INSTITUTE FOR THEORETICAL SCIENCES
理论科学研究院

浙江省杭州市西湖区墩余路 600 号 310030
No. 600 Dunyu Road, Sandun Town, Xihu District, Hangzhou, Zhejiang Province，310030
Email: its_office@westlake.edu.cn

Lecture 9 More on energy

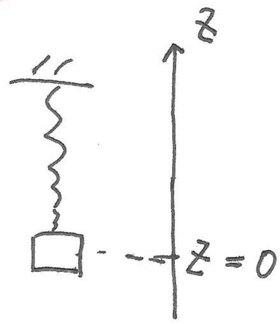
§ Several forces:

- If all of them are conservative forces, each of them gives rise to a potential: $\vec{F}_1 = -\nabla \mathcal{U}_1$, $\vec{F}_2 = -\nabla \mathcal{U}_2$, . .

then $\boxed{E = E_k + \mathcal{U}_1(\vec{r}) + \mathcal{U}_2(\vec{r}) + \cdots \quad \text{is conserved}}$

Example: spring in a gravity filed

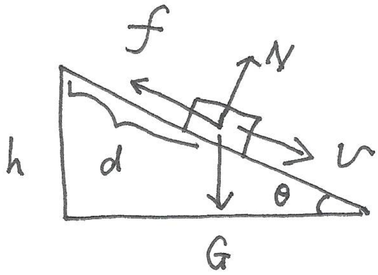
$$E = E_k + mgz + \frac{1}{2} k z^2$$



- non-conservative force

$$\Delta E_k = W_{con} + W_{ncon} = -\Delta \mathcal{U} + W_{ncon}$$

$$\Rightarrow \boxed{\Delta(E_k + \mathcal{U}) = W_{ncon}}$$



example: $\Delta(E_k + \mathcal{U}) = -fd = mg\sin\theta\,\mu\, d$

$E_{k,i} = 0$, $\mathcal{U}_i = mgh = mgd\sin\theta$

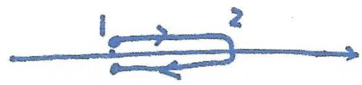
$T_f = ?$ $\quad \mathcal{U}_f = 0$ $\quad \Rightarrow T_f = mgd\sin\theta - mgd\cos\theta\,\mu = \frac{1}{2} m v_f^2$

$$v_f = \left[2gd(\sin\theta - \mu\cos\theta)\right]^{1/2}$$

§ 1D motion : F_x

If F_x is only coordinate-dependent, then F_x is conservative. Thi is because any closed loop in 1D has to come back along the same path

$$\boxed{\int_1^2 dx\, F_x + \int_2^1 dx\, F_x = 0}$$



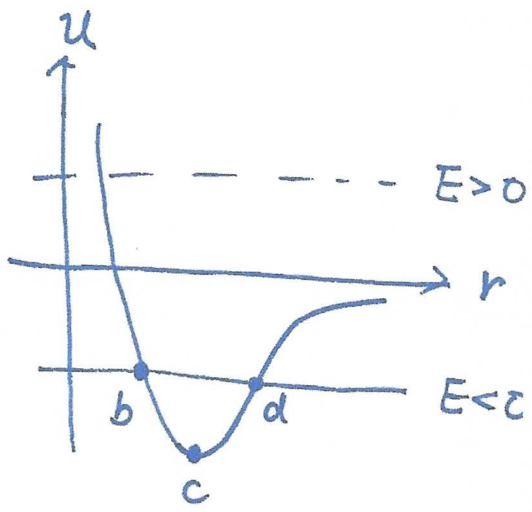
Then the potential energy $U(x)$ can be simply integrated as

$$\boxed{U(x) = -\int_{x_0}^{x} F_x(x')\,dx'}$$

x_0 can be any point

$U(x)$ with different x_0 is up to a constant.

potential energy for a diatomic molecule.



① $E<0$, at b and $d \Rightarrow T=0$, turning points. At c, $\frac{\partial U}{\partial r}=0$, $\frac{\partial^2 U}{\partial r^2} \boxed{>0}$.

c is equilibrium point

$E<0$ — bound states

② $E>0$ — scattering states

we can formally complete solution of motion in 1D

$$T = \frac{1}{2} m \dot{x}^2 = E - U(x) \Rightarrow \dot{x}(x) = \pm \sqrt{\frac{2}{m}} \sqrt{E - U(x)}$$

The direction of $\dot{x}(x)$ can be either right / left mover.

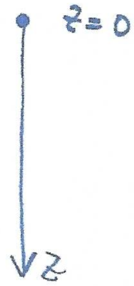
we also have $\dot{x} = \frac{dx}{dt} \Rightarrow dt = \frac{dx}{\dot{x}(x)}$

$$\Rightarrow \int_{t_i}^{t_f} dt = \boxed{\int_{x_i}^{x_f} \frac{dx}{\dot{x}(x)} = t_f - t_i}$$

Suppose $\dot{x}$ is positive, we have $t_f - t_i = \sqrt{\frac{m}{2}} \int_{x_0}^{x} \frac{dx'}{\sqrt{E - U(x')}}$.

