Problem 1: Adiabatic processes

In class, we have derived that during an adiabatic process, the pressure P and volume V of an ideal gas satisfy $PV^\gamma =$ Const. On the other hand, an ideal gas satisfies the equation of state of $PV/T =$ constant. (Here I assume that the thermodynamic temperature T is already defined as you learned in high school. We will learn its deep meaning soon.)

1) Please derive the relation between P and T, and V and T during an adiabatic process.

2) Consider two samples of gases A and B, which are monoatomic and diatomic molecular gases, respectively. Their initial volumes are the same, and their pressure are also the same. They undergo the adiabatic compression, and their volumes are compared to half of their initial volume. Compare the ratios of their final and initial pressures. (Assume that the $\gamma_A = 5/3$ for the monoatomic molecular gas, and $\gamma_B = 7/5$ for the diatomic molecular gas.

3) Calculate the ratio of the works done in the above adiabatic processes of two gases.

Problem 2: Free expansion

A container is divided into two parts by a partition. One of these parts has a volume V; and is filled with a dilute gas; the other part is empty. We assume that there are no interactions between gas molecules. Remove the partition and wait until the final equilibrium condition is attained where the molecules of the gas are uniformly distributed throughout the entire container of volume V_f.

(a) Has the total energy of the gas been changed? Use this result to compare the average energy per molecule and the average speed of a molecule in the equilibrium situations before and after the removal of the partition.

(b) What is the ratio of the pressure exerted by the gas in the final situation to that of the pressure exerted by it in the initial situation?

Problem 3: Equal partition theorem & Maxwell distribution

In Feynman Lecture Vol (I) Lecture 39, it presents a beautiful explanation of the equal partition of energy among different degrees of freedom. The deep reason arises from the complete randomness, which Feynman would like to express but did not express it clearly. Here we examine it in a more general perspective.

Suppose that we do not know the Maxwell or Boltzmann distributions as a priori. We will try to understand why it has to be the case.

There are billions of degrees of freedom in a gas. Let us pick up two of them, and express them in terms of two independent variables a_1, a_2. They are re-scaled, such that they contribute to the energy as

$$E = \tfrac{1}{2}(a_1^2 + a_2^2). \tag{1}$$

For example, it could be $a = p/\sqrt{m}$ for the kinetic energy with p the momentum, or, $\sqrt{K}x$ for the elastic potential energy for intra-molecular vibration with K the spring constant. In the thermal equilibrium, we assume that only $\langle a_i^2 \rangle (i = 1, 2)$ can be nonzero, but the inter-variable correlation should vanish, i.e., $\langle a_1 a_2 \rangle = 0$, where $\langle \rangle$ means thermal average.

1) Now let us reorganize these variables as

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \tag{2}$$

such that $E = \tfrac{1}{2}(b_1^2 + b_2^2)$. Prove that if we assume the determinant of the transformation matrix to be positive, the matrix element can be expressed in terms of an angle θ as

$$A = D = \cos\theta, \quad -B = C = \sin\theta. \tag{3}$$

In other words, it could be viewed as a rotation.

A collision between two atoms with non-equal mass m_1 and m_2 can also be decomposed into the center of mass motion and the relative motion. Figure out the transformation from $\mathbf{p}_1$ and $\mathbf{p_2}$ to the center mass momentum $\mathbf{P}$ and the momentum $\mathbf{p}$ of the relative motion, and the transformation angle.

2) If the system is completely random, then we should have $\langle b_1 b_2 \rangle = 0$ for an arbitrary θ. Otherwise, there still exist certain correlations in a suitably

chosen basis. Prove that this requires that $\langle a_1^2 \rangle = \langle a_2^2 \rangle$, which is equivalent to the equal partition theorem.

3) The probability distribution function $\rho(a_1, a_2)da_1da_2$ gives rise to the relative probability for the system variables lying from a_1 to $a_1 + da_1$ and a_2 to $a_2 + da_2$. Since a_1 and a_2 are independent and are symmetric in Eq. 1, we arrive at

$$\rho(a_1, a_2) = f(a_1^2)f(a_2^2). \tag{4}$$

When expressed in variables b_1 and b_2, we should have

$$f(a_1^2)f(a_2^2) = f(b_1^2)f(b_2^2). \tag{5}$$

Prove that only the exponential functions can satisfy the above relation, i.e.,

$$f(a^2) = e^{-\frac{\beta}{2}a^2}, \tag{6}$$

where β is a common parameter for all degrees of freedom.

4) We define the temperature according to $\frac{1}{2}\langle a^2 \rangle = k_B T$. Based on this definition and properties of the Gaussian integral, prove that $\beta = \frac{1}{k_B T}$.

Problem 4: Diatomic molecular gas pressure

In class, we derived that the pressure $PV = Nk_BT$. Let us consider an atomic gas and a diatomic molecular gas, and examine their similarity and difference. Consider two boxes with the same volume. They are filled with the same numbers of O_2 molecules and He atoms, respectively. The two systems are at the same temperature T.

1) Why these two systems exhibit the same pressure in spite of the fact that the oxygen box actually has twice number of atoms as the helium box? Please examine the collision processes with the box wall carefully.

2) Imagine that we could increase the temperature from room temperature to very very high, say, 10^5K, although no materials can sustain at such a high temperature. How do the pressures change in these two boxes? Are they any differences? Why?