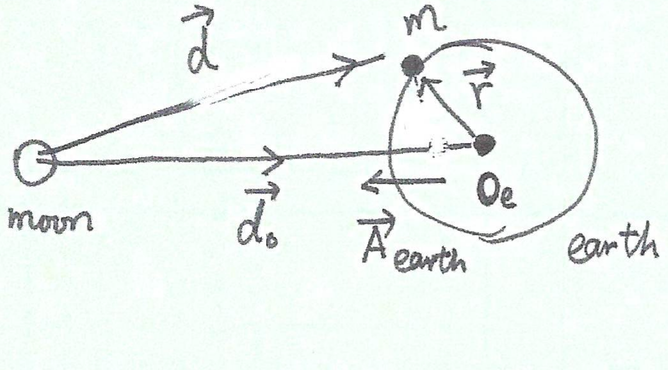
If the earth was an inertial frame, the height of the ocean surface would exhibit dipolar response to moon's gravity.

$\longrightarrow$ a single high tide per day, but this is not true!

$\boxed{\text{why? the earth is a non-inertial frame!}}$

The moon-earth system is orbiting around their center of mass

The acceleration of the earth is $\vec{A}_{earth} = -\frac{G\,m_{moon}}{d_0^2}\hat{d}_0$



The earth is falling to the moon at $\vec{A}_{earth}$.

O_e is the center of earth;

In the earth frame, the mass point

$$\begin{cases} m\ddot{\vec{r}} = \vec{F} - m\vec{A}_{earth} \\ \vec{F} = m\vec{g} + \vec{F}_{other} - \dfrac{G\,M_{moon}\,m\,\hat{d}}{d^2} \end{cases}$$

$$m\ddot{\vec{r}} = m\vec{g} + \vec{F}_{other} + \underbrace{(-GM_{moon}m)\left[\frac{\hat{d}}{d^2} - \frac{\hat{d_o}}{d_o^2}\right]}$$

$$\vec{F}_{tide} = -\frac{GM_{moon}m}{}\left(\frac{\hat{d}}{d^2} - \frac{\hat{d_o}}{d_o^2}\right)$$

$\vec{d} = \vec{d_o} + \vec{r}$ compare to d_o, $\left|\frac{r}{d_o}\right| \sim \frac{6.4\times10^3\,km}{3.8\times10^5\,km} \sim 0.02 \ll 1$

$$\Rightarrow \frac{\hat{d}}{d^2} = \frac{\vec{d}}{d^3} = \frac{\vec{d_o}+\vec{r}}{d_o^3\left(1+\left(\frac{r}{d_o}\right)^2 + \frac{2\vec{r}\cdot\vec{d_o}}{d_o^2}\right)^{3/2}} \simeq \frac{\vec{d_o}+\vec{r}}{d_o^3\left(1+\frac{3\vec{d_o}\cdot\vec{r}}{d_o^2}\right)}$$

$$\simeq \frac{\vec{d_o}}{d_o^3} + \left[\frac{\vec{r}}{d_o^3} - \frac{3\vec{d_o}}{d_o^3}\,\frac{\vec{d_o}\cdot\vec{r}}{d_o^2}\right]$$

$$\Rightarrow \vec{F}_{tide} = \frac{GM_{moon}m}{d_o^2}\left[-\frac{\vec{r}}{d_o} + \frac{3\hat{d_o}(\hat{d_o}\cdot\vec{r})}{d_o}\right]$$

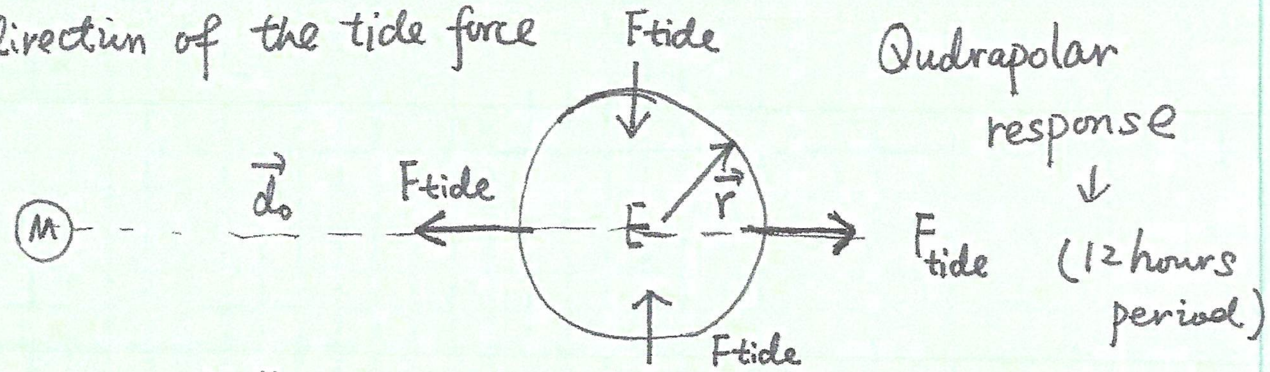
The tide force is the gradient of the gravity force from the moon, i.e. the gravity difference between the $\vec{r}$ and the gravity at the earth center O_e.

