Problem 1: Sailing against the wind

Life is not always a sailing following the wind. Sometimes, you have to sail against the wind to cross the river. You can control the heading direction of the sailing boat, and the direction of the sail. The angle between the directions of the wind and sail is denoted as α, and the angle between the sail and the heading direction is denoted as β, as shown in the figure.

1) Please explain why you could cross the river against the winder after all.

2) Then please determine the optimal values of α and β such that you can cross the river as quickly as possible. You can regard the velocity of the boat is proportional to the force along the heading direction.

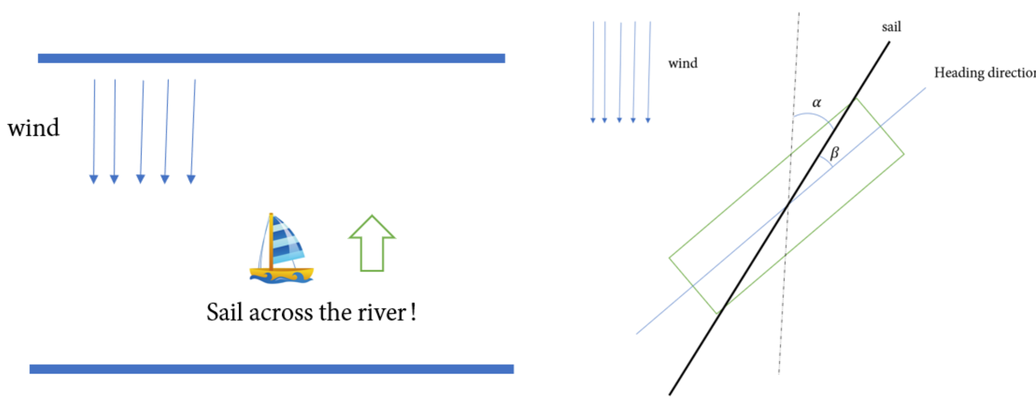


Figure 1: The relative angles α, β between the wind, the sail, and the heading directions.

(Hint: To find the extreme value with respect to two parameters, you can first find an extreme value for one parameter with the other one fixed.)

Problem 2: Vector analysis

1. $\vec{r}_1$ and $\vec{r}_2$ are arbitrary vectors in the 3D space. Try to verify that $\vec{r}_1 \cdot \vec{r}_2$ is invariant under rotations around the x, y, z axes, respectively. If you can prove it for arbitrary rotations, you can get extra points.

2. Verify that $\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{b} \cdot (\vec{c} \times \vec{a}) = \vec{c} \cdot (\vec{a} \times \vec{b})$. Using this, try to verify that the Lorentz force does not do any work.

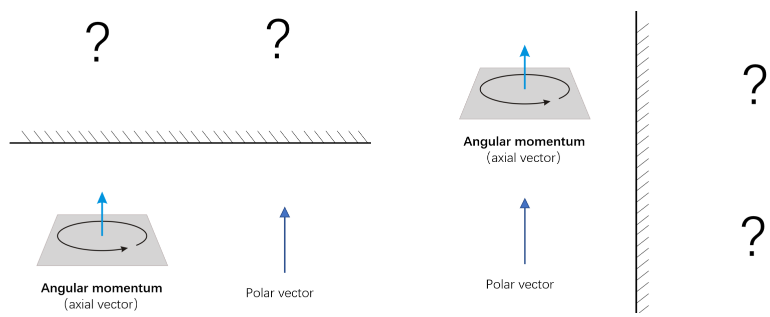
3. Verify that $(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a} \cdot \vec{c})\, \vec{b} - (\vec{b} \cdot \vec{c})\, \vec{a}$.

Problem 3: Polar vector v.s. axial vector
The usual vectors like displacement, velocity, acceleration are called polar vectors. The angular momentum of a particle is defined as the cross product between two polar vectors

$$\vec{L} = m\vec{r} \times \vec{v}, \tag{1}$$

which is called the axial vector.

Under spatial rotation, an axial vector transforms in the same way as a polar vector. Nevertheless, they behave differently under the mirror reflection. Please find out what they look like in a mirror.



(a) (b)

Figure 2: Polar and axial vectors under the mirror reflections. a) The mirror is parallel to the vectors; b) the mirror is perpendicular to the vectors.

Problem 4: Can a projectile reach its maximum height in a finite amount of time or not?

When you throw your coin up, you will observe it goes up and then fall into your hands. But some ancient people were confused about it. They argued in the ascending part of process, when the coin goes upper and upper, its velocity becomes slower and slower. Within the same distance of ΔL, the coin will spend more and more time. When the coin is approaching the peak, its velocity goes to zero. It is unclear whether it could reach its peak at all! The descending process is basically the time-reversal counterpart of the ascending one. It is unclear whether if the coin will fall at all since it starts from the zero velocity.

Please following the similar analysis as we examine the Zeno paradox in class to show that the coin will fall into your hand.

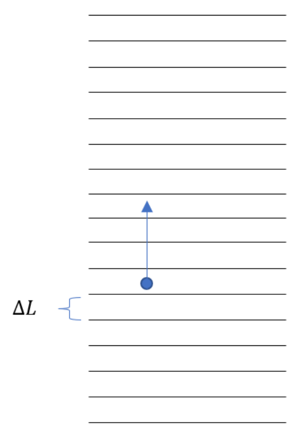


Figure 3: The up-going coin

Problem 5: Tangential and normal accelerations

For a general motion along a curve, the natural coordinate system is more convenient. Around each point at the trajectory, we can use a small part of a circle to approximate the curve, such that the tangential direction and the curvature are maintained. We set the unit vector $\vec{e}_\tau$ along the tangent direction and that of $\vec{e}_n$ along the normal direction. As shown in the figure, the acceleration is be decomposed into the tangential and normal components, accordingly, i.e.,

$$\vec{a} = \frac{dv}{dt}\vec{e}_\tau - \frac{v^2}{\rho}\vec{e}_n, \tag{2}$$

where v is the velocity and ρ is the radius of curvature.

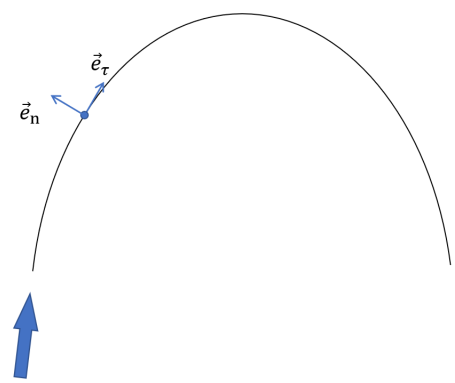


Figure 4: The tangential and normal accelerations.

1. The radius of the curvature can be formulated as

$$\frac{1}{\rho} = |\frac{d^2y}{dx^2}|/(1 + (\frac{dy}{dx})^2)^{3/2}. \tag{3}$$

Please verify that for a circle, it indeed gives the correct results.

2. Consider the projectile motion with an initial velocity $v_x = v_0 \cos\theta$ and $v_y = v_0 \sin\theta$. When the projectile reaches the maximal height, calculate the tangential and normal accelerations at this point. Conform that your results are consistent with the gravity acceleration.