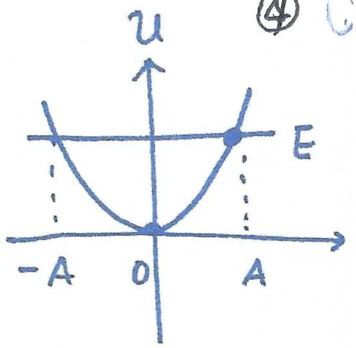
$\dot{x}$ can change directions at turning points, and we can treat by dividing the motion into different regions. In each region, $\dot{x}$'s direction is fixed, and we add the time of each reigon together.

Example: free fall: $U(z) = -mgz$ and $E = 0$ $\left\{ \begin{array}{l} \text{at } z_{in} = 0 \\ v_{in} = 0 \end{array} \right.$



$$\Rightarrow \dot{z}(z) = \sqrt{\frac{2}{m}} \sqrt{E - U(z)} = \sqrt{2gz}$$

$$t = \int_0^z \frac{dz'}{\dot{z}(z')} = \int_0^z \frac{dz'}{\sqrt{2gz'}} = \sqrt{\frac{2z}{g}} \Rightarrow z = \frac{1}{2} g t^2$$



2: harmonic oscillator

$U = \frac{1}{2} k x^2$ with energy E.

The turning points at $\pm A$, with $\frac{1}{2} k A^2 = E$.

Conside at $\begin{cases} t_{in} = 0 \\ x_0 = A \end{cases}$ and at $\begin{cases} t_f = T/4 \\ x_f = 0 \end{cases}$

we have $\dot{x}(x) = -\sqrt{\frac{2}{m}} \left(E - \frac{1}{2} k x^2\right)^{1/2}$

$$\Rightarrow T/4 = +\int_A^0 \frac{dx}{\dot{x}} = \sqrt{\frac{m}{2}} \int_0^A dx \, \frac{1}{\left(E - \frac{1}{2} k x^2\right)^{1/2}}$$

$$= \sqrt{\frac{m}{2}} \left(\frac{k}{2}\right)^{-1/2} \cdot \int_0^A dx \, \frac{1}{A\left(1 - \left(\frac{x}{A}\right)^2\right)^{1/2}}$$

$$= \sqrt{\frac{m}{k}} \int_0^1 dy \, \frac{1}{\left(1 - y^2\right)^{1/2}} = \omega_0^{-1} \, \text{arsin}\, y \Big|_0^1 = \frac{\pi}{2\omega_0}$$

$$\Rightarrow T = \frac{2\pi}{\omega_0} \quad \text{where } \omega_0 = \sqrt{k/m} .$$

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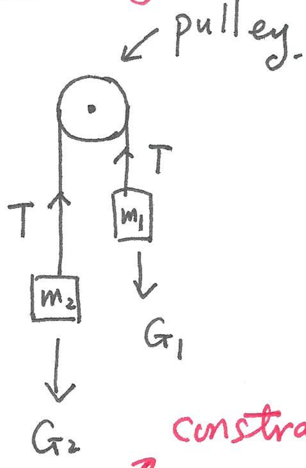
§ Several objects — Atwood machine with constraint

T: Constraining force.



Two masses suspended by a massless inextensible string

$$\Delta(E_{K_1} + U_1) = W_1^T$$

$$\Delta(E_{K_2} + U_2) = W_2^T$$

constraint

$$\boxed{d(S_1 + S_2) = 0}$$

Tensions on m_1 and m_2 are the same, but no elastisity, hence

$$W_1^{ten} + W_2^{ten} = \int dS_1\, W_1^{ten} + \int dS_2\, W_2^{ten} = \int dS_1\, T + \int dS_2\, T$$

$$= \int T(dS_1 + dS_2) = 0$$

$$\Rightarrow \Delta(\underbrace{E_{K_1} + E_{K_2} + U_1 + U_2}_{E}) = 0$$

In general, if a system contains several particles with constrain, if the constraining force does not do the total work on the system, they can be neglected in the total energy.

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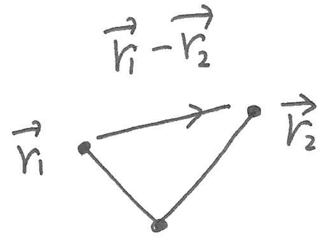
§ Energy of two interacting particles



$$\begin{cases} \vec{F}_{12} = \vec{F}_{12}(\vec{r}_1 - \vec{r}_2) & \text{— translation sym} \\ \vec{F}_{12} = -\vec{F}_{21} \end{cases}$$

interaction only depends on the relative displacement

$\vec{F}_{12}(\vec{r}_1 - \vec{r}_2)$, if for fixed $\vec{r}_2$, is a conservative force for $\vec{r}_1$, i.e.

