Problem 1: The image method

Consider a semi-infinite string, fixed at the origin $x = 0$ and extending far out to the right $x \to +\infty$. Let $f(\xi)$ be a function of a triangular wave localized around the origin.

$$\begin{aligned} f(\xi) &= 0 \qquad (-\infty < \xi < -a) \\ &= \xi + a \quad (-a < \xi < 0) \\ &= -\xi + a \quad (0 < \xi < a) \\ &= 0 \qquad (a < \xi) \end{aligned} \tag{1}$$

(a) Describe the wave given by the function $f(x + ct)$ for a large negative time t_0.

(b) One way to solve for the subsequent motion of this wave on the semi-infinite string is called the method of images and is as follows:

Consider the function $u = f(x + ct) - f(-x + ct)$. (The second term here is called the "image." Can you explain why?) Obviously this satisfies the wave equation for all x and t. Show that it coincides with the given wave of part (a) at the initial time to and everywhere on the semi-infinite string.

Show also that it obeys the boundary condition that $u = 0$ at $x = 0$.

c) It is a fact that there is a unique wave that obeys the wave equation and any given initial and boundary conditions. Therefore the wave of part (b) is the solution for all times (on our semi-infinite string).

Describe the motion on the semi-infinite string for all times. You need to pay special attention to the process when the wave is bounced back from the boundary.

<u>Problem 2: Waves in 3D</u>

The wave equation in 3D is

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}\right) f(x,y,z,t) = (\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2)f(x,y,z,t). = 0 \quad (2)$$

1) Let $f(\xi)$ be any smooth function and let $\hat{\mathbf{n}}$ be an arbitrary fixed unit vector. Show that $\nabla f(\hat{\mathbf{n}}\cdot\mathbf{r}-ct) = \hat{\mathbf{n}}f'(\hat{\mathbf{n}}\cdot\mathbf{r}-ct)$. Hence show that $\nabla f(\mathbf{n}\cdot\mathbf{r}-ct)$ satisfies the three-dimensional wave equation.

Argue that $f(\mathbf{n}\cdot\mathbf{r}-ct)$ represents a signal that is constant in any plane perpendicular to $\hat{n}$ (at any fixed time t) and propagates rigidly with speed c in the direction of $\hat{n}$.

2) Let $f(r)$ be any spherically symmetric function; that is, when expressed in spherical coordinates, (r,θ,φ), it has the form $f(\mathbf{r}) = f(r)$, independent of θ and φ. Using the formula that

$$\nabla^2 f = \frac{1}{r}\frac{\partial^2}{\partial r^2}(rf), \quad (3)$$

find the spherical wave solution to Eq. 2, i.e., $f(x,y,z,t) = f(r,t)$ with $r = \sqrt{x^2+y^2+z^2}$.

3) Can you also find the cylindrical wave solution to Eq. 2, i.e., $f(x,y,z,t) = f(\rho,z,t)$ where $\rho = \sqrt{x^2+y^2}$. You may look on the internet for the formula of ∇^2 in the cylindrical coordinate.

Problem 3: Matter waves – Quantum mechanics

De broglie proposed that a quantum particle are also a wave following $p = \hbar k$ where p is its momentum and k is the wavevector. p and k are bridged by $\hbar = h/(2\pi)$ with h the Planck constant.

Since a particle is also a wave, there must be a wave equation for it. Then such a wave equation was constructed by Schödinger and was named after him – the Schödinger equation. For example, for an electron, it is

$$i\hbar\frac{\partial}{\partial t}\Psi(\mathbf{r},t) = -\frac{\hbar^2}{2m}\nabla^{\mathbf{2}}\Psi(\mathbf{r},t) + V(\mathbf{r})\Psi(\mathbf{r},t), \quad (4)$$

where m is the electron mass, and V is the electric potential energy.

1) Let us consider free space, i.e., $V = 0$. Compare the Schrödinger equation with the 3D sound wave equation. What are the similarities and the differences?

2) Try the plane-wave solution to the Schrödinger equation in free space,

$$\Psi(\mathbf{r},t) = f(\mathbf{r})e^{-i\omega t}. \quad (5)$$

where the spatial part $f(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}}$. What is the relation between the circular frequency ω and the wavevector k? Using the quantum mechanical energy-frequency relation, show that the above relation yields the correct energy-momentum relation of a non-relativistic particle with the mass m.

Schrödinger felt very uncomfortable about the appearance of "i" in his equation since all classic wave equations only involve real numbers. He tried very hard to remove "i" but he failed. Explain why i has to appear in the Schrödinger equation, but not in the wave equation of sound?

3) Consider a simplified problem that an electron lives in a one dimensional potential-well with $V(x) = 0$ for $0 < x < 2a$ and $V(x) = +\infty$ otherwise. Solve the eigen-modes in such a potential well. You will find that the lowest energy is not zero, which is call the zero-point kinetic energy E_{K_0}. How does E_{K_0} scale with a? (Hint: $\Psi(x)$ needs to vanish at $V = +\infty$. You should figure out the correct boundary conditions.)

4) The hydrogen atom Bohr radius a can be estimated in the following way: Electrons are confined with a length scale of size a, and then its momentum is roughly $p = \hbar/a$, and hence, its kinetic energy is $\hbar^2/(2ma^2)$. Then the electric potential energy is roughly $-e^2/a$. Add them together, and minimize

the total energy as a function of a. At what value of a, that the total energy reaches the minimum, and what is the minimum energy?

Problem 4: Lorentz space-time transformation

1) Following the lecture note, derive the Lorentz space-time transformations without using the light velocity invariance as most text books commonly do. What fundamental properties of space, time, and reference frame are used?

2) Prove two characteristic consequences of Lorentz transformations – time-dilation and length contraction.