Here the gravity is from the moon!

Comment: If free free in a uniform gravity field, no tide force. you feel as if completely free, no force.

$$\vec{F} = m\vec{g} - m\vec{A} = m(\vec{g}-\vec{g}) = 0.$$

The direction of the tide force



$$\vec{F}_{tide} = \frac{G M_{moon} m}{d_o^2} \underbrace{\left[-\hat{r} + 3\hat{d_o}(\hat{d_o}\cdot\hat{r}) \right]}_{\text{angular form factor}} \frac{|r|}{d_o}$$

for $\vec{r}$ on the earth, if $\hat{r} \parallel \hat{d_o} \Rightarrow$

$\hat{r} \perp \hat{d_o} \Rightarrow$

$$\begin{cases} \vec{F}_{tide} = \dfrac{GM_m m}{d_o^2}\dfrac{|r|}{d_o} 2\hat{r} \\ \vec{F}_{tide} = \dfrac{GM_m m}{d_o^2}\dfrac{|r|}{d_o}(-\hat{r}) \end{cases}$$

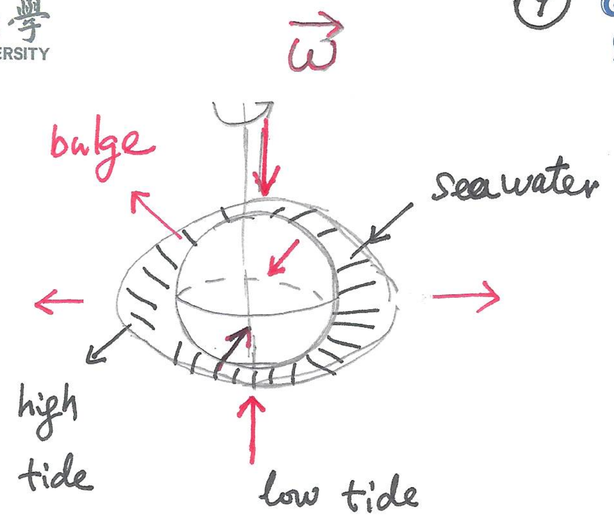
(*) Sun's contribution to the tidal effect

The sun also has the tidal force, Although the Sun's mass is much larger than the moon, but the tidal force decay at $1/r^3$. The solar mass $m_\odot = 2\times10^{30}$ Kg, and the lunar mas $m_{moon} = 7.3\times10^{22}$ Kg. The earth-sun distance $d_{es} = 1.5\times10^8$ km, moon-earth distance $d_{em} = 3.8\times10^5$ km.

$m_\odot/m_{moon} \approx 2.7\times10^7$, $\quad \dfrac{d_{es}}{d_{em}} = 3.9\times10^2 \Rightarrow \left(\dfrac{d_{es}}{d_{em}}\right)^3 \approx 6\times10^7$

hence $$\frac{m_\odot/(d_{es})^3}{m_{moon}/(d_{em})^3} \simeq \frac{2.7}{6} \approx 45\%$$

$\boxed{\text{The contribution to the tidal effects from the Sun is about half of that from the moon!}}$



㊟ The periodicity of the tides

* daily tidal period 12 hours instead of 24 hours

* In homework, you will figure out the alltitude differences between high and low tides $\Delta h_{moon} \simeq 54\,cm$. The sun's effect is about half, i.e. $\Delta h_{sun} \simeq 25\,cm$. They could superpose either constructively or destructively.

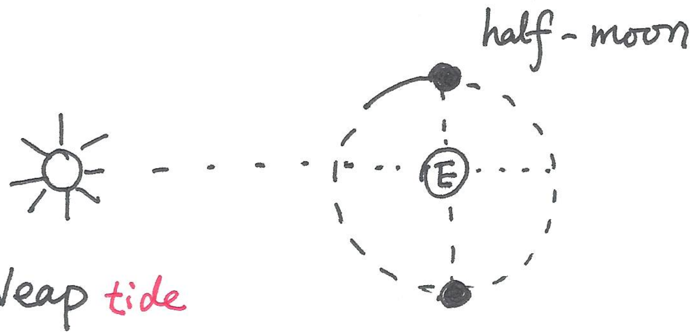
Solar tide

v.s lunar tide

Spring tide

$$\Delta h = \Delta h_{moon} + \Delta h_{sun} = 79\,cm$$

壬戌之秋，七月既望，
苏子与客泛舟于赤壁之下。



Neap tide

$$\Delta h = \Delta h_{moon} - \Delta h_{sun} = 29\,cm$$