$\oint d\vec{r}_1 \cdot \vec{F}_{12}(\vec{r}_1 - \vec{r}_2) = 0$, then we express $\boxed{\vec{F}_{12} = -\nabla_{\vec{r}_1} U_{12}(\vec{r}_1 - \vec{r}_2)}$.

then the same potential can also give rise to

$$\boxed{\vec{F}_{21} = -\nabla_{\vec{r}_2} U_{12}(\vec{r}_1 - \vec{r}_2) = -\vec{F}_{12}} \leftarrow \text{Newton's 3rd law}$$

Now apply the work - kinetic theorem,

$$\left.\begin{aligned} dE_{K_1} &= d\vec{r}_1 \cdot \vec{F}_{12} \\ dE_{K_2} &= d\vec{r}_2 \cdot \vec{F}_{21} \end{aligned}\right\} \Rightarrow \begin{aligned}[t] d(E_{K_1} + E_{K_2}) &= \vec{F}_{12} \cdot (d\vec{r}_1 - d\vec{r}_2) \\ &= d(\vec{r}_1 - \vec{r}_2) \cdot (-\nabla_1 U_{12}(\vec{r}_1 - \vec{r}_2)) \\ &= -d\vec{r} \cdot \nabla U_{12}(\vec{r}) = -dU(\vec{r}) \end{aligned}$$

$\vec{r} = \vec{r}_1 - \vec{r}_2 \leftarrow$ relative coordinate

$$d(\underbrace{E_{K_1} + E_{K_2} + U(\vec{r})}_{E}) = 0$$

In principle, we can also include the external $\underbrace{\text{forces}}_{\text{conservative}}$ on 1 and 2,

and introduce potentials $\mathcal{U}_1^{ex}$ and $\mathcal{U}_2^{ex}$, then

$$E = T_1 + T_2 + \mathcal{U}_1^{ex} + \mathcal{U}_2^{ex} + \mathcal{U}_{12}.$$

This process can be generalized to n-particle conservative systems, with

$$E = T_1 + T_2 + \cdots T_n + \mathcal{U}_1^{ex} + \mathcal{U}_2^{ex} + \cdots \mathcal{U}_n^{ex}$$

$$+ \mathcal{U}_{12} + \cdots \mathcal{U}_{1n} + \mathcal{U}_{23} + \cdots \mathcal{U}_{2n} + \cdots \mathcal{U}_{n-1,n}$$

$$\Rightarrow \boxed{E = \sum_{i=1}^{n} \left(T_i + \mathcal{U}_i^{ex}\right) + \sum_{i<j} \mathcal{U}_{ij}}$$

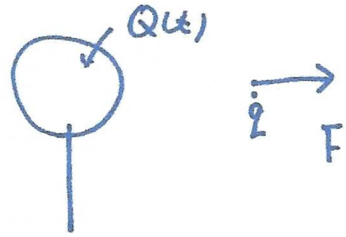
$\sum_{i=1}^{n} \left(T_i + \mathcal{U}_i^{ex}\right) \leftarrow$ single body

$\sum_{i<j} \mathcal{U}_{ij} \leftarrow$ interaction

double couting excluded!

§ Time-dependent potential energy

if $\vec{F}(\vec{r},t)$ satisfies $\oint d\vec{r}\cdot\vec{F}(\vec{r},t)=0$, but it's time-dependent, then we can still write $\vec{F}(\vec{r},t) = -\nabla \mathcal{U}(\vec{r},t)$. Nevertheless $E = T + \mathcal{U}$ is no-longer conserved.



$$F = \frac{kqQ(t)}{r^2}$$

For a changing charge $Q(t)$, we can still define $\mathcal{U}(\vec{r},t) = -\int_{\vec{r}_0}^{\vec{r}} \vec{F}(\vec{r}\,',t)\, d\vec{r}\,'$.

Now check $dT = \frac{dT}{dt}\, dt = \frac{d}{dt}\left(\frac{1}{2}mv^2\right) dt = m\dot{\vec{v}}\cdot\vec{v}\, dt = \vec{F}\cdot d\vec{r}$

$$d\mathcal{U}(\vec{r},t) = \frac{\partial \mathcal{U}}{\partial x} dx + \frac{\partial \mathcal{U}}{\partial y} dy + \frac{\partial \mathcal{U}}{\partial z} dz + \frac{\partial \mathcal{U}}{\partial t} dt$$

$$= \nabla \mathcal{U}\cdot d\vec{r} + \frac{\partial \mathcal{U}}{\partial t} dt = -\vec{F}\cdot d\vec{r} + \frac{\partial \mathcal{U}}{\partial t} dt$$

$$\Rightarrow dT = -d\mathcal{U} + \frac{\partial \mathcal{U}}{\partial t} dt \qquad \Rightarrow \boxed{d(T+\mathcal{U}) = \frac{\partial \mathcal{U}}{\partial t} dt}$$